THE FRACTIONAL $(D_\xi^\alpha G/G)$-EXPANSION METHOD AND ITS APPLICATIONS FOR SOLVING FOUR NONLINEAR SPACE-TIME FRACTIONAL PDES IN MATHEMATICAL PHYSICS

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Abstract. The fractional $(D_\xi^\alpha G/G)$-expansion method is applied in this article to find the exact traveling wave solutions with parameters for four nonlinear space-time fractional partial differential equations (PDEs), namely the space-time fractional Potential Kadomtsev-Petviashvili (PKP) equation, the space-time fractional symmetric regularized long wave (SRLW) equation, the space-time fractional Sharma-Tasso Olver (STO) equation and the space-time fractional Kolmogorov-Petrovskii-Piskunov (KPP) equation. When these parameters are taken special values, we obtain three types of solutions via the solitary, trigonometric and rational solutions. Comparison between our recent results and the well-known results is given. The solutions of these equations with numerical simulations are presented.

Keywords: fractional $(D_\xi^\alpha G/G)$-expansion method; nonlinear space-time fractional PDEs; exact traveling wave solutions; modified Riemann-Liouville derivative.

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1. Introduction

Exact traveling wave solutions for nonlinear fractional partial differential equations (NFPDEs) are of fundamental and important in applied science because they are widely employed to explain some of the nonlinear fractional phenomena and dynamical processes existed in nature world. Fractional partial differential equations have been studied due to their special appearance in different fields, such as physics, biology, engineering, signal processing control theory, the finance and fractal dynamics, see for example the articles [11], [15], [20], [22], [25], [28], [29]. For better realizing the mechanisms of the complicated nonlinear physical phenomena as well as further applications in practical life, the exact solutions of such equations obtained in the articles [2], [16], [19], [27], [36], [37]. In the past several decades, new exact solutions may help to find new phenomena. A variety of powerful methods, such as the finite difference method [17], the finite element method [7], the differential transform method [3], [21], the Adomian decomposition method [4], [5], [12], [23], the variational iteration method [13], [24], [35], the homotopy perturbation method [8], the (G'/G)-expansion method [6], [9], [26], [31], [34], [38], the fractional $(D_\xi^\alpha G/G)$-expansion method [34], [39]-[42], the Jacobi elliptic equation method [39], the fractional sub-equation method [1], [10], [34], [37]-[41], the modified simple equation method [32], the homogeneous balance method [33], the variation of parameters method [30] and so on.

The objective of this paper is to apply the fractional $(D_\xi^\alpha G/G)$-expansion method [34], [39]-[42] for solving the nonlinear fractional NFPDEs, namely the space-time fractional Potential Kadomtsev-Petviashvili (PKP) equation, the space-time fractional symmetric regularized long wave (SRLW) equation, the space-time fractional Sharma-Tasso Olver (STO) equation and the space-time fractional Kolmogorov-Petrovskii-Piskunov (KPP) equation in the sense of the modified Riemann-Liouville derivative obtained in [14], [18]. All these equations have been discussed in [31] using a different technique, namely the fractional complex transformation technique combined with the improved (G'/G)-expansion method.

The modified Riemann-Liouville derivative of order α [14], [18] is defined by the following expression:

$$D_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dt} \displaystyle\int_0^t (t-\eta)^{-\alpha} \left[f(\eta) - f(0)\right] d\eta, & 0 < \alpha \le 1, \\ \left[f^{(n)}(t)\right]^{(\alpha-n)}, & n \le \alpha < n+1,\ n \ge 1. \end{cases} \tag{1.1}$$

We list some important properties for the modified Riemann-Liouville derivative as follows:

$$D_t^\alpha t^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} t^{r-\alpha},\ r > 0 \tag{1.2}$$

$$D_t^\alpha \left[f(t)g(t)\right] = f(t)D_t^\alpha g(t) + g(t)D_t^\alpha f(t) \tag{1.3}$$

$$D_t^\alpha \left[f\left(g(t)\right)\right] = f_g'\left(g(t)\right) D_t^\alpha g(t) = D_g^\alpha f(g(t)) \left[g'(t)\right]^\alpha \tag{1.4}$$

This paper is organized as follows: In Section 2, we give the description of the fractional $(D_\xi^\alpha G/G)$-expansion method. In Section 3, we apply this method to find many exact solutions for the space-time nonlinear fractional PKP equation, the space-time nonlinear fractional SRLW equation, the space-time nonlinear fractional STO equation and the space-time nonlinear fractional KPP equation. In Section 4, conclusions and discussions are obtained.

2. Description the fractional $(D_\xi^\alpha G/G)$-expansion method

Suppose that we have the following nonlinear fractional PDE in the form:

$$F(u, D_t^\alpha u, D_x^\alpha u, D_t^{2\alpha} u, D_x^{2\alpha} u, ...) = 0, \quad 0 < \alpha \le 1, \tag{2.1}$$

where $D_t^\alpha u, D_x^\alpha u, D_t^{2\alpha} u, D_x^{2\alpha} u, ...$ are the modified Riemann Liouville derivatives and F is a polynomial in $u(x,t)$ and its partial fractional derivatives, in which the highest order fractional derivatives and the nonlinear terms are involved. In the following, we give the main steps of this method:

Step 1: using the wave transformation

$$u(x,t) = u(\xi), \quad \xi = kx + ct, \tag{2.2}$$

where k, c are nonzero constants, to reduce equation (2.1) to the following non-linear fractional ODE:

$$P(u, c^\alpha D_\xi^\alpha u, k^\alpha D_\xi^\alpha u, c^{2\alpha} D_\xi^{2\alpha} u, k^{2\alpha} D_\xi^{2\alpha} u, ...) = 0, \tag{2.3}$$

where P is a polynomial in $u(\xi)$ and its total fractional derivatives.

Step 2: Assume that equation (2.3) has the formal solution:

$$u(\xi) = \sum_{i=-N}^{N} a_i \left[\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right]^i, \tag{2.4}$$

where a_i $(i = 0, \pm1, \pm2, ..., \pm N)$ are constants to be determined later, such that $a_N \neq 0$ or $a_{-N} \neq 0$, while the function $G(\xi)$ satisfies the following fractional ordinary differential equation (ODE):

$$D_\xi^{2\alpha} G(\xi) + \lambda D_\xi^\alpha G(\xi) + \mu G(\xi) = 0, \tag{2.5}$$

where λ, μ are arbitrary constants.

Step 3: Determining the positive integer N in (2.4) by using the homogeneous balance between the highest order fractional derivatives and the nonlinear terms in equation (2.3).

Step 4: Substituting (2.4) along with equation (2.5) into equation (2.3), we have a polynomial in $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)$. Equating each coefficient of this polynomial to be zero yields a system of algebraic equations which can be solved by using the Maple or Mathematica to find the values a_i $(i = 0, \pm1, \pm2, ...)$ and k, c.

Step5: It is well-known [34], [39]-[42] that $\left(\frac{D_\xi^\alpha G(\xi)}{G(\xi)}\right)$ has the following forms:

$$\text{(2.6)}\quad \frac{\sqrt{\lambda^2-4\mu}}{2}\left[\frac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)}\right]-\frac{\lambda}{2},\ \lambda^2-4\mu>0,$$

$$\text{(2.7)}\quad \frac{\sqrt{4\mu-\lambda^2}}{2}\left[\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)}\right]-\frac{\lambda}{2},\ \lambda^2-4\mu<0,$$

$$\text{(2.8)}\quad \frac{c_2}{c_1+c_2\eta}-\frac{\lambda}{2}, \qquad \lambda^2-4\mu=0,$$

where $\eta=\dfrac{\xi^\alpha}{\Gamma(1+\alpha)}$, while c_1, c_2 are arbitrary constants.

Step 6: Substituting the values a_i, k, c as well as the values (2.6)-(2.8) into (2.4), we have the exact traveling wave solutions of equation (2.1).

3. Applications

In this section, we apply the above method described in Section 2 to find the exact traveling wave solutions of the following four nonlinear fractional PDEs:

Example 1. The space-time nonlinear fractional PKP equation. This equation is well-known [1], [31] and has the form:

$$\text{(3.1)}\quad \frac{1}{4}D_x^{4\alpha}u+\frac{3}{2}D_x^\alpha u D_x^{2\alpha}u+\frac{3}{4}D_y^{2\alpha}u+D_t^\alpha(D_x^\alpha u)=0.$$

This equation has been discussed in [31] using a different technique, namely the fractional complex transformation technique combined with the improved (G'/G)-expansion method. Let us now solve equation (3.1) using the method of Section 2. To this end, we use the following wave transformation:

$$\text{(3.2)}\quad u(x,y,t)=u(\xi),\quad \xi=k_1x+k_2y+ct,$$

where k_1, k_2 and c are constants, to reduce equation (3.1) to the following fractional ODE

$$\text{(3.3)}\quad k_1^{4\alpha}D_\xi^{3\alpha}u+3k_1^{3\alpha}\left(D_\xi^\alpha u\right)^2+(3k_2^{2\alpha}+4k_1^\alpha c^\alpha)D_\xi^\alpha u=0,$$

By balancing $D_\xi^{3\alpha}u$ with $\left(D_\xi^\alpha u\right)^2$, we have $N=1$. Consequently, equation (3.3) has the formal solution:

$$\text{(3.4)}\quad u(\xi)=a_1\left(\frac{D_\xi^\alpha G}{G}\right)+a_0+a_{-1}\left(\frac{D_\xi^\alpha G}{G}\right)^{-1},$$

where a_1, a_0, a_{-1} are constants to be determined later, such that $a_1 \neq 0$ or $a_{-1} \neq 0$. Substituting (3.4) along with equation (2.5) into equation (3.3), collecting all the terms of the same orders $\left(\frac{D_\xi^\alpha G}{G}\right)^i$, $(i = 0, \pm1, \pm2, ...)$ and setting each coefficient to zero, we have the following set of algebraic equations:

$$\left(\frac{D_\xi^\alpha G}{G}\right)^4 : -3a_1 k_1^{3\alpha}(2k_1^\alpha - a_1) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^3 : -6a_1 \lambda k_1^{3\alpha}(2k_1^\alpha - a_1) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^2 : -k_1^{4\alpha}(8a_1\mu + 7a_1\lambda^2) + 3k_1^{3\alpha}(a_1^2\lambda^2 - 2a_1a_{-1} + 2a_1^2\mu) - a_1(3k_2^{2\alpha} + 4k_1^\alpha c^\alpha) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right) : -k_1^{4\alpha}(8a_1\mu\lambda + a_1\lambda^3) + 3k_1^{3\alpha}(2a_1^2\mu\lambda - 4a_1a_{-1}\lambda) - a_1\lambda(3k_2^{2\alpha} + 4k_1^\alpha c^\alpha) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^0 : k_1^{4\alpha}(-2a_1\mu^2 - a_1\mu\lambda^2 + a_{-1}\lambda^2 + 2a_{-1}\mu) + 3k_1^{3\alpha}(a_{-1}^2 - 4a_1a_{-1}\mu + a_1^2\mu^2 - 2a_1a_{-1}\lambda^2) + (a_{-1} - a_1\mu)(3k_2^{2\alpha} + 4k_1^\alpha c^\alpha) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-1} : a_{-1}\lambda\left[k_1^{4\alpha}(8\mu + \lambda^2) + 3k_1^{3\alpha}(2a_{-1} - 4a_1\mu) + (3k_2^{2\alpha} + 4k_1^\alpha c^\alpha)\right] = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-2} : k_1^{4\alpha}(8a_{-1}\mu^2 + 7a_{-1}\mu\lambda^2) + 3k_1^{3\alpha}(a_{-1}^2\lambda^2 + 2a_{-1}^2\mu - 2a_1a_{-1}\mu^2) + a_{-1}\mu(3k_2^{2\alpha} + 4k_1^\alpha c^\alpha) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-3} : 6a_{-1}\mu\lambda k_1^{3\alpha}(2\mu k_1^\alpha + a_{-1}) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-4} : 3a_{-1}\mu^2 k_1^{3\alpha}(2\mu k_1^\alpha + a_{-1}) = 0.$$

On solving the above algebraic equations with the aid of Maple or Mathematica, we have the following cases:

Case1.

$$\lambda = 0,\ \mu = \mu,\ k_1^\alpha = k_1^\alpha,\ k_2^\alpha = k_2^\alpha,\ c^\alpha = \frac{1}{4k_1^\alpha}(16k_1^{4\alpha}\mu - 3k_2^{2\alpha}),\quad a_{-1} = -2k_1^\alpha\mu,\ a_1 = 2k_1^\alpha. \tag{3.5}$$

Case2.

$$\lambda = 0,\ \mu = \mu,\ k_1^\alpha = k_1^\alpha,\ k_2^\alpha = k_2^\alpha,\ c^\alpha = \frac{1}{4k_1^\alpha}(4k_1^{4\alpha}\mu - 3k_2^{2\alpha}),\quad a_{-1} = 0,\ a_1 = 2k_1^\alpha. \tag{3.6}$$

Case3.

$$\mu = \mu,\ k_1^\alpha = k_1^\alpha,\ k_2^\alpha = k_2^\alpha,\ c^\alpha = \frac{-1}{4k_1^\alpha}(k_1^{4\alpha}(\lambda^2 - 4\mu) + 3k_2^{2\alpha}),\quad a_{-1} = -2k_1^\alpha\mu,\ a_1 = 0. \tag{3.7}$$

Let us now write down the following exact solutions of the space-time fractional PKP equation (3.1) for case 1 (similarly for cases 2 and 3 which are omitted here for simplicity):

(i) If $\mu < 0$ (Hyperbolic function solutions)

In this case, we have the exact wave solution:

$$\begin{aligned} u(x,y,t) &= 2k_1^\alpha\sqrt{-\mu}\left[\frac{c_1\cosh\left(\sqrt{-\mu}\eta\right)+c_2\sinh\left(\sqrt{-\mu}\eta\right)}{c_1\sinh\left(\sqrt{-\mu}\eta\right)+c_2\cosh\left(\sqrt{-\mu}\eta\right)}\right]+a_0 \\ &\quad +2k_1^\alpha\sqrt{-\mu}\left[\frac{c_1\cosh\left(\sqrt{-\mu}\eta\right)+c_2\sinh\left(\sqrt{-\mu}\eta\right)}{c_1\sinh\left(\sqrt{-\mu}\eta\right)+c_2\cosh\left(\sqrt{-\mu}\eta\right)}\right]^{-1}. \end{aligned} \tag{3.8}$$

If we set $c_1 = 0$ and $c_2 \neq 0$ in (3.8) we have the solitary wave solution:

$$u_1(x,y,t) = 2k_1^\alpha\sqrt{-\mu}\left[\coth\left(\sqrt{-\mu}\eta\right)+\tanh\left(\sqrt{-\mu}\eta\right)\right]+a_0, \tag{3.9}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.8) we have the same solitary wave solution (3.9).

(ii) If $\mu > 0$ (Trigonometric function solutions)

In this case, we have the exact wave solution:

$$\begin{aligned} u(x,y,t) &= 2k_1^\alpha\sqrt{\mu}\left[\frac{-c_1\sin\left(\sqrt{\mu}\eta\right)+c_2\cos\left(\sqrt{\mu}\eta\right)}{c_1\cos\left(\sqrt{\mu}\eta\right)+c_2\sin\left(\sqrt{\mu}\eta\right)}\right]+a_0 \\ &\quad -2k_1^\alpha\sqrt{\mu}\left[\frac{-c_1\sin\left(\sqrt{\mu}\eta\right)+c_2\cos\left(\sqrt{\mu}\eta\right)}{c_1\cos\left(\sqrt{\mu}\eta\right)+c_2\sin\left(\sqrt{\mu}\eta\right)}\right]^{-1}. \end{aligned} \tag{3.10}$$

If we set in $c_1 = 0$ and $c_2 \neq 0$ (3.10) we have the periodic wave solution:

$$u_3(x,y,t) = 2k_1^\alpha\sqrt{\mu}\left[\cot\left(\sqrt{\mu}\eta\right)-\tan\left(\sqrt{\mu}\eta\right)\right]+a_0, \tag{3.11}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.10) we have the same periodic wave solution (3.11)

(iii) If $\mu = 0$ (Rational function solutions)

In this case we have the rational solution

$$u(x,y,t) = 2k_1^\alpha\left[\frac{c_2}{c_1+c_2\eta}\right]+a_0, \tag{3.12}$$

where $\eta = \dfrac{\xi^\alpha}{\Gamma(1+\alpha)}$.

Example 2. The space-time nonlinear fractional SRLW equation. This equation is well-known [1], [31] and has the form:

$$D_t^{2\alpha}u + D_x^{2\alpha}u + uD_t^\alpha(D_x^\alpha u) + D_t^\alpha u D_x^\alpha u + D_t^{2\alpha}\left(D_x^{2\alpha}u\right) = 0, \tag{3.13}$$

This equation has been discussed in [31] using a different technique, namely the fractional complex transformation technique combined with the improved $(G\prime/G)$-expansion method. Let us now solve equation (3.13) using the method of Section 2. To this end, we use the following wave transformation:

$$u(x,t) = u(\xi), \quad \xi = kx + ct, \tag{3.14}$$

where k and c are constants, to reduce equation (3.13) to the following fractional ODE

$$(k^{2\alpha} + c^{2\alpha})u + \frac{k^\alpha c^\alpha}{2}u^2 + k^{2\alpha}c^{2\alpha}D_\xi^{2\alpha}u = 0. \tag{3.15}$$

By balancing $D_\xi^{2\alpha}u$ with u^2, we have $N = 2$. Consequently, equation (3.15) has the formal solutions

$$u(\xi) = a_2\left(\frac{D_\xi^\alpha G}{G}\right)^2 + a_1\left(\frac{D_\xi^\alpha G}{G}\right) + a_0 + a_{-1}\left(\frac{D_\xi^\alpha G}{G}\right)^{-1} + a_{-2}\left(\frac{D_\xi^\alpha G}{G}\right)^{-2}, \tag{3.16}$$

where a_2, a_1, a_0, a_{-1}, a_{-2} are constants to be determined later , such that $a_2 \neq 0$ or $a_{-2} \neq 0$. Substituting (3.16) along with equation (2.5) into equation (3.15), collecting all the terms of the same orders $\left(\frac{D_\xi^\alpha G}{G}\right)^i$, $(i = 0, \pm 1, \pm 2, ...)$ and setting each coefficient to be zero, we have the following set of algebraic equations:

$$\left(\tfrac{D_\xi^\alpha G}{G}\right)^4 : \frac{k^\alpha c^\alpha}{2}a_2^2 + 6a_2k^{2\alpha}c^{2\alpha} = 0,$$

$$\left(\tfrac{D_\xi^\alpha G}{G}\right)^3 : k^\alpha c^\alpha a_1a_2 + k^{2\alpha}c^{2\alpha}(2a_1 + 10a_2\lambda) = 0,$$

$$\left(\tfrac{D_\xi^\alpha G}{G}\right)^2 : a_2(k^{2\alpha} + c^{2\alpha}) + \frac{k^\alpha c^\alpha}{2}(a_1^2 + 2a_0a_2) + k^{2\alpha}c^{2\alpha}(8a_2\mu + 4a_2\lambda^2 + 3a_1\lambda) = 0,$$

$$\left(\tfrac{D_\xi^\alpha G}{G}\right) : a_1(k^{2\alpha} + c^{2\alpha}) + k^\alpha c^\alpha(a_2a_{-1} + a_0a_1) + k^{2\alpha}c^{2\alpha}(6a_2\mu\lambda + 2a_1\mu + a_1\lambda^2) = 0,$$

$$\left(\tfrac{D_\xi^\alpha G}{G}\right)^0 : a_0(k^{2\alpha} + c^{2\alpha}) + \frac{k^\alpha c^\alpha}{2}(a_0^2 + 2a_2a_{-2} + 2a_1a_{-1}) + k^{2\alpha}c^{2\alpha}(2a_2\mu^2 + a_1\mu\lambda + a_{-1}\lambda + 2a_{-2}) = 0,$$

$$\left(\tfrac{D_\xi^\alpha G}{G}\right)^{-1} : a_{-1}(k^{2\alpha} + c^{2\alpha}) + k^\alpha c^\alpha(a_1a_{-2} + a_0a_{-1}) + k^{2\alpha}c^{2\alpha}(6a_{-2}\lambda + 2a_{-1}\mu + a_{-1}\lambda^2) = 0,$$

$$\left(\tfrac{D_\xi^\alpha G}{G}\right)^{-2} : a_{-2}(k^{2\alpha} + c^{2\alpha}) + \frac{k^\alpha c^\alpha}{2}(a_{-1}^2 + 2a_0a_{-2}) + k^{2\alpha}c^{2\alpha}(8a_{-2}\mu + 4a_{-2}\lambda^2 + 3a_{-1}\mu\lambda) = 0,$$

$$\left(\tfrac{D_\xi^\alpha G}{G}\right)^{-3} : k^\alpha c^\alpha a_{-1}a_{-2} + k^{2\alpha}c^{2\alpha}(2a_{-1}\mu^2 + 10a_{-2}\mu\lambda) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-4} : \frac{k^\alpha c^\alpha}{2} a_{-2}^2 + 6a_{-2}\mu^2 k^{2\alpha} c^{2\alpha} = 0.$$

On solving the above algebraic equations with the aid of Maple or Mathematica, we have the following cases:

Case1.

$$\begin{aligned} &\lambda = \lambda,\ c^\alpha = c^\alpha,\ k^\alpha = k^\alpha,\ \mu = \frac{1}{4k^{2\alpha}c^{2\alpha}}(\lambda^2 k^{2\alpha}c^{2\alpha} - \left(k^{2\alpha} + c^{2\alpha}\right)), \\ &a_{-1} = \frac{-3\lambda}{k^\alpha c^\alpha}(\lambda^2 k^{2\alpha}c^{2\alpha} - \left(k^{2\alpha} + c^{2\alpha}\right)),\ a_1 = a_2 = 0, \\ &a_{-2} = \frac{-3}{4k^{3\alpha}c^{3\alpha}}(\lambda^2 k^{2\alpha}c^{2\alpha} - \left(k^{2\alpha} + c^{2\alpha}\right))^2, \\ &a_0 = \frac{-1}{k^\alpha c^\alpha}(3\lambda^2 k^{2\alpha}c^{2\alpha} - \left(k^{2\alpha} + c^{2\alpha}\right)). \end{aligned} \tag{3.17}$$

Case2.

$$\begin{aligned} &c^\alpha = c^\alpha,\ k^\alpha = k^\alpha,\ \mu = \frac{-\left(144\left(k^{2\alpha} + c^{2\alpha}\right) - a_1^2\right)}{576k^{2\alpha}c^{2\alpha}},\ \lambda = \frac{-a_1}{12k^\alpha c^\alpha} \\ &a_0 = \frac{1}{48k^\alpha c^\alpha}\left(48\left(k^{2\alpha} + c^{2\alpha}\right) - a_1^2\right),\ a_{-1} = a_{-2} = 0, \\ &a_1 = a_1, a_2 = -12k^\alpha c^\alpha. \end{aligned} \tag{3.18}$$

Let us now write down the following exact solutions of the space-time fractional SRLW equation (3.13) for case 1 (similarly for case 2 which is omitted here for simplicity):

(i) If $\lambda^2 - 4\mu > 0$ (Hyperbolic function solutions)

In this case, we have the exact wave solution:

$$\begin{aligned} u(x,t) = &\frac{-\left[3\lambda^2 k^{2\alpha}c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha} - \frac{3\lambda\left[\lambda^2 k^{2\alpha}c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha} \\ &\times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha} + c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\left(\frac{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right) + c_2\sinh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)}{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right) + c_2\cosh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)}\right)\right]^{-1} \\ &- \frac{3\left[\lambda^2 k^{2\alpha}c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]^2}{4k^{3\alpha}c^{3\alpha}} \\ &\times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha} + c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\left(\frac{c_1\cosh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right) + c_2\sinh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)}{c_1\sinh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right) + c_2\cosh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)}\right)\right]^{-2} \end{aligned} \tag{3.19}$$

If we set $c_1 = 0$ and $c_2 \neq 0$ in (3.19) we have the solitary wave solution:

$$u_1(x,t) = \frac{-\left[3\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha} - \frac{3\lambda\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha} \times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\tanh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-1} - \frac{3\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]^2}{4k^{3\alpha}c^{3\alpha}}\left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\tanh\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-2} \tag{3.20}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.19) we have the solitary wave solution:

$$u_2(x,t) = \frac{-\left[3\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha} - \frac{3\lambda\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha} \times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\coth\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-1} - \frac{3\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]^2}{4k^{3\alpha}c^{3\alpha}}\left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\coth\left(\frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-2} \tag{3.21}$$

If $c_2 \neq 0$ and $c_1^2 < c_2^2$, then we have the solitary wave solution:

$$u_3(x,t) = \frac{-\left[3\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha} - \frac{3\lambda\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha} \times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\coth\left(\xi_1 + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-1} - \frac{3\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]^2}{4k^{3\alpha}c^{3\alpha}} \times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\coth\left(\xi_1 + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-2} \tag{3.22}$$

where $\xi_1 = \tanh^{-1}\left(\frac{c_2}{c_1}\right)$, while if $c_1 \neq 0$ and $c_2^2 < c_1^2$, then we have the solitary wave solution:

$$u_4(x,t) = \frac{-\left[3\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha} - \frac{3\lambda\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]}{k^\alpha c^\alpha} \times \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\tanh\left(\xi_1 + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-1} - \frac{3\left[\lambda^2 k^{2\alpha} c^{2\alpha} - (k^{2\alpha} + c^{2\alpha})\right]^2}{4k^{3\alpha}c^{3\alpha}} \left[-\frac{\lambda}{2} + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\tanh\left(\xi_1 + \frac{1}{2}\sqrt{\frac{k^{2\alpha}+c^{2\alpha}}{k^{2\alpha}c^{2\alpha}}}\eta\right)\right]^{-2} \tag{3.23}$$

where $\xi_1 = \coth^{-1}\left(\frac{c_2}{c_1}\right)$ and $\eta = \dfrac{\xi^\alpha}{\Gamma(1+\alpha)}$.

Example 3. The space-time nonlinear fractional STO equation. This equation is well-known [31], [40] and has the form:

$$D_t^\alpha u + 3\beta\left(D_x^\alpha u\right)^2 + 3\beta u^2 D_x^\alpha u + 3\beta u D_x^{2\alpha} u + \beta D_x^{3\alpha} u = 0, \tag{3.24}$$

where $0 < \alpha \leq 1$. equation (3.24) has been investigated in [40] using the fractional sub-equation method. It is also discussed in [31] using a different technique, namely the fractional complex transformation technique combined with the improved (G'/G)-expansion method. Let us now solve equation (3.24) using the method of Section 2. To this end, we use the wave transformation (3.14) to reduce equation (3.24) to the following fractional ODE:

$$c^\alpha u + 3\beta k^{2\alpha} u D_\xi^\alpha u + \beta k^\alpha u^3 + \beta k^{3\alpha} D_\xi^{2\alpha} u = 0. \tag{3.25}$$

By balancing $D_\xi^{2\alpha} u$ with u^3, we have $N = 1$. Consequently, equation (3.25) has the formal solution:

$$u(\xi) = a_1\left(\frac{D_\xi^\alpha G}{G}\right) + a_0 + a_{-1}\left(\frac{D_\xi^\alpha G}{G}\right)^{-1}, \tag{3.26}$$

where are a_1, a_0, a_{-1} constants to be determined later, such that $a_1 \neq 0$ or $a_{-1} \neq 0$. Substituting (3.26) along with equation (2.5) into equation (3.25), collecting all the terms of the same orders $\left(\frac{D_\xi^\alpha G}{G}\right)^i$, $(i = 0, \pm 1, \pm 2, ...)$ and setting each coefficient to zero, we have the following set of algebraic equations:

$$\left(\frac{D_\xi^\alpha G}{G}\right)^3 : a_1\beta k^\alpha(-3a_1k^\alpha + a_1^2 + 2k^{2\alpha}) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^2 : -3\beta k^{2\alpha}(a_1^2\lambda + a_0a_1) + 3a_0a_1^2\beta k^\alpha + 3a_1\lambda\beta k^{3\alpha} = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right) : a_1c^\alpha - 3\beta k^{2\alpha}(a_1^2\mu + a_0a_1\lambda) + \beta k^\alpha\left(3a_0^2a_1 + 3a_1^2a_{-1}\right) + \beta k^{3\alpha}\left(a_1\lambda^2 + 2a_1\mu\right) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^0 : a_0c^\alpha + 3\beta k^{2\alpha}(a_0a_{-1} - a_0a_1\mu) + \beta k^\alpha\left(a_0^3 + 6a_0a_1a_{-1}\right) + \beta k^{3\alpha}\left(a_1\mu\lambda + a_{-1}\lambda\right) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-1} : a_{-1}c^\alpha + 3\beta k^{2\alpha}(a_{-1}^2 + a_0a_{-1}\lambda) + \beta k^\alpha\left(3a_0^2a_{-1} + 3a_{-1}^2a_1\right) + \beta k^{3\alpha}\left(a_{-1}\lambda^2 + 2a_{-1}\mu\right) = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-2} : 3\beta k^{2\alpha}(a_{-1}^2\lambda + a_0a_{-1}\mu) + 3a_0a_{-1}^2\beta k^\alpha + 3a_{-1}\mu\lambda\beta k^{3\alpha} = 0,$$

$$\left(\frac{D_\xi^\alpha G}{G}\right)^{-3} : a_{-1}\beta k^\alpha(-3a_{-1}\mu k^\alpha + a_{-1}^2 + 2\mu^2k^{2\alpha}) = 0.$$

On solving the above algebraic equations with the aid of Maple or Mathematica, we have the following cases:

Case1.

$$\begin{gathered}\lambda = \lambda,\ \mu = \mu,\ \beta = \beta,\ k^\alpha = k^\alpha,\ c^\alpha = -\beta k^{3\alpha}(\lambda^2 - 4\mu), \\ a_1 = 2k^\alpha, a_0 = \lambda k^\alpha,\ a_{-1} = 0.\end{gathered} \tag{3.27}$$

Case2.

$$\begin{aligned}&\lambda=\lambda,\ \mu=\mu,\ \beta=\beta,\ k^\alpha=k^\alpha,\ c^\alpha=-\beta k^{3\alpha}(\lambda^2-4\mu),\\&a_1=0,a_0=-\lambda k^\alpha,\ a_{-1}=-2\mu k^\alpha.\end{aligned} \tag{3.28}$$

Let us now write down the following exact wave solutions of the space-time fractional STO equation (3.24) for case 1 (similarly for case 2 which is omitted here for simplicity):

(i) If $\lambda^2-4\mu>0$ (Hyperbolic function solutions)

In this case, we have the exact wave solution:

$$u(x,t)=k^\alpha\sqrt{\lambda^2-4\mu}\left[\frac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right)}\right]. \tag{3.29}$$

If we set $c_1=0$ and $c_2\neq 0$ in (3.29) we have the solitary wave solution:

$$u_1(x,t)=k^\alpha\sqrt{\lambda^2-4\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right). \tag{3.30}$$

while if we set $c_2=0$ and $c_1\neq 0$ in (3.29) we have the solitary wave solution:

$$u_2(x,t)=k^\alpha\sqrt{\lambda^2-4\mu}\coth\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right). \tag{3.31}$$

If $c_2\neq 0$ and $c_1^2<c_2^2$, then we have the solitary wave solution:

$$u_3(x,t)=k^\alpha\sqrt{\lambda^2-4\mu}\coth\left(\xi_1+\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right), \tag{3.32}$$

where $\xi_1=\tanh^{-1}\left(\frac{c_2}{c_1}\right)$, while if $c_1\neq 0$ and $c_2^2<c_1^2$, then we have the solitary wave solution:

$$u_4(x,t)=k^\alpha\sqrt{\lambda^2-4\mu}\tanh\left(\xi_1+\frac{\sqrt{\lambda^2-4\mu}}{2}\eta\right), \tag{3.33}$$

where $\xi_1=\coth^{-1}\left(\frac{c_2}{c_1}\right)$.

(ii) If $\lambda^2-4\mu<0$ (Trigonometric function solutions)

In this case, we have the exact wave solution:

$$u(x,t)=k^\alpha\sqrt{4\mu-\lambda^2}\left[\frac{-c_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\eta\right)}\right], \tag{3.34}$$

If we set $c_1 = 0$ and $c_2 \neq 0$ in (3.34) we have the periodic wave solution:

$$u_1(x,t) = k^\alpha \sqrt{4\mu - \lambda^2} \cot\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\eta\right), \tag{3.35}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.34) we have the periodic wave solution:

$$u_2(x,t) = -k^\alpha \sqrt{4\mu - \lambda^2} \tan\left(\frac{\sqrt{4\mu - \lambda^2}}{2}\eta\right). \tag{3.36}$$

If $c_2 \neq 0$ and $c_1^2 < c_2^2$, then we have the periodic wave solution:

$$u_3(x,t) = k^\alpha \sqrt{4\mu - \lambda^2} \cot\left(\xi_1 + \frac{\sqrt{4\mu - \lambda^2}}{2}\eta\right), \tag{3.37}$$

where $\xi_1 = \tan^{-1}\left(\frac{c_1}{c_2}\right)$, while if $c_1 \neq 0$ and $c_2^2 < c_1^2$, then we have the periodic wave solution:

$$u_4(x,t) = k^\alpha \sqrt{4\mu - \lambda^2} \tan\left(\xi_1 + \frac{\sqrt{4\mu - \lambda^2}}{2}\eta\right), \tag{3.38}$$

where $\xi_1 = \cot^{-1}\left(\frac{c_1}{c_2}\right)$ and $\eta = \frac{\xi^\alpha}{\Gamma(1+\alpha)}$.

Example 4. The space-time nonlinear fractional KPP equation. This equation is well-known [6], [8], [31], [32] and has the form:

$$D_t^\alpha u - D_x^{2\alpha} u + \mu_1 u + \gamma u^2 + \delta u^3 = 0, \tag{3.39}$$

where $0 < \alpha \leq 1$ and μ_1, γ, δ are nonzero constants. This equation is important in the physical fields and it includes the fisher equation. Huxlay equation, Burgers equation, Chaffee-Infanfe eqution and Fitzhugh-Nagumo equation. When $\alpha = 1$ equation (3.39) has been discussed in [6] by using the -expansion method and in [32] using the modified simple equation method. Equation (3.39) has been studied in [8] by using the homotopy perturbation method. This equation has been discussed in [31] by using a different technique, namely the fractional complex transformation technique combined with the improved (G'/G)-expansion method. Let us now solve equation (3.39) by using the method of Section 2. To this end, we use the wave transformation (3.14) to reduce equation (3.39) to the following fractional ODE:

$$c^\alpha D_\xi^\alpha u - k^{2\alpha} D_\xi^{2\alpha} u + \mu_1 u + \gamma u^2 + \delta u^3 = 0. \tag{3.40}$$

By balancing u^3 with $D_\xi^{2\alpha} u$, we have $N = 1$. Consequently, equation (3.40) has the formal solutions:

$$u(\xi) = a_1\left(\frac{D_\xi^\alpha G}{G}\right) + a_0 + a_{-1}\left(\frac{D_\xi^\alpha G}{G}\right)^{-1}, \tag{3.41}$$

where are a_1, a_0, a_{-1} constants to be determined later, such that $a_1 \neq 0$ or $a_{-1} \neq 0$. Substituting (3.41) along with equation (2.5) into equation (3.40), collecting all the terms of the same orders $\left(\frac{D_\xi^\alpha G}{G}\right)^i$, $(i = 0, \pm1, \pm2, ...)$ and setting each coefficient to zero, we have the following set of algebraic equations:

$\left(\frac{D_\xi^\alpha G}{G}\right)^3 : -2a_1k^{2\alpha} + \delta a_1^3 = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right)^2 : -a_1c^\alpha - 3a_1\lambda k^{2\alpha} + a_1^2\gamma + 3a_0a_1^2\delta = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right) : -a_1\lambda c^\alpha - k^{2\alpha}(a_1\lambda^2 + 2a_1\mu) + a_1\mu_1 + 2a_0a_1\gamma + \delta\left(3a_0^2a_1 + 3a_1^2a_{-1}\right) = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right)^0 : c^\alpha(a_{-1}-a_1\mu)-k^{2\alpha}(a_1\mu\lambda+a_{-1}\lambda)+a_0\mu_1+\gamma\left(a_0^2+2a_1a_{-1}\right)+\delta\left(a_0^3+6a_0a_1a_{-1}\right) = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right)^{-1} : a_{-1}\lambda c^\alpha-k^{2\alpha}(a_{-1}\lambda^2+2a_{-1}\mu)+a_{-1}\mu_1+2a_0a_{-1}\gamma+\delta\left(3a_1a_{-1}^2+3a_0^2a_{-1}\right) = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right)^{-2} : a_{-1}\mu c^\alpha - 3a_{-1}\mu\lambda k^{2\alpha} + a_{-1}^2\gamma + 3a_0a_{-1}^2\delta = 0,$

$\left(\frac{D_\xi^\alpha G}{G}\right)^{-3} : -2a_{-1}\mu^2k^{2\alpha} + \delta a_{-1}^3 = 0.$

On solving the above algebraic equations with the aid of Maple or Mathematica, we have the following cases:

Case1.

$$\begin{aligned} &\lambda = 0,\ \gamma = \gamma,\ k^\alpha = k^\alpha,\ \delta = \delta,\ \mu = \frac{-1}{32\delta k^{2\alpha}}(\gamma^2 - 4\delta\mu_1), \\ &a_{-1} = \frac{-(\gamma^2 - 4\delta\mu_1)}{32\delta k^\alpha}\sqrt{\frac{2}{\delta}},\ a_1 = -k^\alpha\sqrt{\frac{2}{\delta}},\ c^\alpha = \frac{k^\alpha\gamma}{\sqrt{2\delta}},\ a_0 = \frac{-\gamma}{2\delta}. \end{aligned} \tag{3.42}$$

Case2.

$$\begin{aligned} &\lambda = \lambda,\ \mu = \mu,\ k^\alpha = k^\alpha,\ \delta = \delta,\ \mu_1 = \frac{1}{4\delta}\left[-2\delta k^{2\alpha}(\lambda^2 - 4\mu) + \gamma^2\right], \\ &a_{-1} = 0,\ a_0 = \left[\lambda k^\alpha\sqrt{\tfrac{1}{2\delta}} - \tfrac{\gamma}{2\delta}\right],\ a_1 = 2k^\alpha\sqrt{\tfrac{1}{2\delta}},\ c^\alpha = -k^\alpha\gamma\sqrt{\tfrac{1}{2\delta}}. \end{aligned} \tag{3.43}$$

Let us now write down the following exact solutions of the space-time fractional KPP equation (3.39) for case 1 (similarly for case 2 which is omitted here for simplicity):

(i) If $\mu < 0$ (Hyperbolic function solutions)

In this case, we have the exact wave solution:

$$\begin{aligned} u(x,t) &= -k^\alpha\sqrt{\frac{-2\mu}{\delta}}\left[\frac{c_1\cosh(\sqrt{-\mu}\eta) + c_2\sinh(\sqrt{-\mu}\eta)}{c_1\sinh(\sqrt{-\mu}\eta) + c_2\cosh(\sqrt{-\mu}\eta)}\right] - \frac{\gamma}{2\delta} \\ &\quad -\frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha}\sqrt{\frac{-1}{2\mu\delta}}\left[\frac{c_1\cosh(\sqrt{-\mu}\eta) + c_2\sinh(\sqrt{-\mu}\eta)}{c_1\sinh(\sqrt{-\mu}\eta) + c_2\cosh(\sqrt{-\mu}\eta)}\right]^{-1}. \end{aligned} \tag{3.44}$$

If we set $c_1 = 0$ and $c_2 \neq 0$ in (3.44) we have the solitary wave solution:

$$\begin{aligned} u_1(x,t) = & -k^\alpha\sqrt{\frac{-2\mu}{\delta}}\tanh\left(\sqrt{-\mu}\eta\right) - \frac{\gamma}{2\delta} \\ & - \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha}\sqrt{\frac{-1}{2\mu\delta}}\coth\left(\sqrt{-\mu}\eta\right), \end{aligned} \tag{3.45}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.44) we have the solitary wave solution:

$$\begin{aligned} u_2(x,t) = & -k^\alpha\sqrt{\frac{-2\mu}{\delta}}\coth\left(\sqrt{-\mu}\eta\right) - \frac{\gamma}{2\delta} \\ & - \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha}\sqrt{\frac{-1}{2\mu\delta}}\tanh\left(\sqrt{-\mu}\eta\right), \end{aligned} \tag{3.46}$$

If $c_2 \neq 0$ and $c_1^2 < c_2^2$, then we have the solitary wave solution:

$$\begin{aligned} u_3(x,t) = & -k^\alpha\sqrt{\frac{-2\mu}{\delta}}\tanh\left(\xi_1 + \sqrt{-\mu}\eta\right) - \frac{\gamma}{2\delta} \\ & - \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha}\sqrt{\frac{-1}{2\mu\delta}}\coth\left(\xi_1 + \sqrt{-\mu}\eta\right), \end{aligned} \tag{3.47}$$

where $\xi_1 = \tanh^{-1}\left(\frac{c_1}{c_2}\right)$.

(ii) If $\mu > 0$ (Trigonometric function solutions)

In this case, we have the exact wave solutions:

$$\begin{aligned} u(x,t) = & -k^\alpha\sqrt{\frac{2\mu}{\delta}}\left[\frac{-c_1\sin\left(\sqrt{\mu}\eta\right) + c_2\cos\left(\sqrt{\mu}\eta\right)}{c_1\cos\left(\sqrt{\mu}\eta\right) + c_2\sin\left(\sqrt{\mu}\eta\right)}\right] - \frac{\gamma}{2\delta} \\ & - \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha}\sqrt{\frac{1}{2\mu\delta}}\left[\frac{-c_1\sin\left(\sqrt{\mu}\eta\right) + c_2\cos\left(\sqrt{\mu}\eta\right)}{c_1\cos\left(\sqrt{\mu}\eta\right) + c_2\sin\left(\sqrt{\mu}\eta\right)}\right]^{-1}. \end{aligned} \tag{3.48}$$

If we set $c_1 = 0$ and $c_2 \neq 0$ in (3.48) we have the periodic wave solution:

$$u_1(x,t) = -k^\alpha\sqrt{\frac{2\mu}{\delta}}\cot\left(\sqrt{\mu}\eta\right) - \frac{\gamma}{2\delta} - \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha}\sqrt{\frac{1}{2\mu\delta}}\tan\left(\sqrt{\mu}\eta\right), \tag{3.49}$$

while if we set $c_2 = 0$ and $c_1 \neq 0$ in (3.48) we have the periodic wave solution:

$$u_2(x,t) = k^\alpha\sqrt{\frac{2\mu}{\delta}}\tan\left(\sqrt{\mu}\eta\right) - \frac{\gamma}{2\delta} + \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha}\sqrt{\frac{1}{2\mu\delta}}\cot\left(\sqrt{\mu}\eta\right). \tag{3.50}$$

If $c_2 \neq 0$ and $c_1^2 < c_2^2$, then we have the periodic wave solution:

$$\begin{aligned} u_3(x,t) = & -k^\alpha\sqrt{\frac{2\mu}{\delta}}\cot\left(\xi_1 + \sqrt{\mu}\eta\right) - \frac{\gamma}{2\delta} \\ & - \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha}\sqrt{\frac{1}{2\mu\delta}}\tan\left(\xi_1 + \sqrt{\mu}\eta\right), \end{aligned} \tag{3.51}$$

where $\xi_1 = \tan^{-1}\left(\frac{c_1}{c_2}\right)$, while if $c_1 \neq 0$ and $c_2^2 < c_1^2$, then we have the periodic wave solution:

$$u_4(x,t) = -k^\alpha\sqrt{\frac{2\mu}{\delta}}\tan\left(\xi_1 + \sqrt{\mu}\eta\right) - \frac{\gamma}{2\delta} - \frac{(\gamma^2 - 4\delta\mu_1)}{16\delta k^\alpha}\sqrt{\frac{1}{2\mu\delta}}\cot\left(\xi_1 + \sqrt{\mu}\eta\right). \tag{3.52}$$

where $\xi_1 = \cot^{-1}\left(\frac{c_1}{c_2}\right)$ and $\eta = \dfrac{\xi^\alpha}{\Gamma(1+\alpha)}$.

(iii) If $\mu = 0$ (Rational function solution)

In this case we have the rational solution

$$u(x,t) = -k^\alpha\sqrt{\frac{2}{\delta}}\left[\frac{c_2}{c_1 + c_2\eta}\right] - \frac{\gamma}{2\delta} - \frac{1}{32\delta k^\alpha}\sqrt{\frac{2}{\delta}}\left[\frac{c_2}{c_1 + c_2\eta}\right]^{-1}. \tag{3.53}$$

where $\eta = \dfrac{\xi^\alpha}{\Gamma(1+\alpha)}$.

4. Physical explanations of our obtained solutions

Solitary, periodic and rational wave solutions can be obtained from the exact solutions by setting particular values in its unknown parameters. In this section, we have presented some graphs for solitary and periodic wave solutions constructed by taking suitable values of involved unknown parameters to visualize the underlying mechanism of the original equations (3.1), (3.13), (3.24) and (3.39). By using the mathematical software Maple, the plots of some solutions have been shown in Figs.1-5 as follows:

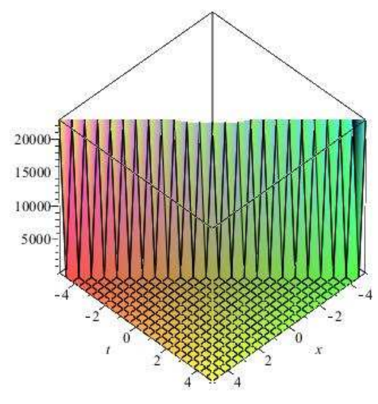

Fig.1. The plot of $u_1(x,0,t)$ of (3.9) with $k_1 = 1,\ \mu = -1,\ a_0 = 0,\ c = 1,\ \alpha = \frac{1}{2}$

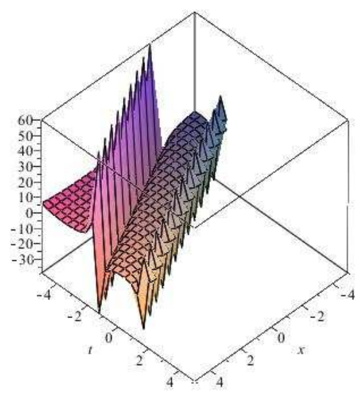

Fig.2. The plot of $u_3(x,0,t)$ of (3.11) with $k_1 = 1,\ \mu = 1,\ a_0 = 0,\ c = -1,\ \alpha = \frac{1}{2}$

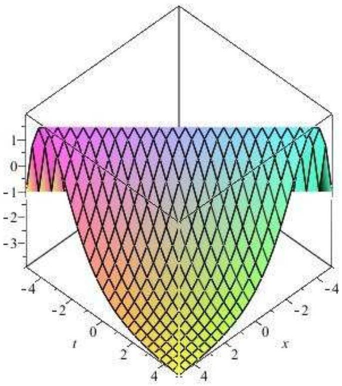

Fig.3. The plot of (3.20) with $k = 1$, $c = 1,\ \lambda = 2,\ \alpha_0 = \frac{2}{3}$

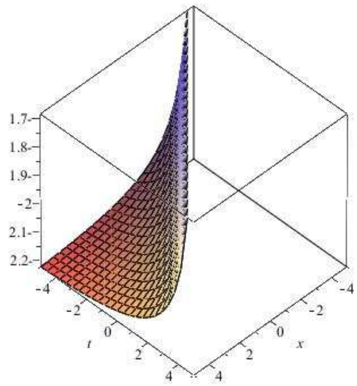

Fig.4. The plot of (3.45) with $k = 1$, $c = -1,\ \mu = -1,\ \mu_1 = 3,\ \delta = 1,$ $\gamma = 2,\ \alpha = \frac{3}{8}$

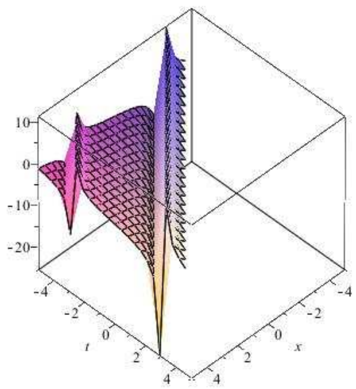

Fig.5. The plot of (3.50) with $k = 1,\ c = -1,$ $\mu = 1,\ \mu_1 = -1,\ \delta = 1,\ \gamma = 2,\ \alpha = \frac{1}{2}$

5. Conclusions and discussions

The fractional $(D_\xi^\alpha G/G)$-expansion method was applied in this paper to construct new exact traveling wave solutions for four nonlinear space- time fractional partial differential equations (PDEs) namely, the space-time fractional Potential Kadomtsev Petviashvili (PKP) equation, the space-time fractional symmetric regularized long wave (SRLW) equation, the space-time fractional Sharma-Tasso Olver (STO) equation and the space-time fractional Kolmogorov-Petrovskii-Piskunov (KPP) equation. The graphical representations of some solutions of these equations have been presented. In [31] these equations have been discussed by using the fractional complex transformation technique combined with the improved (G'/G)-expansion method for finding the exact solutions of these equations. On comparing our results in this paper with those obtained in [31] we deduce that our results are new and different. This method can be applied to many other nonlinear fractional partial differential equations (NFPDEs) in the mathematical physics. Finally, with help of Maple or Mathematica, we have made sure that our new solutions obtained in this article satisfy the original four space- time fractional PDEs.

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