ROUGH FUZZY (FUZZY ROUGH) STRONG h-IDEALS OF HEMIRINGS

Jianming Zhan
Qi Liu

Department of Mathematics
Hubei University for nationalities
Enshi, Hubei Province 445000
China
e-mails: zhanjianming@hotmail.com
390181718@qq.com

Hee Sik Kim

Department of Mathematics
Research Institute for Natural Sciences
Hanyang University
Seoul, 133-791
Korea
e-mail: heekim@hanyang.ac.kr

Abstract. By means of Dubois and Prade's idea, we apply rough fuzzy sets and fuzzy rough sets to algebraic structures. The concepts of rough fuzzy strong h-ideals (rough fuzzy prime ideals) and fuzzy rough strong h-ideals (fuzzy rough prime ideals) of hemirings are introduced, respectively. The relationships between them are investigated. Some characterizations of these two kinds of rough set theory of hemirings are explored.

Keyword: rough set; strong h-ideal; rough fuzzy strong h-ideal; fuzzy rough strong h-ideal; rough fuzzy prime ideal; fuzzy rough prime ideal; hemiring.

2000 Mathematics Subject Classification: MSC 2010; 16Y60; 16Y99.

1. Introduction

Rough set theory, a new mathematical approach to deal with inexact and uncertain knowledge, was originally proposed by Pawlak [11]. There are at least two methods for the development of this theory, the constructive and axiomatic approaches. In constructive methods, lower and upper approximations are constructed by basic concepts, such as equivalence relations on a universal set and neighborhood systems [12]. The Pawlak approximation operators are defined by an equivalence relation. However, an equivalence relation in Pawlak rough set models seem to be a very restrictive condition. Hence, some more general models have been put forward, such as [10], [15], [20], [22]. In 1990, Dubois and Prade [4] introduced the concepts of fuzzy rough sets and rough fuzzy sets. After that, some researchers put forward some generalized fuzzy rough sets, such as [21]. Nowadays, this theory has been applied successfully to many areas, such as knowledge discovery, machine learning, information sciences, and intelligent systems, and so on.

It naturally leads to a question of what happen if we substitute an algebraic system instead of a universe set. In 1997, Kuroki [7] studied the rough sets in semigroups and put forward two new algebraic structures, called rough semigroups and rough ideals. Further, Davvaz [2], [3] introduced the concept of rough ideals with respect to an ideal of rings. After that, many researches investigated rough set theory in different algebraic structures, for example, see [1], [6].

We know that the ideals of semirings play a crucial role in the structure theory, but ideals in semirings do not in general coincide with the usage of ideals in semirings is somewhat limited. To overcome this difficulty, Torre [13] studied h-ideals and k-ideals of hemirings. In 2004, Jun [5] applied fuzzy set theory to hemirings. Zhan and Dudek [18] introduced the concept of h-hemiregular hemirings and investigated this kind of hemirings by fuzzy h-ideals. Further, some characterization of h-hemiregular and h-intra-henmiregular henmirings were investigated by Yin [16], [17]. In particular, some generalized fuzzy h-ideals of hemirings were studied by Ma [8], [9].

The paper is organized as follows. In Section 2, we recall some concepts and results on rough sets and hemirings. In Section 3, we introduce the concept of rough fuzzy strong h-ideals and rough fuzzy prime ideals of hemirings and investigate some related properties. Finally, some characterizations of fuzzy rough strong h-ideals and fuzzy rough prime ideals of hemiring are investigated in Section 4.

2. Preliminaries

In this section, we divide into three parts. Some basic concepts and results on hemirings, strong h-ideals, congruence relations and rough sets are pointed out in these three subsections.

2.1. (Prime) strong h-ideals

By zero of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring with a zero and a commutative semiring $(S, +)$ is called a hemiring[13]. In this paper, S always denotes a hemiring.

A non-empty subset A of S is called a subhemiring of S if A is closed under addition and multiplication. A non-empty subset A of S is a left(right) ideal of S if A closed under addition and $SA \subseteq A(AS \subseteq A)$. An ideal A of S is prime ideal such that $xy \in A$ for some $x, y \in S$ implies $x \in A$ or $y \in A$. A subhemiring

A of S is called an h-subhemiring if $x, z \in S, a, b \in A$ and $x + a + z = b + z$, implies $x \in A$. Left(right) h-ideals are defined similarly. A subhemiring A of S is called a strong h-subhemring if $x, y, z \in S, a, b \in A$ and $x + a + z = y + b + z$ implies $x \in y + A$. Strong h-ideals are defined similarly. Clearly, every strong h-subhemiring (h-ideal) is an h-subhemiring (h-ideal).

2.2. Congruence relations

Recall that an equivalence relation θ on S is a reflexive, symmetric and transitive binary relation on S. If θ is an equivalence relation on S, then the equivalence class of $x \in S$ is the set $\{y \in S|(x, y) \in \theta\}$, denoted by $[x]_\theta$. An equivalence relation θ on S is called a congruence relation if $(a, b) \in \theta$ implies $(a+x, b+x) \in \theta$ and $(ax, bx) \in \theta$ and $(xa, xb) \in \theta$ for all $x \in S$.

A congruence relation θ on S is called complete if $[ab]_\theta = \{xy|x \in [a]_\theta, y \in [b]_\theta\}$ for all $a, b \in S$.

Let I be a strong h-ideal of S, $x, y \in S$. We say x is congruent to y modulo I, denoted by $x \equiv y(\text{mod } I)$, if and only if there exist $a, b \in I$ and $z \in S$ such that $x + a + z = y + b + z$. Clearly, $x \equiv y(\text{mod } I)$ is a congruence relation on S.

Lemma 2.1 [17] *Let I be a strong h-ideal of S. If $x, y \in S$, then*

(1) $x \in [y]_I$ *if and only if* $x \in y + I$,

(2) $[x]_I + [y]_I = [x + y]_I$,

(3) $\{ab|a \in [x]_I, b \in [y]_I\} \subseteq [xy]_I$.

Remark 2.2 If I is an h-ideal of S, then the assertions (1) and (2) in above lemma may not be true as shown in the following.

Example 2.3 The set $\mathbb{N}_0$ of all non-negative integers with usual addition and multiplication operations is a hemiring, let $I =< 3 >$, then I is an h-ideal, but it is not a strong h-ideal of S, since $1+6+1 = 4+3+1$, but $1 \notin 4+I$. By calculations, $[4]_I = \{1, 4, 7, 10, ...\}$, $[5]_I = \{2, 5, 8, 11, ...\}$ and $[9]_I = \{0, 3, 6, 9, 12, ...\}$. Thus, we have

(1) $1 \in [4]_I$, but $1 \notin 4 + I$.

(2) $[4]_I + [5] \neq [9]_I$.

Proposition 2.4 *Let I be an idempotent ($II = I$) strong h-ideal of S. If $x, y \in S$, then*

$$\{ab|a \in [x]_I, b \in [y]_I\} = [xy]_I.$$

Proof. Let $c \in [xy]_I$, by Lemma 2.1 (1), we have $c \in xy + I = xy + II \subseteq xy + xI + Iy + II = (x + I)(y + I) = [x]_I[y]_I$, that is, $[xy]_I \subseteq [x]_I[y]_I$. Combing Lemma 2.1(3), we have $\{ab|a \in [x]_I, b \in [y]_I\} = [xy]_I$. ∎

2.3. Rough strong h-ideals

Definition 2.5 [11] For an approximation space(U,θ), by a rough approximation in (U,θ), we mean a mapping $Apr : P(U) \to P(U) \times P(U)$ defined by for any $X \in P(U)$, $Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where $\underline{Apr}(X) = \{x \in U | [x]_\theta \subseteq X\}$ and $\overline{Apr}(X) = \{x \in U | [x]_\theta \sqcap X \neq \phi\}$. $\underline{Apr}(X)(\overline{Apr}(X))$ is called a lower (upper)-rough approximation of X in (U,θ). $Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$ is called a rough set if $\underline{Apr}(X) \neq \overline{Apr}(X)$.

Let I be a strong h-ideal of S and $A \subseteq S$. Then the sets
$\underline{Apr}_I(A) = \{x \in S | [x]_I \subseteq A\} = \{x \in S | x + I \subseteq A\}$ and
$\overline{Apr}_I(A) = \{x \in S | [x]_I \cap A \neq \emptyset\} = \{x \in S | x + I \cap A \neq \emptyset\}$
are called, resp., lower and upper approximations of A with respect to (briefly, w.r.t.) I.

Clearly, $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are subsets of S and $\underline{Apr}_I(A) \subseteq A \subseteq \overline{Apr}_I(A)$. We call $Apr_I(A) = (\underline{Apr}_I(A), \overline{Apr}_I(A))$ is a rough set on S if $\underline{Apr}_I(A) \neq \overline{Apr}_I(A)$.

Definition 2.6 Let I be a strong h-ideal of S and $A \subseteq S$. Then A is called a lower(upper) rough strong h-ideal w.r.t I of S if $\underline{Apr}_I(A)(\overline{Apr}_I(A))$ is strong h-ideal of S. Moreover, $Apr_I(A)$ is called a rough strong h-ideal w.r.t. I of S if both $\underline{Apr}_I(A)$ and $\overline{Apr}_I(A)$ are strong h-ideals of S.

Example 2.7 Let $S = \mathbb{Z}_8$, $I = \{0,2,4,6\}$ and $A = \{0,1,2,4,6\}$. By calculations, $\overline{Apr}_I(A) = S$ and $\underline{Apr}_I(A) = \{0,1,2,4,6\}$. Hence $Apr_I(A)$ is a rough strong h-ideal w.r.t. I of S.

Lemma 2.8 [17] *Let I be a strong h-ideal of S and A any non-empty subset in S. Then $\overline{Apr}_I(A) = I + A$.*

Lemma 2.9 [17] *Let I and A be any two strong h-ideals of S. Then $\overline{Apr}_I(A)$ is a strong h-ideal of S.*

Lemma 2.10 [17] *Let I and A be any two strong h-ideals of S. If $\underline{Apr}_I(A) \neq \emptyset$, then $\underline{Apr}_I(A) = A$ and $\underline{Apr}_I(A)$ is a strong h-ideal of S.*

Lemma 2.11 *Let I be any idempotent strong h-ideal of S and A be any prime ideal of S. Then $\overline{Apr}_I(A)$ is a prime ideal of S.*

Proof. Assume that I is a strong h-ideal of S and A is a prime ideal of S. Then we have

(1) $\overline{Apr}_I(A) + \overline{Apr}_I(A) = I + A + I + A = I + A = \overline{Apr}_I(A)$.

(2) $S\,\overline{Apr}_I(A) = S(I + A) = SI + SA \subseteq I + A = \overline{Apr}_I(A)$ and $\overline{Apr}_I(A)\,S = (I + A)S = IS + AS \subseteq I + A = \overline{Apr}_I(A)$.

(3) Let $xy \in \overline{Apr}_I(A)$ for some $x, y \in S$, then $[xy]_I \cap A = [x]_I[x]_I \cap A \neq \emptyset$. So there exist $x' \in [x]_I$ and $y' \in [y]_I$ such that $x'y' \in A$. Thus $[x]_I \cap A \neq \emptyset$ or $[y]_I \cap A \neq \emptyset$, and so $x \in \overline{Apr}_I(A)$ or $y \in \overline{Apr}_I(A)$. Therefore $\overline{Apr}_I(A)$ is a prime ideal of S. ∎

Lemma 2.12 *Let I be an idempotent strong h-ideal of S and A a prime strong h-ideal of S. Then $\overline{Apr}_I(A)$ is a prime strong h-ideal of S.*

Proof. Combining Lemmas 2.9 and 2.11, we obtain easily. ■

Lemma 2.13 *Let I be any idempotent strong h-ideal of S and A be any prime ideal of S. If $\underline{Apr}_I(A) \neq \emptyset$, then $\underline{Apr}_I(A)$ is a prime ideal of S.*

Proof. (1) Let A be any prime ideal of S and $x, y \in \underline{Apr}_I(A)$. Then $[x]_I \subseteq A$ and $[y]_I \subseteq A$. Since I is an h-ideal of S, so $[x+y]_I = [x]_I + [y]_I \subseteq A + A \subseteq A$. That is $x + y \in \underline{Apr}_I(A)$.

(2) Let $s \in S$ and $x \in \overline{Apr}_I(A)$, then $[x]_I \subseteq A$. Since I is an idempotent h-ideal of S, then $[sx]_I = [s]_I[x]_I \subseteq [s]_I A \subseteq A$, this implies $sx \in \underline{Apr}_I(A)$. Similarly, we can get $xs \in \underline{Apr}_I(A)$.

(3) Suppose that $\underline{Apr}_I(A)$ is not a prime ideal, then there exist $x, y \in S$ such that $xy \in \underline{Apr}_I(A)$, but $x \notin \underline{Apr}_I(A)$ and $y \notin \underline{Apr}_I(A)$, that is, $[x]_I[y]_I = [xy]_I \subseteq A$ but $[x]_I \nsubseteq A$ and $[y]_I \nsubseteq A$, then there exist $x' \in [x]_I$ and $y'[y]_I$ such that $x' \notin A$, $y' \notin A$ and $x'y' \in [x]_I[y]_I \subseteq A$. Since A is a prime ideal, we have $x' \in A$ or $y' \in A$, this contradiction the assumption. Hence $\underline{Apr}_I(A)$ is a prime ideal of S. ■

The following lemma follows from Lemma 2.10.

Lemma 2.14 *Let I be a strong h-ideals of S and A a prime strong h-ideal of S. If $\underline{Apr}_I(A) \neq \emptyset$, then $\underline{Apr}_I(A)$ is a prime strong h-ideal of S.*

3. Rough fuzzy strong h-ideals

In this section, we introduce the concept of rough fuzzy strong h-ideals of hemirings and investigate some related properties.

A fuzzy set μ in S of the form

$$\mu(y) = \begin{cases} r & \text{if } y = x, \\ 0 & \text{if } y \neq x. \end{cases}$$

is called a fuzzy point with support x and r, and is denoted by x_r. In particular, if $r = 1$, we denote x_1.

Definition 3.1 [17] Let μ and ν be any fuzzy sets of S, we define the sum, denoted by $\mu + \nu$, of μ and ν by

$$(\mu + \nu)(x) = \bigvee_{x=a+b} \mu(a) \wedge \nu(b)$$

for all $x \in S$.

In particular, for any $x \in S$, define $x + \mu$ by

$$(x + \mu)(y) = \begin{cases} \bigvee_{y=x+a} \mu(a) & \text{if } \exists a \in S. \text{ s.t. } y = x + a, \\ 0 & \text{otherwise.} \end{cases}$$

For all $x \in S$, we have $x + \mu = x_1 + \mu$.

Definition 3.2 A fuzzy set μ of S is called a fuzzy ideal of S if for all $x, y \in S$, if it satisfies:

(1) $\mu(x+y) \geq \mu(x) \wedge \mu(y)$,

(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$.

Definition 3.3 A fuzzy ideal μ of S is called a fuzzy prime ideal of S if $\mu(xy) = \mu(x)$ or $\mu(xy) = \mu(y)$ for all $x, y \in S$.

Definition 3.4 [17] A fuzzy ideal μ of S is called a fuzzy strong h-ideal of S if $x + a + z = y + b + z \to (y_1 + \mu)(x) \geq \mu(a) \wedge \mu(b)$ for all $x, y, z, a, b \in S$.

A fuzzy strong h-ideal μ of S is called normal if $\mu(0) = 1$.

Example 3.5 Let $R[x]$ be a polynomial ring in a real number field R. Define $A = \{x^2 f(x) | f(x) \in R[x]\}$ and $B = \{g(x) = a_1 + a_2 x + ... \in R[x] \mid a_1 > 0,\, a_i \in R,\, i = 1, 2...\}$. Let $S = A \cup B$. Then it is a hemiring. One can check that A is a strong h-ideal of S. Define a fuzzy set μ by

$$\mu(x) = \begin{cases} 0.8 & \text{if } x \in A, \\ 0.4 & \text{if } x \in B. \end{cases}$$

One can check that μ is a fuzzy strong h-ideal of S.

Let μ be a fuzzy set of S and $r \in [0,1]$. Then the sets $\mu_r = \{x \in S \mid \mu(x) \geq r\}$ and $\mu_r^s = \{x \in S \mid \mu(x) > r\}$ are called r-level subset and r-strong level of μ, respectively.

Definition 3.6 A fuzzy ideal μ of S is called a fuzzy prime strong h-ideal of S if μ is both a fuzzy prime ideal and a fuzzy strong h-ideal of S.

Theorem 3.7 [17] *A fuzzy set μ of S is a fuzzy strong h-ideal of S if and only if non-empty subset μ_r (μ_r^s) is a strong h-ideal of S for all $r \in [0,1]$.*

Theorem 3.8 *A fuzzy set μ of S is a fuzzy prime ideal of S if and only if non-empty subset μ_r (μ_r^s) is a prime ideal of S for all $r \in [0,1]$.*

Proof. We only prove the case for μ_r. The proof of μ_r^s is similar.

Let μ be a fuzzy prime ideal of S, $x, y \in \mu_r$ and $a \in S$. Then $\mu(x+y) \geq \mu(x) \wedge \mu(y) \geq r, \mu(ax) \geq \mu(a) \vee \mu(x) \geq r$, and so $x + y, ax \in \mu_r$. Similarly, we get $xa \in \mu_r$, hence μ_r is an ideal of S.

Now, let $x, y \in \mu_r$ for some $x, y \in S$, then $\mu(xy) \geq r$. Since μ is a fuzzy prime ideal of S, we have $\mu(xy) = \mu(x) \geq r$ or $\mu(xy) = \mu(y) \geq r$. Thus $x \in \mu_r$ or $y \in \mu_r$. Therefore, μ_r is a prime ideal of S.

Conversely, assume that the given conditions hold. Let $x', y' \in S$. If possible, let $\mu(x' + y') < \mu(x') \wedge \mu(y')$. Choose r such that $\mu(x' + y') < r < \mu(x') \wedge \mu(y')$. Then $x', y' \in \mu_r$, but $x' + y' \notin \mu_r$, a contradiction. Hence $\mu(x+y) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in S$. Similarly, we have $\mu(xy) \geq \mu(x) \vee \mu(y)$ for all $x, y \in S$.

We can make clear from the above discussion that $\mu(xy) \geq \mu(x)$ and $\mu(xy) \geq \mu(y)$. Now assume there exist $x', y' \in S$ such that $\mu(x'y') \neq \mu(x')$ and $\mu(x'y') \neq \mu(y')$, then we have $\mu(x'y') > \mu(x')$ and $\mu(x'y') > \mu(y')$, choose r such that $\mu(x'y') > r > \mu(x')$ and $\mu(x'y') > r > \mu(y')$. Then $x'y' \in \mu_r$ but $x' \notin \mu_r$ and $y' \notin \mu_r$, a contradiction. Hence $\mu(xy) = \mu(x)$ or $\mu(xy) = \mu(y)$ for all $x, y \in S$. This implies that μ is a fuzzy prime ideal of S. ∎

Now, we introduce the concepts of rough fuzzy strong h-ideals (rough fuzzy prime ideals and rough fuzzy prime strong h-ideals) of hemirings.

Definition 3.9 Let I be a strong h-ideal of S and μ a fuzzy set of S. Then we define the two fuzzy sets $\underline{Apr}_I(\mu)$ and $\overline{Apr}_I(\mu)$ as follows:

$$\underline{Apr}_I(\mu)(x) = \bigwedge_{y \in [x]_I} \mu(y)$$

and

$$\overline{Apr}_I(\mu)(x) = \bigvee_{y \in [x]_I} \mu(y),$$

for all $x \in S$.

The fuzzy sets $\underline{Apr}_I(\mu)$ and $\overline{Apr}_I(\mu)$ are called, resp., the lower and upper approximations of μ w.r.t. I of S. Moreover, $Apr_I(\mu) = (\underline{Apr}_I(\mu), \overline{Apr}_I(\mu))$ is called a rough fuzzy set w.r.t. I of S if $\underline{Apr}_I(\mu) \neq \overline{Apr}_I(\mu)$.

Definition 3.10 Let I be a strong h-ideal of S and μ a fuzzy set of S. Then μ is called a lower(upper) rough fuzzy strong h-ideal (rough fuzzy prime ideal, rough fuzzy prime strong h-ideal) w.r.t. I of S if $\underline{Apr}_I(\mu)$ ($\overline{Apr}_I(\mu)$) is a fuzzy strong h-ideal (fuzzy prime ideal, fuzzy prime strong h-ideal) of S. Moreover, $Apr_I(\mu) = (\underline{Apr}_I(\mu), \overline{Apr}_I(\mu))$ is called a rough fuzzy strong h-ideal (rough fuzzy prime ideal, rough fuzzy prime strong h-ideal) w.r.t. I of S if both $\underline{Apr}_I(\mu)$ and $\overline{Apr}_I(\mu)$ are fuzzy strong h-ideals (fuzzy prime ideals, fuzzy prime strong h-ideals) of S.

Example 3.11 Consider a hemiring $S = \{0, a, b, c\}$ is the Klein's four group with the multiplication $xy = c$ if $x, y \in \{b, c\}$ and $xy = 0$ otherwise.

Let $I = \{0, a\}$, it is a strong h-ideal of S. Moreover, $[0]_I = \{0, a\}$ and $[b]_I = \{b, c\}$.

Define a fuzzy set μ of S by $\mu(0) = 0.8$ and $\mu(a) = \mu(b) = \mu(b) = 0.6$.

By calculations, we have

$$\underline{Apr}_I(\mu) = \frac{0.6}{0} + \frac{0.6}{a} + \frac{0.6}{b} + \frac{0.6}{c}$$

and

$$\overline{Apr}_I(\mu) = \frac{0.8}{0} + \frac{0.8}{a} + \frac{0.6}{b} + \frac{0.6}{c}.$$

This implies that $Apr_I(\mu)$ is a rough fuzzy prime strong h-ideal w.r.t. I of S.

Now, we give the (strong) level subset of lower and upper rough approximations of a fuzzy set μ w.r.t. I of hemirings.

Theorem 3.12 *Let I be a strong h-ideal of S. If μ is a fuzzy set of S and $r \in [0,1]$. Then*

(1) $(\underline{Apr}_I(\mu))_r = \underline{Apr}_I(\mu_r)$,

(2) $(\overline{Apr}_I(\mu))_r^s = \overline{Apr}_I(\mu_r^s)$.

Proof. For any $x \in S$, we have

$$\text{(i)}\quad x \in (\underline{Apr}_I(\mu))_r \Leftrightarrow \underline{Apr}_I(\mu)(x) \geq r \Leftrightarrow \bigwedge_{y\in[x]_I} \mu(y) \geq r \Leftrightarrow \forall y \in [x]_I, \mu(y) \geq r \Leftrightarrow [x]_I \subseteq \mu_r \Leftrightarrow x \in \underline{Apr}_I(\mu_r).$$

$$\text{(ii)}\quad x \in (\overline{Apr}_I(\mu))_r^s \Leftrightarrow \overline{Apr}_I(\mu)(x) > r \Leftrightarrow \bigvee_{y\in[x]_I} \mu(y) > r \Leftrightarrow \exists y \in [x]_I, \mu(y) > r \Leftrightarrow [x]_I \cap \mu_r^s \neq \emptyset \Leftrightarrow x \in \overline{Apr}_I(\mu_r^s).$$

This completes the proof. ∎

Theorem 3.13 *Let I be a strong h-ideal of S. If μ is a fuzzy strong h-ideal of S, then $\overline{Apr}_I(\mu)$ is a fuzzy strong h-ideal of S.*

Proof. Let μ be a fuzzy strong h-ideal of S. For any $r \in [0,1]$, by Theorem 3.12(2), $(\overline{Apr}_I(\mu))_r^s = \overline{Apr}_I(\mu_r^s)$. By Theorem 3.7, we know that μ_r^s is a strong h-ideal of S. Hence, by Lemma 2.9, $\overline{Apr}_I(\mu_r^s)$ is a strong h-ideal of S, and so $(\overline{Apr}_I(\mu))_r^s$ is also a strong h-ideal of S. Then, by Theorem 3.7, $\overline{Apr}_I(\mu)$ is a fuzzy strong h-ideal of S. ∎

Theorem 3.14 *Let I be a strong h-ideal of S. If μ is a fuzzy strong h-ideal of S and $\underline{Apr}_I(\mu) \neq \emptyset$, then $\underline{Apr}_I(\mu)$ is a fuzzy strong h-ideal of S.*

Proof. Let μ be a fuzzy strong h-ideal of S. Since $\underline{Apr}_I(\mu) \neq \emptyset$, there exists $r \in [0,1]$ such that $(\underline{Apr}_I(\mu))_r = \underline{Apr}_I(\mu_r) \neq \emptyset$ by Theorem 3.12(1). Let r be any value that fulfills the above property. Then it is clear that $\mu_r \neq \emptyset$, and we know from Theorem 3.7 that μ_r is a strong h-ideal of S. Hence, by Lemma 2.10, $\underline{Apr}_I(\mu_r)$ is a strong h-ideal of S, and so $(\underline{Apr}_I(\mu))_r$ is also a strong h-ideal of S. Then, by Theorem 3.7, $\underline{Apr}_I(\mu)$ is a fuzzy strong h-ideal of S. ∎

Corollary 3.15 *Let I be a strong h-ideal of S. If μ is a fuzzy strong h-ideal of S and $\underline{Apr}_I(\mu) \neq \emptyset$, then $Apr_I(\mu)$ is a rough fuzzy strong h-ideal w.r.t. I of S.*

Remark 3.16 The converse of Corollary 3.15 may not be true as shown in the following :

Example 3.17 Let $S = \mathbb{Z}_8$ and $I = \{0, 2, 4, 6\}$. Then I is a strong h-ideal of S. Clearly, $[0]_I = \{0, 2, 4, 6\}$ and $[1]_I = \{1, 3, 5, 7\}$.

Define a fuzzy set μ of S by $\mu(0) = 1, \mu(1) = \mu(2) = \mu(4) = \mu(6) = 0.8$ and $\mu(3) = \mu(5) = \mu(7) = 0.6$. Then we can check that μ is not a fuzzy strong h-ideal of S.

By calculations, we have

$$\overline{Apr}_I(\mu) = \frac{1}{0} + \frac{0.8}{1} + \frac{1}{2} + \frac{0.8}{3} + \frac{1}{4} + \frac{0.8}{5} + \frac{1}{6} + \frac{0.8}{7}$$

and

$$\underline{Apr}_I(\mu) = \frac{0.8}{0} + \frac{0.6}{1} + \frac{0.8}{2} + \frac{0.6}{3} + \frac{0.8}{4} + \frac{0.6}{5} + \frac{0.8}{6} + \frac{0.6}{7}.$$

This means that $\overline{Apr}_I(\mu)$ and $\underline{Apr}_I(\mu)$ are both fuzzy strong h-ideals of S, and so $Apr_I(\mu) = (\underline{Apr}_I(\mu), \overline{Apr}_I(\mu))$ is a rough fuzzy strong h-ideal w.r.t. I of S, but μ is not a fuzzy strong h-ideal of S.

Theorem 3.18 *Let I be an idempotent strong h-ideal of S. If μ is a fuzzy prime ideal of S, then $\overline{Apr}_I(\mu)$ is a fuzzy prime ideal of S.*

Proof. Let μ be a fuzzy prime ideal of S. For any $r \in [0, 1]$, by Theorem 3.12(2), $(\overline{Apr}_I(\mu))_r^s = \overline{Apr}_I(\mu_r^s)$. By Theorem 3.8, we know that μ_r^s is a prime ideal of S. Then by Lemma 2.11, $\overline{Apr}_I(\mu_r^s)$ is a prime ideal of S, and so $(\overline{Apr}_I(\mu))_r^s$ is also a prime ideal of S. Then, by Theorem 3.8, $\overline{Apr}_I(\mu)$ is a fuzzy prime ideal of S. ■

Theorem 3.19 *Let I be an idempotent strong h-ideal of S. If μ is a fuzzy prime ideal of S and $\underline{Apr}_I(\mu) \neq \emptyset$, then $\underline{Apr}_I(\mu)$ is a fuzzy prime ideal of S.*

Proof. Let μ be a fuzzy prime ideal of S. Since $\underline{Apr}_I(\mu) \neq \emptyset$, there exists $r \in [0, 1]$ such that $(\underline{Apr}_I(\mu))_r = \underline{Apr}_I(\mu_r) \neq \emptyset$ by Theorem 3.12(1). Let r be any value that fulfills the above property. Then it is clear that $\mu_r \neq \emptyset$, and we know from Theorem 3.8 that μ_r is a prime ideal of S. And then by Lemma 2.13, $\underline{Apr}_I(\mu_r)$ is a prime ideal of S, and so $(\underline{Apr}_I(\mu))_r$ is also a prime ideal of S. Then, by Theorem 3.8, $\underline{Apr}_I(\mu)$ is a fuzzy prime ideal of S. ■

Combining Theorems 3.18 and 3.19, we can obtain the following result:

Corollary 3.20 *Let I be a strong h-ideal of S. If μ is a fuzzy prime ideal of S and $\underline{Apr}_I(\mu) \neq \emptyset$, then $Apr_I(\mu)$ is a rough fuzzy prime ideal w.r.t. I of S.*

4. Fuzzy rough strong h-ideals

In this section, we investigate fuzzy rough strong h-ideals (fuzzy rough prime ideals) of hemirings and show that every rough fuzzy strong h-ideal (rough fuzzy prime ideal) is a fuzzy rough strong h-ideal (fuzzy rough prime ideal) of hemirings.

Definition 4.1 Let μ be a normal fuzzy strong h-ideal of S and ν a fuzzy set of S. Define two fuzzy sets $\underline{Apr}_\mu(\nu)$ and $\overline{Apr}_\mu(\nu)$ in S by

$$\underline{Apr}_\mu(\nu)(x) = \bigwedge_{y\in S}(1 - \bigvee_{x+a+z=y+b+z} \mu(a) \wedge \mu(b)) \vee \nu(y)$$

and

$$\overline{Apr}_\mu(\nu)(x) = \bigvee_{y\in S}(\bigvee_{x+a+z=y+b+z} \mu(a) \wedge \mu(b)) \wedge \nu(y)$$

for all $x \in S$.

For any fuzzy set ν of S, $Apr_\mu(\nu) = (\underline{Apr}_\mu(\nu), \overline{Apr}_\mu(\nu))$ is called a fuzzy rough set w.r.t. μ of S if $\underline{Apr}_\mu(\nu) \neq \overline{Apr}_\mu(\nu)$.

Remark 4.2

(1) If μ is a crisp strong h-ideal of S and ν is a fuzzy set of S, then $\underline{Apr}_\mu(\nu)(x) = \bigwedge_{y\in[x]_\mu} \nu(y)$ and $\overline{Apr}_\mu(\nu)(x) = \bigvee_{y\in[x]_\mu} \nu(y)$, for all $x \in S$.
This means that every rough fuzzy set is a fuzzy rough set.

(2) If μ is a crisp strong h-ideal of S and ν is a non-empty subset of S, then $\underline{Apr}_\mu(\nu) = \{x \in S | x + \mu \subseteq \nu\}$ and $\overline{Apr}_\mu(\nu) = \{x \in S | (x + \mu) \cap \nu \neq \emptyset\}$.
This means that $(\underline{Apr}_\mu(\nu), \overline{Apr}_\mu(\nu))$ is a Pawlak rough set.

Definition 4.3 Let μ be a normal fuzzy strong h-ideal of S and ν a fuzzy set of S. Then ν is called a lower (upper) fuzzy rough strong h-ideal (fuzzy rough prime ideal, fuzzy rough prime strong h-ideal) w.r.t μ of S if $\underline{Apr}_\mu(\nu)(\overline{Apr}_\mu(\nu))$ is a fuzzy strong h-ideal (fuzzy prime ideal, fuzzy prime strong h-ideal) of S. Moreover, $Apr_\mu(\nu) = (\underline{Apr}_\mu(\nu), \overline{Apr}_\mu(\nu))$ is called a fuzzy rough strong h-ideal (fuzzy rough prime ideal, fuzzy rough prime strong h-ideal) w.r.t μ of S if both $\underline{Apr}_\mu(\nu)$ and $\overline{Apr}_\mu(\nu)$ are fuzzy strong h-ideals (fuzzy prime ideals, fuzzy prime strong h-ideals) of S.

Example 4.4 Let $S = Z_8$. Define two fuzzy sets μ and ν of S by

$\mu(0) = 1, \mu(2) = \mu(4) = \mu(6) = 0.8$ and $\mu(1) = \mu(3) = \mu(5) = \mu(7) = 0.4$.
$\nu(0) = 1, \nu(1) = \nu(2) = \nu(4) = \nu(6) = 0.7$ and $\nu(3) = \nu(5) = \nu(7) = 0.3$.

By calculations, $\underline{Apr}_\mu(\nu)$ and $\overline{Apr}_\mu(\nu)$ are fuzzy strong h-ideals of S, and so $Apr_\mu(\nu) = (\underline{Apr}_\mu(\nu), \overline{Apr}_\mu(\nu))$ is a fuzzy rough strong h-ideal w.r.t. μ of S.

Now, we give strong level subsets of lower and upper rough approximations of a fuzzy set w.r.t. μ of hemirings.

Theorem 4.5 *Let μ be a normal fuzzy strong h-ideal and ν a fuzzy set of S and $r \in [0,1]$. Then*

(1) $(\overline{Apr}_{\mu}(\nu))_r^s = \overline{Apr}_{\mu_r^s}(\nu_r^s)$,

(2) $(\underline{Apr}_{\mu}(\nu))_r^s = \underline{Apr}_{\mu_{1-r}}(\nu_r^s)$.

Proof. For any $x \in S$, we have

(1)
$$\begin{aligned}
& x \in \overline{Apr}_{\mu}(\nu)_r^s \\
&\Leftrightarrow \bigvee_{y\in S}\left(\bigvee_{x+a+z=y+b+z} \mu(a)\wedge\mu(b)\right)\wedge \nu(y) > r \\
&\Leftrightarrow \exists y,z,a,b \in S \text{ with } x+a+z=y+b+z \text{ s.t } \mu(a)\wedge\mu(b)\wedge\mu(y) > r \\
&\Leftrightarrow \exists y,z,a,b \in S \text{ with } x+a+z=y+b+z \text{ s.t } a,b \in \mu_r^s \text{ and } y \in \nu_r^s \\
&\Leftrightarrow x \in \mu_r^s + y \text{ and } y \in \nu_r^s \\
&\Leftrightarrow x \in \mu_r^s + \nu_r^s \\
&\Leftrightarrow x \in \overline{Apr}_{\mu_r^s}(\nu)_r^s.
\end{aligned}$$

(2)
$$\begin{aligned}
& x \in (\underline{Apr}_{\mu}(\nu))_r^s \\
&\Leftrightarrow \bigwedge_{y\in S}\left(1-\bigvee_{x+a+z=y+b+z} \mu(a)\wedge\mu(b)\right)\vee \nu(y) > r \\
&\Leftrightarrow \forall y \in S,\ \left(1-\bigvee_{x+a+z=y+b+z} \mu(a)\wedge\mu(b)\right)\vee \nu(y) > r \\
&\Leftrightarrow \forall y \in S,\ \left(1-\bigvee_{x+a+z=y+b+z} \mu(a)\wedge\mu(b)\right)\leq r \Rightarrow \nu(y) > r \\
&\Leftrightarrow \forall y \in S,\ \bigvee_{x+a+z=y+b+z} \mu(a)\wedge\mu(b) \geq 1-r \Rightarrow \nu(y) > r \\
&\Leftrightarrow \forall y \in S, \text{ if there exist } a,b,z \in S \text{ such that } x+a+z=y+b+z \\
&\qquad \text{and } \mu(a)\wedge\mu(b)\geq 1-r, \text{then } \nu(y) > r, \\
&\Leftrightarrow \forall y \in S, y \in [x]_{\mu_{1-r}} \Rightarrow y \in \nu_r^s \\
&\Leftrightarrow [x]_{\mu_{1-r}} \subseteq \nu_r^s \Leftrightarrow x \in \underline{Apr}_{\mu_{1-r}}(\nu_r^s).
\end{aligned}$$

■

Now, we investigate the properties of fuzzy rough strong h-ideals of hemirings.

Theorem 4.6 *Let μ be a normal fuzzy strong h-ideals of S and ν a fuzzy strong h-ideal of S, then ν is an upper fuzzy rough strong h-ideal w.r.t. μ of S.*

Proof. Let ν be a fuzzy strong h-ideal of S. For any $r \in [0,1]$, by Lemma 4.5, $(\overline{Apr}_{\mu}(\nu))_r^s = \overline{Apr}_{\mu_r^s}(\nu_r^s)$. By Theorem 3.7, we know that μ_r^s and ν_r^s are both strong h-ideals of S. Then by Lemma 2.9, $\overline{Apr}_{\mu_r^s}(\nu_r^s)$ is a strong h-ideal of S, and so

$(\overline{Apr}_{\mu}(\nu))_r^s$ is also a strong h-ideal of S. Then, by Theorem 3.7, $\overline{Apr}_{\mu}(\nu)$ is a fuzzy strong h-ideal of S. Thus, ν is an upper fuzzy rough strong h-ideal w.r.t. μ of S. ∎

Similarly, we can obtain the following:

Theorem 4.7 *Let μ be a normal fuzzy strong h-ideal of S and ν a fuzzy strong h-ideal of S and $\underline{Apr}_{\mu}(\nu) \neq \emptyset$, then ν is a lower fuzzy rough strong h-ideal w.r.t. μ of S.*

Combining Theorems 4.6 and 4.7, we can obtain the following result:

Corollary 4.8 *Let μ be a normal fuzzy strong h-ideal of S, if ν is a fuzzy strong h-ideal of S and $\underline{Apr}_{\mu}(\nu) \neq \emptyset$, then $Apr_{\mu}(\nu) = (\underline{Apr}_{\mu}(\nu), \overline{Apr}_{\mu}(\nu))$ is a fuzzy rough strong h-ideal w.r.t μ of S.*

Remark 4.9 The converse of the above theorem may not be true as shown in the following example.

Example 4.10 Consider Example 4.4. We know that $Apr_{\mu}(\nu)$ is a fuzzy rough strong h-ideal w.r.t. μ of S, but ν is not a fuzzy strong h-ideal of S.

Theorem 4.11 *Let μ be a normal fuzzy strong h-ideals of S and $\mu_r\mu_r{=}\mu_r$ for all $r \in (0, 1]$, ν a fuzzy prime ideal (fuzzy prime strong h-ideal) of S, then ν is an upper fuzzy rough prime ideal (upper fuzzy rough prime strong h-ideal) w.r.t. μ of S.*

Proof. Let μ be a fuzzy strong h-ideal of S. For any $r \in [0, 1]$, by Theorem 4.5(1), $(\overline{Apr}_{\mu}(\nu))_r^s = \overline{Apr}_{\mu_r^s}(\nu_r^s)$. By Theorems 3.7 and 3.8, we know that μ_r^s is a strong h-ideal of S and ν_r^s is a prime ideal of S. Since $\mu_r\mu_r = \mu_r$, then by Lemma 2.11, $\overline{Apr}_{\mu_r^s}(\nu_r^s)$ is a prime ideal of S, and so $(\overline{Apr}_{\mu}(\nu))_r^s$ is also a prime ideal of S. And then, by Theorem 3.8, $\overline{Apr}_{\mu}(\nu)$ is a fuzzy prime ideal of S. Thus, ν is an upper fuzzy rough strong h-ideal w.r.t. μ of S. Similarly, we can prove the upper fuzzy rough prime strong h-ideal. ∎

Theorem 4.12 *Let μ be a normal fuzzy strong h-ideal of S and $\mu_r\mu_r = \mu_r$ for all $r \in (0, 1]$, ν a fuzzy prime ideal (fuzzy prime strong h-ideal) of S and $\underline{Apr}_{\mu}(\nu) \neq \emptyset$, then ν is a lower fuzzy rough prime ideal (fuzzy rough prime strong h-ideal) w.r.t. μ of S.*

Proof. Let μ be a normal fuzzy strong h-ideal of S. For any $r \in [0, 1]$, by Theorem 4.5 (2), $(\underline{Apr}_{\mu}(\nu))_r^s = \underline{Apr}_{\mu_{1-r}}(\nu_r^s)$. Let r be any value that fulfills the above property, then it is clear that $\mu_r \neq \emptyset$. By Theorems 3.7 and 3.8, we know that μ_{1-r} is a strong h-ideal of S and ν_r^s is a prime ideal of S. Since $\mu_r\mu_r = \mu_r$, then by Lemma 2.13, $\underline{Apr}_{\mu_{1-r}}(\nu_r^s)$ is a prime ideal of S, and so $(\underline{Apr}_{\mu}(\nu))_r^s$ is also a prime ideal of S. Then, by Theorem 3.8, $\underline{Apr}_{\mu}(\nu)$ is a fuzzy prime ideal of S. Thus, ν is an lower fuzzy rough prime ideal w.r.t. μ of S. Similarly, we can prove the lower fuzzy rough prime strong h-ideal. ∎

From the above discussion, we get the following result:

Corollary 4.13 *Let μ be a normal fuzzy strong h-ideal of S, if ν is a fuzzy prime ideal (fuzzy prime strong h-ideal) of S and* $\underline{Apr}_{\mu}(\nu) \neq \emptyset$, *then* $Apr_{\mu}(\nu) = (\underline{Apr}_{\mu}(\nu), \overline{Apr}_{\mu}(\nu))$ *is a fuzzy rough prime ideal (fuzzy rough prime strong h-ideal) w.r.t μ of S.*

Acknowledgements. This research is partially supported by a grant of National Natural Science Foundation of China (11461025) and Key Subject of Hubei Province (Mathematics).

References

[1] Ali, M.I., Shabir, M., Tanveer, S., *Roughness in hemirings*, Neural Comput. Appl., 21 (2012), 171-180.

[2] Davvaz, B., *Roughness in rings*, Inform. Sci., 164 (2004), 147-163.

[3] Davvaz, B., *Roughness based on fuzzy ideals*, Inform. Sci., 176 (2006), 2417-2437.

[4] Dubois, D., Prade, H., *Rough fuzzy sets and fuzzy rough sets*, Int. J. General Syset., 17 (1990), 191-209.

[5] Jun, Y.B., Öztürk, M.A., Sung, S.Z., *On fuzzy h-ideals in hemirings*, Inform. Sci., 162 (2004), 211-216.

[6] Kazanci, O., Davvaz, B., *On the structure of rough prime(primary) ideals and rough fuzzy prime (primary) ideals in commutative rings*, Inform. Sci., 178 (2008), 1343-1354.

[7] Kuroki, N., *Rough ideals in semigroups*, Inform. Sci., 100 (1997), 139-163.

[8] Ma, X., Yin, Y., Zhan, J., *Characterizations of h-intra- and h-quasi-hemiregular hemirings*, Comput. Math. Appl., 63 (2012), 783-793.

[9] Ma, X., Zhan, J., *Generalized fuzzy h-bi-ideals of h-bi-ideals and h-quasi-ideals of hemirings*, Inform. Sci., 179 (2009), 1249-1268.

[10] Liu, G., Zhu, W., *The algebraic structures of generalized rough set theory*, Inform. Sci., 178 (2008), 4105-4113.

[11] Pawlak, Z., *Rough sets*, Int. J. Comput Inform. Sci., 11 (1982), 341-365.

[12] Wu, W.Z., Zhang, W.X., *Neighborood operator systems and approximations*, Inform. Sci., 144 (2012), 201-217.

[13] La Torre, D.R., *On h-ideals and k-ideals in hemirings*, Publ. Math. Debrecen., 12 (1965), 219-226.

[14] Wu, W.Z., Zhang, W.X., *Constructive and axiomatic approaches of fuzzy approximation operators*, Inform. Sci., 159 (2004), 233-254.

[15] Yao, Y.Y., *Three-way decisions with probabilistic rough sets*, Inform. Sci., 180(2010), 341-353.

[16] Yin, Y., Li, H., *The characterization of h-hemiregular hemirings and h-intra-hemiregular hemirings*, Inform. Sci., 178 (2008), 3451-3464.

[17] Yin, Y., Wang, J., *Fuzzy Hemirings*, Science Press, 2010.

[18] Zhan, J., Dudek, W.A., *Fuzzy h-ideals of hemirings*, Inform. Sci., 177 (2007), 876-886.

[19] Zhan, J., Liu, Q., Davvaz, B., *A new rough set theory: rough soft hemirings*, J. Intell. Fuzzy Systems, 28 (2015), 1687-1697.

[20] Zhang, X.H., Dai, J.H., Yu, Y.C., *On the union and intersection operations of rough sets based on various approximation spaces*, Inform. Sci., 292 (2015), 214-229.

[21] Zhang, X.H., Zhou, B., Li, P., *A general frame for intuitionistic fuzzy rough sets*, Inform. Sci., 216 (2012), 34-49.

[22] Zhu, W., *Relationships among basic concepts in covering-based rough sets*, Inform. Sci., 179 (2009), 2479-2486.

Accepted: 12.03.2015