A MONGE-AMPÈRE TYPE OPERATOR IN 2-DIMENSIONAL SPECIAL LAGRANGIAN GEOMETRY

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Abstract. In this paper, we construct a Monge-Ampère type operator in 2-dimensional special Lagrangian geometry based on the calibrated geometry developed by Harvey and Lawson. We give a special Lagrangian version of the Chern-Levine-Nirenberg estimate for complex Monge-Ampère operator, which enables us to define the Monge-Ampère type operator on continuous ϕ-plurisubharmonic functions on a domain in $\mathbb{C}^2$.

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1. Introduction

In [8]-[11], Harvey and Lawson introduce several concrete calibrations, and study operators, which define the notion of plurisubharmonic functions in calibrated geometry. These functions generalize the classical plurisubharmonic functions from complex geometry and enjoy their important properties. Based on that fact, we present a operator in 2-dimensional special Lagrangian geometry which has similar properties as complex Monge-Ampère operator.

Recall the definitions in [8], [10]. A *calibration* ϕ of degree p is a closed p-form on a Riemannian manifold X with the property that $\phi(\xi) \leq 1$ for all unit simple tangent p-vectors ξ on X. A unit simple tangent p-vectors ξ on X satisfying

$$\phi(\xi) = 1 \tag{1}$$

is called ϕ-plane. We denote by $G(\phi)$ the set of all ϕ-planes on X. If the covariant derivative of a calibration is zero, then it is called parallel. In [9], Harvey and Lawson give the definition of ϕ-plurisubharmonic function for general calibrations.

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Here we only need the definition for parallel calibrations. Let's recall it. For any smooth function f on X, the d^ϕ-operator is defined as

$$d^\phi f := \nabla f \rfloor \phi,$$

where $\rfloor$ is the interior product of a differential form and ∇f is the gradient of f on X. For a parallel calibration ϕ, a function $f \in C^\infty(X)$ is called *ϕ-plurisubharmonic* if

$$dd^\phi f(\xi) \ \geq \ 0, \quad \text{for each } \xi \in G(\phi).$$

Let $D'(X)$ be the dual of the space of smooth functions on X. A distribution $f \in D'(X)$ is called *ϕ-plurisubharmonic* if

$$dd^\phi f(\xi)(\lambda) \ \geq 0$$

for every smooth section $\xi \in G(\phi)$ and every smooth compactly supported non-negative multiple λ of the volume form on X. It is easy to see this definition is compatible with the definition of $f \in C^\infty(X) \subset D'(X)$. Denote $PSH(X, \phi)$ both the smooth functions and the distributions which are ϕ-plurisubharmonic on X. In n-dimensional complex Euclidean space $\mathbb{C}^n$ with coordinates $z = (z_1, z_2, ..., z_n)$, the closed n-form $\phi = RedZ = Re(dz_1 \wedge dz_2 \wedge ... \wedge dz_n)$ is a parallel calibration, called *special Lagrangian calibration.* Consider the grassmannian $G(n, 2n)$ of oriented real n-planes in $\mathbb{C}^n$. $\xi \in G(n, 2n)$ is called *Lagrangian* if $Ju \perp \xi$ for all $u \in \xi$, where J is the complex structure on $\mathbb{R}^{2n}$. Let SU_n be the special unitary group. If Lagrangian n-plane ξ satisfies

$$\xi = A\xi_0,$$

where $A \in SU_n$ and $\xi_0 \equiv span_{\mathbb{R}}\{e_1, e_3, ..., e_{2n-1}\} \cong \mathbb{R}^n$, then ξ is called *special Lagrangian.* Here $e_1, ..., e_{2n}$ are orthonormal basis for $\mathbb{R}^{2n}$, e_{2j-1} is a vector with the jth position 1 and others 0, e_{2j} is a vector with the $(n + j)$th position 1 and others 0. We know that a unit simple tangent vector ξ is a ϕ-plane with $\phi = RedZ$ if and only if ξ is special Lagrangian, see Proposition 2.2, or Theorem 1.10 in Section 3 of [8].

For the calibration $\phi = Redz_1 \wedge dz_2$ on special Lagrangian geometry $\mathbb{C}^2$, let f_1, f_2 be two smooth ϕ-plurisubharmonic functions on $\mathbb{C}^2$. We define the Monge-Ampère type operator on f_1 and f_2 as

$$dd^\phi f_1 \wedge dd^\phi f_2.$$

That operator has similar properties as complex Monge-Ampère operator. Especially, it has an estimate, which is similar to the Chern-Levine-Nirenberg estimate for complex Monge-Ampère operator established in [3]-[5], [14] etc. Now we give that estimate for $dd^\phi f_1 \wedge dd^\phi f_2$ on special Lagrangian geometry $\mathbb{C}^2$.

Theorem 1.1. *Let Ω be an open neighborhood of a compact set $K \subseteq \mathbb{C}^2$ and U be a compact neighborhood of $K \subset \Omega$. For any $\psi \in C_0^\infty(\Omega)$, there exist a constant $C > 0$, which depends on U, Ω and $\|\psi\|_{C^2}$, such that for smooth functions $f_1, f_2 \in PSH(\Omega, \phi)$ with $\phi = Re(dz_1 \wedge dz_2)$, the following estimate holds,*

$$\left| \int_K \psi dd^\phi f_1 \wedge dd^\phi f_2 \right| \leq C \|f_1\|_K \|f_2\|_U. \tag{2}$$

Based on that estimate, we can define the Monge-Ampère type operator on continuous ϕ-plurisubharmonic functions.

Theorem 1.2. *Let f_1, f_2 be continuous ϕ-plurisubharmonic functions on a domain $\Omega \subseteq \mathbb{C}^2$. Let $f_{1,N}, f_{2,N}$ be two sequences of twice continuously differentiable ϕ-plurisubharmonic functions converging to f_1 and f_2 uniformly on compact subsets of Ω respectively. Then $dd^\phi f_{1,N} \wedge dd^\phi f_{2,N}$ weakly converges to a distribution on Ω. This distribution depends only on f_1 and f_2, not on the choice of approximating sequences $f_{1,N}$ and $f_{2,N}$.*

We denote by $dd^\phi f_1 \wedge dd^\phi f_2$ the limit in Theorem 1.2. So the Monge-Ampère type operator is well defined on continuous ϕ-plurisubharmonic functions f_1 and f_2, though the currents $dd^\phi f_1$ and $dd^\phi f_2$ can't do exterior product on the form.

The complex Monge-Ampère operator is a positive distribution, so it is a measure. But we only know that the special Lagrangian version Monge-Ampère operator is a distribution. Hence, perhaps it doesn't enjoy some deeper results on complex Monge-Ampère operator. For example, Alesker [1], [2] deals with boundary value problem for Monge-Ampère equation. In [1], he first proves that Monge-Ampère operator is well-defined as a measure, and then he proves the uniqueness for the boundary value problem. In addition, in [2], he also proves the existence of the solution. So, next step, we need to investigate the special Lagrangian version Monge-Ampère operator from some other points of view.

The proof of Theorem 1.2 roughly follows the lines of the classical proof. In the process of proving, we need a fact that the linear combinations of delta-functions of special Lagrangian n-planes in $\mathbb{C}^n$ are dense in the space of all generalized functions, which can be induced by the fact that Radon transform over special Lagrangian n-planes is injective, see Proposition 3.1. Hence, we show the Radon transform over the special Lagrangian n-planes in section **??**.

2. Monge-Ampère type operator on $\mathbb{C}^2$

In this section, we give the representation of $dd^\phi f$ for special Lagrangian calibration ϕ and a symmetrical property on the operator $dd^\phi f_1 \wedge dd^\phi f_2$ in $\mathbb{C}^2$, which enables us to get the estimate in Theorem 1.1.

Consider the special Lagrangian calibration $\phi = \; RedZ$ on $\mathbb{C}^n$. Let Z_{ij} be the form obtained from $dz_1 \wedge dz_2 \wedge ... \wedge dz_n$ by replacing dz_i with $d\bar{z}_j$ (in the ith position). For a smooth function f, we have the following proposition.

Proposition 2.1.

$$dd^{\phi}f = 2Re\left\{\sum_{k,j=1}^{n}\frac{\partial^2 f}{\partial\bar{z}_k\partial\bar{z}_j}Z_{kj}\right\} + \frac{1}{2}(\Delta f)Re(dZ). \tag{3}$$

Proof. Given a smooth ϕ-plurisubharmonic function f. Since

$$d^{\phi}f{=}\nabla f\rfloor\phi{=}\sum_{k=1}^{n}(-1)^{k-1}\left(\frac{\partial f}{\partial\bar{z}_k}dz_1\wedge...\wedge\widehat{dz_k}\wedge...\wedge dz_n+\frac{\partial f}{\partial z_k}d\bar{z}_1\wedge...\wedge\widehat{d\bar{z}_k}\wedge...\wedge d\bar{z}_n\right),$$

where $dz_1\wedge...\wedge\widehat{dz_k}\wedge...\wedge dz_n$ denotes the form obtained from $dz_1\wedge dz_2\wedge...\wedge dz_n$ by removing dz_k. We have,

$$\begin{aligned}
dd^{\phi}f = d(\nabla f\rfloor\phi) &= \sum_{k=1}^{n}\frac{\partial^2 f}{\partial\bar{z}_k\partial z_k}dz_1\wedge...dz_n + \sum_{k,j=1}^{n}\frac{\partial^2 f}{\partial\bar{z}_k\partial\bar{z}_j}Z_{kj}\\
&+\frac{\partial^2 f}{\partial z_k\partial z_j}\overline{Z}_{kj} + \frac{\partial^2 f}{\partial\bar{z}_k\partial z_k}d\bar{z}_1\wedge...d\bar{z}_n\\
&= 2Re\left\{\sum_{k,j=1}^{n}\frac{\partial^2 f}{\partial\bar{z}_k\partial\bar{z}_j}Z_{kj}\right\} + \frac{1}{2}(\Delta f)Re(dZ).
\end{aligned}$$

Particularly, for $n=2$, we have

$$\begin{aligned}
dd^{\phi}f &= \frac{1}{2}\left(\frac{\partial^2 f}{\partial x_1^2} - \frac{\partial^2 f}{\partial y_1^2}\right)(dx_1\wedge dx_2{+}dy_1\wedge dy_2)\\
&-\frac{\partial^2 f}{\partial x_1\partial y_1}(dx_1\wedge dy_2{-}dy_1\wedge dx_2)\\
&+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x_2^2} - \frac{\partial^2 f}{\partial y_2^2}\right)(dx_1\wedge dx_2 + dy_1\wedge dy_2)\\
&+\frac{\partial^2 f}{\partial x_2\partial y_2}(dx_1\wedge dy_2 - dy_1\wedge dx_2)\\
&-\left(\frac{\partial^2 f}{\partial x_1\partial y_2} + \frac{\partial^2 f}{\partial x_2\partial y_1}\right)dx_2\wedge dy_2 + \left(\frac{\partial^2 f}{\partial x_2\partial y_1} + \frac{\partial^2 f}{\partial y_2\partial x_1}\right)dx_1\wedge dy_1 \qquad (4)\\
&+\frac{1}{2}\sum_{k=1}^{2}\left(\frac{\partial^2 f}{\partial x_k^2} + \frac{\partial^2 f}{\partial y_k^2}\right)(dx_1\wedge dx_2 - dy_1\wedge dy_2)\\
&= \left(\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f_2}{\partial x_2^2}\right)dx_1\wedge dx_2 - \left(\frac{\partial^2 f}{\partial y_1^2} + \frac{\partial^2 f}{\partial y_2^2}\right)dy_1\wedge dy_2\\
&+\left(\frac{\partial^2 f}{\partial x_2\partial y_2} - \frac{\partial^2 f}{\partial x_1\partial y_1}\right)(dx_1\wedge dy_2 - dy_1\wedge dx_2)\\
&+\left(\frac{\partial^2 f}{\partial x_1\partial y_2} + \frac{\partial^2 f}{\partial x_2\partial y_1}\right)(dx_1\wedge dy_1 - dx_2\wedge dy_2).
\end{aligned}$$

■

Proposition 2.2. *For the calibration $\phi = RedZ$ in $\mathbb{C}^n$, with $dZ = dz_1 \wedge dz_2 \wedge ... \wedge dz_n$, the unit simple n-plane ξ is a ϕ-plane if and only if ξ is special Lagrangian.*

Before proving the proposition, we need two lemmas. The proofs of those lemmas have appeared in [8], so we cite them without proofs.

Lemma 2.1. (Theorem 1.7 in [8]) *For any $\xi \in G(n, 2n)$,*

$$| dZ(\xi) |^2 = \big(RedZ(\xi)\big)^2 + \big(ImdZ(\xi)\big)^2 = | \xi \wedge J\xi |.$$

Lemma 2.2. (Lemma 1.9 in [8]) $|\xi \wedge J\xi| \leq |\xi|^2$, *for any $\xi \in G(n, 2n)$, with equality if and only if ξ is Lagrangian.*

Proof of Proposition 2.2. Let ξ be a ϕ-plane. Then $|\xi| = 1$ and

$$\phi(\xi) = RedZ(\xi) = 1. \tag{5}$$

Denote $\varepsilon_1, ..., \varepsilon_n$ an oriented orthonormal basis of ξ. Then by Lemma 2.1 and Lemma 2.2, $\big(RedZ(\xi)\big)^2 + \big(ImdZ(\xi)\big)^2 = |\xi \wedge J\xi| \leq |\xi|^2 = 1$. So $ImdZ(\xi) = 0$ and $|\xi \wedge J\xi| = |\xi|^2$ by (5). Hence, ξ is special Lagrangian.

For the inverse, let ξ be a unit simple vector, i.e., $|\xi| = 1$. Suppose ξ is special Lagrangian and $\varepsilon_1, ..., \varepsilon_n$ is an oriented basis for $\xi \in G(n, 2n)$. Denote $e_1, ..., e_n, Je_1, ..., Je_n$ the standard basis for $\mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{C}^n$, and A the linear map sending e_j to ε_j and Je_j to $J\varepsilon_j$. Then $det_{\mathbb{C}} A = 1$ and so $ImdZ(\xi) = Im(det_{\mathbb{C}} A) = 0$. Then, by Lemma 2.1 and Lemma 2.2 again,

$$\big(RedZ(\xi)\big)^2 = \big(RedZ(\xi)\big)^2 + \big(ImdZ(\xi)\big)^2 = |\xi \wedge J\xi| = 1.$$

So ξ is a ϕ-plane. ∎

Let $z = (z_1, z_2)$ be the coordinates of $\mathbb{C}^2$, where $z = x + iy$ with $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Now we give a proposition of $dd^\phi f_1 \wedge dd^\phi f_2$ in $\mathbb{C}^2$. We follow the approach to quaternionic Monge-Ampère operator of S. Alesker in [1].

Proposition 2.3. *Let f_0, f_1, f_2 be real valued compactly supported smooth functions on $\mathbb{C}^2$. Then the 3-linear functional*

$$L(f_0, f_1, f_2) = \int_{\mathbb{C}^2} f_0 \; dd^\phi f_1 \wedge dd^\phi f_2, \tag{6}$$

is symmetric with respect to f_0, f_1, f_2.

Proof. Since $dd^\phi f_1$ is a 2-form, we have $dd^\phi f_1 \wedge dd^\phi f_2 = dd^\phi f_2 \wedge dd^\phi f_1$, i.e., L is symmetric with respect to f_1 and f_2. Thus it is sufficient to check that

$$L(f_0, f_1, f_2) = L(f_1, f_0, f_2), \tag{7}$$

for any smooth compactly supported functions f_0, f_1, f_2. Both sides of (7) make sense if f_0 is a generalized function. Since linear combinations of delta-functions of points δ_z are dense in the space of all the generalized functions, it is sufficient to prove (7) for $f_0 = \delta_0$, namely

$$dd^\phi f_1 \wedge dd^\phi f_2|_{z=0} = \int_{\mathbb{C}^2} f_1 dd^\phi \delta_0 \wedge dd^\phi f_2. \tag{8}$$

Clearly the right hand side of (8) depends only on derivatives at 0 of f_1, f_2, up to order 2. Consider the terms of the Taylor series of f_1 at 0:

$$\begin{aligned} f_1(z) &= f_1(0) + \sum_{i=1}^{2} f'_{1x_i}(0)x_i + f'_{1y_i}(0)y_i \\ &+ \frac{1}{2!}\sum_{i,j=1}^{2}\left(f''_{1x_ix_j}(0)x_ix_j + f''_{1y_iy_j}(0)y_iy_j + 2f''_{1x_iy_j}(0)x_iy_j\right) + O(|z|^3) \\ &= g(z) + h(z) + O(|z|^3), \end{aligned}$$

where g is a polynomial of degree one and h is a quadratic term. So it is sufficient to prove the following two statements:

Case 1. $f_0 = \delta_0$, $f_1 = h$ is a smooth compactly supported function which is a homogeneous polynomial of degree 2 in a neighborhood of 0.

Case 2. $f_0 = \delta_0$, $f_1 = g$ is a smooth compactly supported function which is a polynomial of degree 1 in a neighborhood of 0.

For *Case* 1. Write down $L(h, \delta_0, f_2) = \int_{\mathbb{C}^2} h(z) dd^\phi \delta_0 \wedge dd^\phi f_2$ as a polynomial in $\frac{\partial^2 f_2}{\partial x_i \partial x_j}, \frac{\partial^2 f_2}{\partial x_i \partial y_j}$ etc. and in $\frac{\partial^2 \delta_0}{\partial x_i \partial x_j}, \frac{\partial^2 \delta_0}{\partial x_i \partial y_j}$ etc. Then we see that the derivatives of δ_0 enter at each monomial only once because of linearity of L with respect to each arguments. For example, consider a monomial containing $\frac{\partial^2 \delta_0}{\partial x_i \partial x_j}$,

$$\begin{aligned} \int_{\mathbb{C}^2} h(z)\frac{\partial^2 \delta_0}{\partial x_i \partial x_j} \cdot \partial^2 f_2 &= \frac{\partial^2}{\partial x_i \partial x_j}(h(z) \cdot \partial^2 f_2)|_{z=0} = \frac{\partial^2 h(z)}{\partial x_i \partial x_j}(0) \cdot \partial^2 f_2(0) \\ &= \frac{\partial^2 h(z)}{\partial x_i \partial x_j} \cdot \partial^2 f_2|_{z=0}. \end{aligned}$$

The second identity holds since h and the first derivatives of h at 0 vanish. Thus in each monomial the term $h \cdot \frac{\partial^2 \delta_0}{\partial x_i \partial x_j} \partial^2 f_2$ is just replaced by $\frac{\partial^2 h(z)}{\partial x_i \partial x_j}(0)\partial^2 f_2(0)$. Hence the final expression is

$$\int_{\mathbb{C}^2} h(z) dd^\phi \delta_0 \wedge dd^\phi f_2 = dd^\phi h(z) \wedge dd^\phi f_2|_{z=0}.$$

This proves *Case* 1.

Before proving *Case* 2, we claim that for any smooth compactly supported function g which is equal to a polynomial of degree 1 inside a fixed neighborhood U of origin and a generalized function f_3 with support contained in U, we have

$$\int_{\mathbb{C}^n} g(z) dd^\phi f_3 \wedge dd^\phi f_2 = 0, \tag{9}$$

where f_2 is a smooth function. Since g is a polynomial of degree 1, we have $dd^\phi g$=0. Let f_3 be δ_0 in equation (9). By using that claim, we have $\int_{\mathbb{C}^2} g dd^\phi \delta_0 \wedge dd^\phi f_2 = 0$. Hence,

$$0 = dd^\phi g \wedge dd^\phi f_2|_{z=0} = \int_{\mathbb{C}^2} g dd^\phi \delta_0 \wedge dd^\phi f_2.$$

This proves *Case* 2.

Let's prove the claim. By the following Proposition 3.1, we know that the linear combinations of delta-functions of special Lagrangian 2-planes are dense in the space of all generalized functions in $\mathbb{C}^2$. Hence it is sufficient to prove the claim for $f_3 = \delta_{\xi_0}$, where ξ_0 is a certain special Lagrangian 2-plane through 0 satisfying $y_1 = 0, y_2 = 0$. Since δ_{ξ_0} is invariant with respect to translations in directions $x_1 = 0, x_2 = 0$, we have $\frac{\partial \delta_{\xi_0}}{\partial x_i} = 0$, $\frac{\partial^2 \delta_{\xi_0}}{\partial x_i \partial x_j} = 0$, $\frac{\partial^2 \delta_{\xi_0}}{\partial x_i \partial y_j} = 0$ for $i, j = 1, 2$. By (4) in Proposition 2.1, we have $dd^\phi \delta_{\xi_0} = -\left(\frac{\partial^2 \delta_{\xi_0}}{\partial y_1^2} + \frac{\partial^2 \delta_{\xi_0}}{\partial y_2^2}\right) dy_1 \wedge dy_2$. Hence, by Proposition 2.1, we have

$$\int_{\mathbb{C}^2} g(z) dd^\phi \delta_{\xi_0} \wedge dd^\phi f_2 = \int_{\mathbb{C}^2} g(z)(\frac{\partial^2 \delta_{\xi_0}}{\partial y_1^2} + \frac{\partial^2 \delta_{\xi_0}}{\partial y_2^2})(\frac{\partial^2 f_2}{\partial x_1^2} + \frac{\partial^2 f_2}{\partial x_2^2}) dV, \tag{10}$$

where $dV = dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$ is the volume element of $\mathbb{C}^2$. Without loss of generality, we can assume $g(z)$ is a polynomial of degree one on z_1.

Denote

$$A_i = \int_{\mathbb{C}^2} g(z_1) \frac{\partial^2 \delta_{\xi_0}}{\partial y_1^2} \cdot \frac{\partial^2 f_2}{\partial x_i^2}, \quad B_i = \int_{\mathbb{C}^2} g(z_1) \frac{\partial^2 \delta_{\xi_0}}{\partial y_2^2} \cdot \frac{\partial^2 f_2}{\partial x_i^2}, i = 1, 2.$$

Let us consider monomials in the right side of (10). For $i = 1, 2$,

$$\begin{aligned}
A_i &= \int_{\mathbb{C}^2} \delta_{\xi_0} \frac{\partial^2}{\partial y_1^2}\left(g(z_1)\frac{\partial^2 f_2}{\partial x_i^2}\right) \\
&= \int_{\mathbb{C}^2} \delta_{\xi_0}\left(g(z_1)\frac{\partial^4 f_2}{\partial x_i^2 \partial y_1^2} + 2\frac{\partial g(z_1)}{\partial y_1} \cdot \frac{\partial^3 f_2}{\partial x_i^2 \partial y_1}\right) \\
&= \int_{\mathbb{C}^2} \frac{\partial \delta_{\xi_0}}{\partial x_i}\left(g(z_1) \cdot \frac{\partial^3 f_2}{\partial x_i \partial y_1^2}\right) - \int_{\mathbb{C}^2} \delta_{\xi_0} \frac{\partial g(z_1)}{\partial x_i} \cdot \frac{\partial^3 f_2}{\partial x_i \partial y_1^2} + 2C_1 \int_{\mathbb{C}^2} \delta_{\xi_0} \cdot \frac{\partial^3 f_2}{\partial x_i^2 \partial y_1} \\
&= 0 - \int_{\mathbb{C}^2} \frac{\partial \delta_{\xi_0}}{\partial x_i}\left(\frac{\partial g(z_1)}{\partial x_i} \cdot \frac{\partial^2 f_2}{\partial y_1^2}\right) + \int_{\mathbb{C}^2} \delta_{\xi_0}\left(\frac{\partial^2 g(z_1)}{\partial x_i^2} \cdot \frac{\partial^2 f_2}{\partial y_1^2}\right) + 2C_1 \int_{\mathbb{C}^2} \delta_{\xi_0} \frac{\partial^3 f_2}{\partial x_i^2 \partial y_1} \\
&= 2C_1 \int_{\mathbb{C}^2} \frac{\partial \delta_{\xi_0}}{\partial x_i} \frac{\partial^2 f_2}{\partial x_i \partial y_1} = 0.
\end{aligned}$$

Here $C_1 = \frac{\partial g(z_1)}{\partial y_1}|_{\xi_0}$. The first identity is by the definition of generalized function δ_{ξ_0}. The second one holds since $\frac{\partial^2 g(z_1)}{\partial^2 y_1} = 0$.

Similarly,

$$\begin{aligned}B_i &= \int_{\mathbb{C}^2} \delta_{\xi_0} \frac{\partial^2}{\partial y_2^2}\Big(g(z_1)\frac{\partial^2 f_2}{\partial x_i^2}\Big) = \int_{\mathbb{C}^2} \delta_{\xi_0} g(z_1)\frac{\partial^4 f_2}{\partial y_2^2 \partial x_i^2}\\ &= \int_{\mathbb{C}^2} \frac{\partial \delta_{\xi_0}}{\partial x_i}\Big(g(z_1)\cdot \frac{\partial^3 f_2}{\partial y_2^2 \partial x_i}\Big) - \int_{\mathbb{C}^2} \delta_{\xi_0} \frac{\partial g(z_1)}{\partial x_i}\cdot \frac{\partial^3 f_2}{\partial y_2^2 \partial x_i}\\ &= 0 - \int_{\mathbb{C}^2} \frac{\partial \delta_{\xi_0}}{\partial x_i}\Big(\frac{\partial g(z_1)}{\partial x_i}\cdot \frac{\partial^2 f_2}{\partial y_2^2}\Big) + \int_{\mathbb{C}^2} \delta_{\xi_0}\Big(\frac{\partial^2 g(z_1)}{\partial x_i^2}\cdot \frac{\partial^2 f_2}{\partial y_2^2}\Big) = 0.\end{aligned}$$

Hence $\int_{\mathbb{C}^2} g(z) dd^\phi \delta_{\xi_0} \wedge dd^\phi f_2 = A_1 + A_2 + B_1 + B_2 = 0.$ ■

Before proving Theorem 1.1 and Theorem1.2, we need to show that a function f is ϕ-plurisubharmonic if and only if $dd^\phi f$ is a ϕ-positive current for special Lagrangian calibration ϕ.

Let $\Lambda_p T_x X$ be the vector space of p-vectors at x in a Riemannian manifold X. The corresponding bundle is denoted by $\Lambda_p TX$. Denote $\Lambda^p T^* X$ the dual of $\Lambda_p TX$. Recall [6] that a current T is *representable by integration* if T has measure coefficients when expressed as a generalized differential form. Equivalently, the mass norm $M_K(T)$ of T on each compact set K is finite. Associated with such a current T is a Radon measure $\|T\|$ and a generalized tangent space $\overrightarrow{T}_x \in \wedge_p T_x X$ defined for $\|T\|$ almost every point x. Recall that each $\overrightarrow{T}_x$ has mass norm one. For any p-form α with compact support, we have $T(\alpha) = \int \alpha(\overrightarrow{T}) d\|T\|$. Let $\wedge(\phi)$ be the span of $G(\phi) \subset \Lambda_p TX$ and $\wedge_+(\phi) \subset \wedge(\phi)$ be the convex cone on $G(\phi)$ with vertex the origin. Note that $\wedge_+(\phi)$ is just the cone on $\mathrm{ch}G(\phi)$. The following lemma is needed for a robust understanding of the definition of ϕ-positive current.

Lemma 2.3. (Lemma 5.4 in [9]) *The following conditions are equivalent:*

(1) $\overrightarrow{T} \in \wedge_+(\phi)$, $\|T\| - a.e.$,

(2) $\overrightarrow{T} \in chG(\phi)$, $\|T\| - a.e.$,

(3) $\phi(\overrightarrow{T}) = 1$, $\|T\| - a.e.$

A *ϕ-positive current* is a p-dimensional current T which is representable by integration and for which the equivalent conditions of Lemma 2.3 are satisfied.

Let $\wedge^+(\phi) \subset \Lambda^p T^* X$ be the polar cone of $\wedge_+(\phi) \subset \Lambda_p TX$. By definition, this is the set of $\alpha \in \Lambda^p T^* X$ such that $\alpha(\xi) \geq 0$ for all $\xi \in \wedge_+(\phi)$, or equivalently,

$$\wedge^+(\phi) := \{\alpha \in \Lambda^p T^* X : \alpha(\xi) \geq 0 \quad \text{for all} \quad \xi \in \wedge_+(\phi)\}.$$

A *$\wedge_+(\phi)$-positive current* is a p-dimension current T satisfying $T(\alpha) \geq 0$, for all p-forms $\alpha \in \wedge^+(\phi)$ with compact support.

Theorem 2.1. (Theorem 5.13 in [9]) *A current T is $\wedge_+(\phi)$-positive if and only if it is ϕ-positive.*

Proposition 5.19 in [9], the appendix: the reduced ϕ-Hessian, is on the relation of ϕ-plurisubharmonic function and ϕ-positive current. Here we only need the version for parallel calibration ϕ. We can rewrite it as follows.

Proposition 2.4. *For a parallel calibration ϕ in Riemannian manifold X, a function $f \in D'(X)$ is ϕ-plurisubharmonic if and only if $dd^{\phi} f$ is a ϕ-positive current.*

Now, based on Propositions 2.3 and 2.4, we can give the proof of Theorem 1.1.

Proof of Theorem 1.1. Given smooth 2-forms $\omega, \eta \in \Lambda^2 T^*\mathbb{C}^2$. Let $\langle \omega, \eta \rangle_{\Lambda_2}$ be the inner product of ω and η. Let $*$ be the Hodge $*$-operator, (cf. P155 in [15]). Define $\| * \overline{\omega} \|_{\Omega} := \sup_{z\in\Omega, \xi\in\wedge_{+}\phi} | * \overline{\omega}(z)(\xi)|$. Note that there exists a constant C satisfying $C \geq \| * \overline{\omega} \|_{\Omega}$, such that $C\phi(\xi) - *\overline{\omega}(\xi) \geq 0$ for any ξ in the convex hull of $G(\phi)$, i.e., $C\phi - *\overline{\omega} \in \wedge^{+}(\phi)$. Since $*\phi = \phi$, we have

$$C\phi - *\overline{\omega} = *(C(*\phi) - \overline{\omega}) \in \wedge^{+}(\phi).$$

$dd^{\phi} f_2$ is a ϕ-positive current since f_2 is ϕ-plurisubharmonic by Proposition 2.4. Then we have

$$\begin{aligned} \int_{\Omega} dd^{\phi} f_2 \wedge (C(*\phi) - \omega) &= \int_{\Omega} dd^{\phi} f_2 \wedge *(\overline{*(C(*\phi) - \overline{\omega})}) \\ &= \langle\ dd^{\phi} f_2, \quad * (C(*\phi) - \overline{\omega})\ \rangle_{\Lambda_2} \\ &= dd^{\phi} f_2 \Big(* (C(*\phi) - \overline{\omega}) \Big) \geq 0. \end{aligned}$$

The first identity holds since ω is a 2-form, $*(*\overline{\omega}) = \overline{\omega}$, and the fact $*\phi = \phi$, $\overline{\phi} = \phi$. The second identity is by the fact $\langle \alpha, \beta \rangle_{\Lambda_2} = \alpha \wedge *\overline{\beta}$ vol, for $\alpha, \beta \in \Lambda^2 T^*\mathbb{C}^2$, cf. P156 in [15]. The third one is by the definition of ϕ-positive current. Hence, for $\omega \in \Lambda^2 T^*\mathbb{C}^2$, there exists a constant $C > 0$ depending on ω, such that

$$\int_{\Omega} dd^{\phi} f_2 \wedge \omega \leq C \int_{\Omega} dd^{\phi} f_2 \wedge (*\phi).$$

Similarly, there exists a constant $C > 0$ depending on ω, such that $-\int_{\Omega} dd^{\phi} f_2 \wedge \omega \leq C \int_{\Omega} dd^{\phi} f_2 \wedge (*\phi)$. Then

$$| \int_{\Omega} dd^{\phi} f_2 \wedge \omega | \leq C \int_{\Omega} dd^{\phi} f_2 \wedge (*\phi). \tag{11}$$

Since $\phi \in \wedge^{+}(\phi)$ and $dd^{\phi} f_2$ is ϕ-positive current, we have

$$dd^{\phi} f_2(\phi) = \int_{\Omega} dd^{\phi} f_2 \wedge (*\phi) > 0.$$

A direct calculation shows that

$$*\phi = \frac{1}{4} dd^{\phi} \|z\|^2.$$

Choose a non-negative function $\psi_0 \in C_0^\infty(\Omega)$, satisfying $\psi_0 \mid_K \equiv 1$ and vanishing on $\Omega \setminus U$.

For any $\psi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned}
&\left|\int_K \psi dd^\phi f_1 \wedge dd^\phi f_2\right| = \left|\int_K f_1 dd^\phi \psi \wedge dd^\phi f_2\right| \leq \|f_1\|_K \int_K C_1(*\phi) \wedge dd^\phi f_2 \\
&= \frac{1}{4} C_1 \|f_1\|_K \int_K dd^\phi \|z\|^2 \wedge dd^\phi f_2 \leq \frac{1}{4} C_1 \|f_1\|_K \int_\Omega \psi_0 dd^\phi \|z\|^2 \wedge dd^\phi f_2 \\
&= \frac{1}{4} C_1 \|f_1\|_K \int_\Omega f_2 dd^\phi \|z\|^2 \wedge dd^\phi \psi_0 \leq \frac{1}{4} C_1 C_2 \|f_1\|_K \|f_2\|_U \int_\Omega dd^\phi \|z\|^2 \wedge *\phi \\
&= C_1 C_2 \|f_1\|_K \|f_2\|_U \int_\Omega *\phi \wedge *\phi = C_1 C_2 \|f_1\|_K \|f_2\|_U \int_\Omega 2dV \\
&= C \|f_1\|_K \|f_2\|_U,
\end{aligned}$$

where C_1 and C_2 are chosen to satisfy $C_1 \geq \|dd^\phi \psi\|_K$ and $C_2 \geq \|dd^\phi \psi_0\|_K$, $C = C_1 C_2 \int_\Omega 2dV$. The first and third equations are due to Proposition 2.3. The first and third inequalities are by equation (11). ∎

Proof of Theorem 1.2. By Theorem 1.1, we see that for any compact subset $K \subseteq \Omega$, the sequence of $dd^\phi f_{1,N} \wedge dd^\phi f_{2,N}|_K$ is bounded. Thus it is sufficient to show that for any continuous compactly supported function φ the sequence $\int_\Omega \varphi dd^\phi f_{1,N} \wedge dd^\phi f_{2,N}$ is a Cauchy sequence. Fix $\varepsilon > 0$, and a function $\psi \in C_0^\infty(\Omega)$ such that $\|\varphi - \psi\|_\Omega < \varepsilon$. Fix an arbitrary compact subset $K \subseteq \Omega$ and a compact neighborhood U of K in Ω. We have

$$\begin{aligned}
&\left|\int_K (\psi - \varphi)(dd^\phi f_{1,M} \wedge dd^\phi f_{2,M} - dd^\phi f_{1,N} \wedge dd^\phi f_{2,N})\right| \\
&= \left|\int_K \frac{1}{2}(\psi - \varphi)(dd^\phi(f_{1,M} - f_{1,N}) \wedge dd^\phi(f_{2,M} + f_{2,N}) + dd^\phi(f_{1,M} + f_{1,N}) \wedge dd^\phi(f_{2,M} - f_{2,N}))\right| \\
&\leq C\|\psi - \varphi\|_\Omega (\|f_{1,M} - f_{1,N}\|_K \|f_{2,M} + f_{2,N}\|_U + \|f_{1,M} + f_{1,N}\|_K \|f_{2,M} - f_{2,N}\|_U) \\
&\leq C\varepsilon (\|f_{1,M} - f_{1,N}\|_K \|f_{2,M} + f_{2,N}\|_U + \|f_{1,M} + f_{1,N}\|_K \|f_{2,M} - f_{2,N}\|_U).
\end{aligned}$$

For sufficient large M and N, the last expression can be estimated by $4C\varepsilon \|f_1\|_K \|f_2\|_U$. Hence it is sufficient to prove that for any function $\psi \in C_0^\infty(\Omega)$, the sequence $\int_\Omega \psi dd^\phi f_{1,N} \wedge dd^\phi f_{2,N}$ is a Cauchy sequence. By using Theorem 1.1 again, we get

$$\begin{aligned}
&\left|\int_\Omega \psi (dd^\phi f_{1,M} \wedge dd^\phi f_{2,M} - dd^\phi f_{1,N} \wedge dd^\phi f_{2,N})\right| \\
&= \left|\int_{K'} \frac{1}{2} \psi (dd^\phi(f_{1,M} - f_{1,N}) \wedge dd^\phi(f_{2,M} + f_{2,N}) + dd^\phi(f_{1,M} + f_{1,N}) \wedge dd^\phi(f_{2,M} - f_{2,N}))\right| \\
&\leq C(\|f_{1,M} - f_{1,N}\|_{K'} \|f_{2,M} + f_{2,N}\|_{U'} + \|f_{1,M} + f_{1,N}\|_{K'} \|f_{2,M} - f_{2,N}\|_{U'}),
\end{aligned}$$

where $K' = supp\psi \subseteq \Omega$ and U' is a compact neighborhood of K' in Ω. Hence, $dd^\phi f_{1,N} \wedge dd^\phi f_{2,N}$ has weak limit. Choose another two sequences $g_{1,M}$ and $g_{2,M}$

which are also two twice continuous differentiable ϕ-plurisubharmonic functions converging to f_1 and f_2 uniformly on compact subsets of Ω respectively. We have

$$\begin{aligned}&\left|\int_\Omega \psi(dd^\phi f_{1,M}\wedge dd^\phi f_{2,M}-dd^\phi g_{1,M}\wedge dd^\phi g_{2,M})\right|\\ =&\left|\int_{K'}\frac{1}{2}\psi(dd^\phi(f_{1,M}-g_{1,M})\wedge dd^\phi(f_{2,M}+g_{2,M})+dd^\phi(f_{1,M}+g_{1,M})\wedge dd^\phi(f_{2,M}-g_{2,M}))\right|\\ \le& C'(\|f_{1,M}-g_{1,M}\|_{K'}\|f_{2,M}+g_{2,M}\|_{U'}+\|f_{1,M}+g_{1,M}\|_{K'}\|f_{2,M}-g_{2,M}\|_{U'}).\end{aligned}$$

When M tends to $+\infty$, the right of the inequality tends to 0. Hence they have the same limit. We denote this limit by $dd^\phi f_1\wedge dd^\phi f_2$. The theorem is proved. ■

3. Radon Transform over special Lagrangian n-planes in $\mathbb{C}^n$

In this section, we give the definition of Radon transform over the special Lagrangian n-planes in $\mathbb{C}^n$. We prove that Radon transform is injective, which enables us to prove the Theorem 1.2.

The theory of Radon transforms associated to a double fibration

$$\begin{array}{ccc} & G/(H_X\cap H_\Xi) & \\ & \swarrow \qquad \searrow & \\ X=G/H_X & & \Xi=G/H_\Xi,\end{array} \tag{12}$$

is introduced in Helgason [12,13], where X and Ξ are two left coset spaces of G, H_X and H_Ξ are closed subgroups of G. Two elements $x\in X$, $\xi\in\Xi$ are said to be *incident* if as cosets in G they intersect. Let

$$\begin{aligned}\check{x}&=\{\xi\in\Xi: x \text{ and } \xi \text{ incident}\},\\ \hat{\xi}&=\{x\in X: x \text{ and } \xi \text{ incident}\}.\end{aligned} \tag{13}$$

The *Radon transform* $f\to\hat{f}$ associated to the double fibration (12) is defined as

$$\hat{f}(\xi)=\int_{\hat{\xi}} f(x)d\mu_{\hat{\xi}}(x),\quad \text{for any } \xi\in\Xi, \tag{14}$$

where f is a rapidly decreasing function on Ξ, $d\mu_{\hat{\xi}}$ is the normalized H_Ξ-invariant measure on $\hat{\xi}$. The *dual transform* $\psi\to\check{\psi}$ for rapidly decreasing function ψ on Ξ is

$$\check{\psi}(x)=\int_{\check{x}}\psi(\xi)d\mu_{\check{x}}(\xi), \tag{15}$$

where $d\mu_{\check{x}}$ is the normalized H_X-invariant measure on $\check{x}$. The integrals (14) and (15) are well-defined since f and ψ are rapidly decreasing.

Let $SLAG_0\subseteq G(n,2n)$ be the set of all special Lagrangian n-planes through 0. By definition, SU_n acts transitively on $SLAG_0$, and the isotropic subgroup of SU_n

at the point $\xi_0 = span_{\mathbb{R}}\{e_1, e_3, ..., e_{2n-1}\}$ is SO_n acting diagonally on $\mathbb{R}^n \oplus \mathbb{R}^n$. Thus $SLAG_0 \cong SU_n/SO_n$.

Let $SLAG \equiv \{(\xi, v) \mid \xi \in SLAG_0,\ v \in \mathbb{R}^{2n}\}/SO_n \rtimes \mathbb{R}^n$, it is the set of all planes of the form $\xi + v$, $\xi \in SLAG_0$, $v \in \mathbb{R}^{2n}$. Namely, $SLAG \cong SU_n \rtimes \mathbb{R}^{2n}/SO_n \rtimes \mathbb{R}^n$. Here $\mathbb{R}^n = span_{\mathbb{R}}\{e_1, e_3, ..., e_{2n-1}\}$ and the production on Lie group $SU_n \rtimes \mathbb{R}^{2n}$ is defined as, for $(A_1, v_1), (A_2, v_2) \in SU_n \rtimes \mathbb{R}^{2n}$, $(A_1, v_1)(A_2, v_2) = (A_1A_2, A_1v_2 + v_1)$. Thus we have the following double fibration,

$$\begin{array}{ccc} & SU_n \rtimes \mathbb{R}^{2n}/SO_n & \\ & \swarrow \qquad \searrow & \\ \mathbb{R}^{2n} \cong SU_n \rtimes \mathbb{R}^{2n}/SU_n & & SLAG \cong SU_n \rtimes \mathbb{R}^{2n}/SO_n \rtimes \mathbb{R}^n. \end{array} \tag{16}$$

Now we can define the Radon transform and its dual associated to the double fibration (16). The Radon transform $\hat{f}$ of a rapidly decreasing function f on $\mathbb{R}^{2n}$ is

$$\hat{f}(\xi) = \int_{\hat{\xi}} f(x) d\mu_{\hat{\xi}}(x), \quad \text{for any } \xi \in SLAG, \tag{17}$$

where $d\mu_{\hat{\xi}}$ is the normalized $SO_n \rtimes \mathbb{R}^n$-invariant measure on $\hat{\xi}$. The dual transform $\psi \to \check{\psi}$ for rapidly decreasing function ψ on $SLAG$ is

$$\check{\psi}(x) = \int_{\check{x}} \psi(\xi) d\mu_{\check{x}}(\xi), \tag{18}$$

where $d\mu_{\check{x}}$ is the normalized SU_n-invariant measure on $\check{x}$. The integrals (17) and (18) are well-defined since f and ψ are rapidly decreasing.

In our case, we know $\hat{\xi}$ are the points in $\mathbb{R}^{2n}$ that lie in ξ by the second equation of (13), i.e.

$$\hat{\xi} = \xi.$$

The $SO_n \rtimes \mathbb{R}^n$-invariant measure $d\mu_{\hat{\xi}}$ on $\hat{\xi}$ is the Lebesgue measure on ξ up to a constant factor. So the definition of Radon transform (14) can be written as

$$\hat{f}(\xi) = \int_{\xi} f(x) dm(x), \quad \text{for any } \xi \in SLAG, \tag{19}$$

where dm is the Lebesgue measure on ξ.

The following inversion formula (20) has appeared in Grinberg [7], section 8, we cite this theorem without proof.

Theorem 3.1. *Let G be a subgroup of the group of isometries of $M = \mathbb{R}^n$. Assume that G acts transitively on M and that M is still a two-point homogeneous space of G. Let X be a fixed k-plane in M and let R be the k-plane transform restricted to the set of planes GX. Then R is invertible with inversion formula:*

$$c_{k,n} \Delta^{\frac{k}{2}} R^t R = I. \tag{20}$$

Here $\Delta^{\frac{1}{2}}$ is the pseudodifferential operator on $\mathbb{R}^n$, R^t is the dual of this Radon transform R, and $c_{k,n}$ is a constant.

Let G in Theorem 3.1 be the group SU_n, M in Theorem 3.1 be S^{2n-1} and X in Theorem 3.1 be $\mathbb{R}^n$, a fixed n-plane in S^{2n-1}. By using the fact that SU_n acts transitively on the unit sphere S^{2n-1}, and Theorem 3.1, we can get the Radon transform over special Lagrangian n-planes is injective.

Similar to [1], we have the following proposition.

Proposition 3.1. *The linear combinations of delta-functions of special Lagrangian n-planes in $\mathbb{C}^n$ are dense in the space of distributions.*

Proof. Let $\mathcal{S}(\mathbb{C}^n)$ be the rapidly decreasing functions on $\mathbb{C}^n$ and $\mathcal{S}'(\mathbb{C}^n)$ be continuous dual of $\mathcal{S}(\mathbb{C}^n)$ with the weak topology. Denote Z the closure of all linear combinations of delta-functions of special Lagrangian n-planes in $\mathbb{C}^n$ in the weak topology. Thus $Z \subset \mathcal{S}'(\mathbb{C}^n)$. Assume $Z \neq \mathcal{S}'(\mathbb{C}^n)$, then there exists $u \in \mathcal{S}'(\mathbb{C}^n) \setminus Z$. By the Hahn-Banach theorem, there exists a continuous linear functional l on $\mathcal{S}'(\mathbb{C}^n)$ such that

$$l(u) \neq 0, \tag{21}$$

$$l(Z) = 0. \tag{22}$$

But any continuous linear functional a on $\mathcal{S}'(\mathbb{C}^n)$ is given by an element of $\mathcal{S}(\mathbb{C}^n)$, namely there exists an element $f \in \mathcal{S}(\mathbb{C}^n)$ such that

$$a(\psi) = \psi(f), \quad \text{for any } \psi \in \mathcal{S}'(\mathbb{C}^n).$$

Let us apply this fact for our functional l. That means there exists $g \in \mathcal{S}(\mathbb{C}^n)$ such that $l(\psi) = \psi(g)$ for any $\psi \in \mathcal{S}'(\mathbb{C}^n)$. (21) implies that

$$l(\psi) = \psi(g) \neq 0,$$

so g is not identically 0. (22) means that for any $\delta_\xi \in Z$, ξ is a special Lagrangian n-planes, we have

$$l(\delta_\xi) = \delta_\xi(g) = 0.$$

This means that the Radon transform over special Lagrangian n-planes of g vanishes. By the injectivity of this Radon transform, we know $g = 0$. This is a contradiction. ∎

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