FINITE p-GROUPS IN WHICH NORMAL CLOSURES FOR EVERY NONNORMAL SUBGROUPS ARE MINIMAL NONABELIAN

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Abstract. The authors study finite p-groups G such that A^G is minimal non-abelian for all non-normal subgroup $A < G$. This topic is Problem 805 posed by Berkovich and Janko in [4]. The authors give the complete classification of such kind of p-groups.

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1. Introduction

Let G be a finite group, H a subgroup of G. It is well-known fact that the normal closure H^G of H in G plays a very important role in determining the structure of the group G, especially for a p-group. For example, let G be a p-group, for every $a \in G$ if $\langle a\rangle^G$ is abelian, then the nilpotence class of G is less or equal to 3. Moreover the nilpotence class of $\langle a\rangle^G$ for every $a \in G$ is equal or less than 2 if and only if G is 3-Engel group. In [1] and [2], the authors studied p-groups G such that $\langle a\rangle^G$ having a cyclic subgroup maximal subgroup for any $a \in G$.

Berkovich and Janko in [4] posed an open Problem 805: Study the p-groups G such that A^G is minimal nonabelian for all nonnormalabelian $A < G$.

In this paper, we study the above problem, i.e., finite p-groups G such that A^G is a minimal non-abelian group for all non-normal subgroup $A < G$ and come to the classification of such kinds of p-groups. For convenience, we say such a p-group satisfies A^G-MNA-property.

All notations are the same as in [3] and [6].

2. Preliminaries

Lemma 2.1 *Let G be a p-group satisfying A^G-MNA-property. Then the following holds:*

(1) *If an abelian subgroup $A \lhd G$, then the subgroup of A is normal in G;*

(2) *If G is not a Dedekind group, then $Cl(G) \geq 3$;*

(3) *For every $a \in G$, $\langle a^p\rangle$ is normal in G;*

(4) *If $p \geq 3$, then every element of order p is contained $Z(G)$, i.e., $\Omega_1(G) \leq Z(G)$.*

Proof. (1) If there exists a subgroup $B < A$ such that B is non-normal G, then $B^G \leq A$ is a minimal non-abelian subgroup, a contradiction.

(2) If $Cl(G) \leq 2$, then $G' \leq Z(G)$. Since G is not a Dedekind group, then there exists a cyclic subgroup A of G such that $A \ntrianglelefteq G$. It is easy to see that $A^G \leq AG'$, hence A^G is abelian, a contradiction.

(3) Suppose that $\langle a^p\rangle$ is not a normal subgroup in G. Then $\langle a\rangle \ntrianglelefteq G$. Thus $\langle a\rangle^G$ is a minimal non-abelian subgroup. We have that $\langle a^p\rangle \leq \Phi(\langle a\rangle^G) \trianglelefteq G$, where $\Phi(\langle a\rangle^G)$ is the Frattini subgroup of $\langle a\rangle^G$. Clearly $\Phi(\langle a\rangle^G)$ is abelian. Hence $\langle a^p\rangle \trianglelefteq G$, a contradiction.

(4) It is enough to show $< a > \lhd G$ for every element a of order p. Otherwise, there exists $\langle a\rangle$ not normal in G. Hence $\langle a\rangle^G$ is a minimal non-abelian subgroup and $|\langle a\rangle^G| = p^3$. Since $p \geq 3$, it follows that $\langle a\rangle^G$ has an abelian G-invariant subgroup N of type (p, p) by Lemma 1.4 in [3]. Now, by (1), we get $N \leq Z(G)$ and $\langle a\rangle^G$ is abelian, a contradiction. ∎

Lemma 2.2 *Suppose that G is not a Dedekind p-group and satisfies A^G-MNA-property. Then $Cl(G) \leq 3$ if $p \neq 3$, but $Cl(G) \leq 4$ if $p = 3$.*

Proof. Consider $\overline{G} = G/Z(G)$. Let $\langle \overline{x} \rangle \ntrianglelefteq \overline{G}$, where $\overline{x} \in \overline{G}$. Obviously, $\langle x \rangle$ is not normal in G. Then $\langle x \rangle^G$ is a minimal non-abelian group. If $[x, x^g] \neq 1$ for $g \in G$, then $\langle x \rangle^G = \langle x, x^g \rangle = H$. Since $H' = [x, x^g]$ is of order p, it follows that $[x, x^g] \in Z(G)$. Hence $\langle \overline{x} \rangle^{\overline{G}}$ is abelian, thus $\overline{G}$ is a 2-Engel group. Therefore, $Cl(\overline{G}) \leq 2$ if $p \neq 3$, but $Cl(\overline{G}) \leq 3$ if $p = 3$, which concludes the lemma. ∎

Proposition 2.3 *Suppose that S is not a Dedekind p-group G and satisfies A^G-MNA-property. Then $p = 2$ and $Cl(G) = 3$.*

Proof. At first we show that $p = 2$. Otherwise, let $\langle y \rangle \ntrianglelefteq G$. Then $\langle y \rangle^G$ is a minimal non-abelian group. By Lemma 2.1(4), we have $|y| \geq p^2$. But by Lemma 2.1(3) it follows that $\langle y^p \rangle \trianglelefteq G$, hence $\langle y^p \rangle^g = \langle y^p \rangle = \langle (y^g)^p \rangle$. Therefore $(y^g)^p = y^{dp}$, where $(d, p) = 1$. By regularity of $\langle y \rangle^G$, we have $(y^{-d} y^g)^p = 1$. Let $y_1 = y^{-d} y^g$, then there exists an element y_1 of order p such that $\langle y \rangle^G = \langle y, y^g \rangle = \langle y, y_1 \rangle$. By Lemma 2.1(4), we have $y_1 \in Z(G)$, consequently $\langle y \rangle^G = \langle y, y_1 \rangle$ is abelian, a contradiction.

Now, by Lemma 2.1(2) and Lemma 2.2, we have $Cl(G) = 3$ and $p = 2$. ∎

3. Classification of p-groups satisfying A^G-MNA-property

Theorem 3.1 *Assume that a p-group G is not a Dedekind group and satisfies A^G-MNA-property. Then one of the following holds:*

(1) $G \cong D_{2^4}$;

(2) $G \cong Q_{2^4}$;

(3) $G \cong SD_{2^4}$;

(4) $G \cong \langle x, y \rangle$ *and* $|G| = 2^5$, *where* $|x| = 8$ *and* $|y| = 4$, $\langle x \rangle \cap \langle y \rangle = 1$, $Cl(G) = 3$.

Proof. (a) We assume $\Omega_1(G) \nleq Z(G)$ and prove (1) or (3) holds.

At first we have $p = 2$ by Lemma 2.1 (4). Since G is not a Dedekind group, there exists $\langle x \rangle \ntrianglelefteq G$. Hence $H = \langle x \rangle^G$ is a minimal non-abelian subgroup by G satisfying A^G-MNA-property, which implies $H \cong D_8$. Because $Aut(D_8) \cong D_8$ and $H/Z(H) \cong Inn(H) \cong C_2 \times C_2$, we have $|G/HC_G(H)| \leq 2$.

If $|G/HC_G(H)| = 2$. We assert that $C_G(H) \leq H$. Otherwise, there exists $y \in C_G(H) - H$. Let $A = \langle x, y \rangle$, then it must hold that $A \ntrianglelefteq G$. In fact, if $A \trianglelefteq G$ then $\langle x \rangle \trianglelefteq G$ for A is abelian and by Lemma 2.1(1), a contradiction. Hence $A^G \leq HC_G(H)$ is a minimal non-abelian subgroup. Since $H \leq A^G$, we have $H = A^G$, and then $y \in H$, which contradicts the fact $y \in C_G(H) \setminus H$. Therefore $G = HC_G(H)$, consequently $G \cong D_{2^4}$ or $G \cong SD_{2^4}$, i.e., (1) or (3) holds.

(b) Now, we assume $\Omega_1(G) \leq Z(G)$ and shall prove (2) or (4) holds.

(I) At first, we assume $|\Omega_1(G)| = 2$. Since G has no abelian G-invariant subgroups of type (p,p), G is a 2-group maximal class by Lemma 1.4 in [3]. Then, $G \cong Q_{2^4}$, that is, G is as in (2).

(II) Now assume $|\Omega_1(G)| \geq 4$. We have divided the proof into two subcases.

(i) If $exp(G) \leq 4$. By $\Omega_1(G) \leq Z(G)$ and G is not a Dedekind group, there exists some $\langle a \rangle \ntrianglelefteq G$ and $|a| = 4$. Let $\langle a \rangle^G = \langle a, a^b \rangle$. Since $\langle a^2 \rangle \trianglelefteq G$ by Lemma 2.1 (3), we get that $(a^2)^b = a^2 = (a^b)^2$. Hence $|\langle a \rangle^G| = 8$. But $\langle a \rangle^G$ is a minimal non-abelian subgroup by hypothesis, it follows that $\langle a \rangle^G \cong D_8$. Also $\langle a \rangle^G$ can be generated by two elements of order 2 and $\Omega_1(G) \leq Z(G)$, so that $\langle a \rangle^G$ is abelian, a contradiction. Therefore G is a Dedekind group, a last contradiction.

(ii) Suppose $exp(G) \geq 8$. At first we claim that if $x \in G$ of order ≥ 8 then $\langle x \rangle \trianglelefteq G$.

If there exists some $x \in G$ such that $|x| \geq 8$ and $\langle x \rangle \ntrianglelefteq G$, then we may set $\langle x \rangle^G = \langle x, x_g \rangle = H$ for some $g \in G$. It follows that $\langle x^2 \rangle \trianglelefteq G$ by Lemma 2.1 (3), hence $(\langle x^2 \rangle)^g = \langle x^2 \rangle = \langle {x_1}^2 \rangle$. Therefore there exists k such that $x^2 = {x_1}^{2k}$ where $(k,2) = 1$. By Hall-Petrescu formula and H is a minimal non-abelian subgroup, we have $(x{x_1}^{-k})^4 = x^4({x_1}^{-k})^4[x, {x_1}^{-k}]^6 = 1$. If $|x{x_1}^{-k}| = 2$. Then it follows by $H = \langle x{x_1}^{-k}, x_1 \rangle$ and $\Omega_1(G) \leq Z(G)$ that H is abelian, a contradiction. Thus $|x{x_1}^{-k}| = 4$. Let $x_2 = x x_1^{-k}$, then $H = \langle x, x_2 \rangle$. Since $x_2^2 = (x x_1^{-k})^2 \in \langle x \rangle$ and $|x_2| = 4$, we come to $x_2^2 = [x, x_1]$ and $\langle x_2 \rangle \trianglelefteq H$. Because $Cl(H) = 2$ and $exp(H) = |x|$, one has that $x_2^2 \in \langle x \rangle \cap \langle x_2 \rangle \neq 1$. Hence $\langle x \rangle$ is a cyclic subgroup of H having index 2 in G, so there exists x_3 of order 2 such that $H = \langle x, x_3 \rangle$. But $x_3 \in Z(G)$, we get that H is abelian, a contradiction. The claim follows.

Let x be an element of order 8, and $y \in G$ an element of order 4 such that $\langle y \rangle$ is not normal in G. Then $\langle x \rangle \trianglelefteq G$ by above argument, thus $K = \langle x, y \rangle = \langle x \rangle \langle y \rangle$ is a subgroup of order at most 32. In the following, we discuss the structure of K case by case.

Case 1. Assume that $\langle x \rangle \cap \langle y \rangle \neq 1$. By $\Omega_1(G) \leq Z(G)$, we have $\langle y^2 \rangle \trianglelefteq G$. If $Cl(K) \leq 2$, since $[x,y]^2 = [x, y^2] = [x^2, y] = 1$, then K is minimal non-abelian. By $\langle x \rangle \cap \langle y \rangle \neq 1$, then there exists l such that $x^{4l} = y^2$ where $(l,2) = 1$. It follows that $(x^{-2l}y)^2 = x^{-4l}y^2 = 1$. Hence $|x^{-2l}y| = 2$. Obviously, since $K = \langle x, x^{-2l}y \rangle$ is minimal non-abelian, we get $\langle x^{-2l}y \rangle \ntrianglelefteq G$, which contradicts $\Omega_1(G) \leq Z(G)$. Therefore, $Cl(K) = 3$. Notice that K has a maximal and cyclic subgroup $\langle x \rangle$ and $\Omega_1(G) \leq Z(G)$, we get that $K \cong Q_{2^4}$. It follows that $\langle y \rangle^G = \langle y \rangle^K \cong Q_8 \trianglelefteq G$. Let $T = \langle y, \Omega_1(G) \rangle$, since $\langle y \rangle^G \leq \langle T \rangle^G$, we have that $\langle T \rangle^G$ is a non-abelian subgroup and $T \ntrianglelefteq G$. Now because T is abelian, it follows that $\langle T \rangle^G$ is a minimal non-abelian subgroup. Hence $\langle y \rangle^G = \langle T \rangle^G$. Otherwise, $\langle y \rangle^G$ is abelian, a contradiction. By $\Omega_1(G) \leq \langle T \rangle^G$ and $|\Omega_1(\langle T \rangle^G)| = |\Omega_1(\langle y \rangle^G)| = |\Omega_1(Q_8)| = 2$, it follows that $|\Omega_1(G)| = 2$, which contradicts $|\Omega_1(G)| \geq 4$.

Case 2. Now, assume that $\langle x \rangle \cap \langle y \rangle = 1$. If $Cl(K) = 3$. Take $a \in G \setminus K$, set $T = \langle y, a \rangle$. Because $|K| = 2^5$ and $K' \leq \langle x \rangle$, if $|K : K'| = 4$, then K is of maximal nilpotent class, which contradicts $Cl(K) = 3$. Thus $K' = \langle x^2 \rangle$. If $|\langle y \rangle^K| = 2^3$, then $\langle y \rangle^K$ is abelian by $\langle y \rangle^K \leq \langle y \rangle K'$, so K is a 2-Engle group. Thus $Cl(K) \leq 2$, a contradiction. It follows that $\langle y \rangle^K = \langle x^2, y \rangle$ is a minimal non-abelian subgroup. If $\langle y \rangle^K < \langle y \rangle^G$, since $\langle y \rangle^G$ is a minimal non-abelian subgroup, then $\langle y \rangle^K$ is abelian, a contradiction too. Hence $\langle y \rangle^K = \langle y \rangle^G = \langle x^2, y \rangle$. In order to prove that $G = K$, we divide the following proof into two subcases.

Subcase 1. While $|a| = 2$. By $\Omega_1(G) \leq Z(G)$, it follows that T is abelian. If $T \trianglelefteq G$, then $\langle y \rangle$ is normal in G by Lemma 2.1(1), a contradiction. Hence $T \ntrianglelefteq G$. Consequently T^G is a minimal non-abelian subgroup, it follows that $\langle y \rangle^G = T^G = \langle a \rangle \langle y \rangle^G = \langle y \rangle^K$, so that $a \in K$, a contradiction.

Subcase 2. While $|a| = 4$. Since $T^G = \langle a \rangle^G \langle y \rangle^G = \langle a \rangle^T \langle y \rangle^T = T$, then $T \trianglelefteq G$. Thus $T = \langle a \rangle \langle y \rangle^G$. By $a^2 \in \Omega_1(\langle y \rangle^G)$, it follows $|T| = 2^5$. If $exp(G) = 4$, then $Cl(G) \leq 2$ by the same argument in the proof of (i). Since $T' = \langle [a, y] \rangle$ and $[a, y]^2 = [a^2, y] = 1$, we have $|T'| = 2$. It follows that T is a minimal non-abelian subgroup. It follows by $\langle y \rangle^G < T$ that $\langle y \rangle^G$ is abelian, a contradiction. Hence there exists $a_1 \in T \setminus \langle y \rangle^G$ with $|a_1| = 8$. If $a_1 \in K$, then $|\langle a_1, \langle y \rangle^G \rangle| = 2^5$. It follows that $\langle a_1, \langle y \rangle^G \rangle = T \leq K$, which contradicts $a \in G \setminus K$. Thus $a_1 \notin K$. Let $H = \langle a_1, x \rangle$. Because of $\langle a_1 \rangle \trianglelefteq G$ and $\langle x \rangle \trianglelefteq G$, we get $Cl(H) \leq 2$ by Theorem 21 in [3]. By $|T : \langle y \rangle^G| = 2$, it follows that ${a_1}^2 \in \langle y \rangle^G$. Hence $|\langle a_1, x, y \rangle| = 2^6$.

If $\langle a_1 \rangle \cap \langle x \rangle = 1$, then $|\langle a_1, x \rangle| = 2^6$. We get $\langle a_1, x \rangle = \langle a_1, x, y \rangle$, which contradicts $Cl(\langle x, y \rangle) = 3$. Therefore, $\langle a_1 \rangle \cap \langle x \rangle \neq 1$.

Now, assume that $\langle a_1 \rangle \cap \langle x \rangle = \langle {a_1}^2 \rangle$. Since $Cl(\langle a_1, x \rangle) \leq 2$, one has $\langle a_1, x \rangle = \langle a_2, x \rangle$, where $|a_2| = 2$. By $a_2 \in \Omega_1(\langle a_1, x, y \rangle) = \Omega_1(\langle x, y \rangle)$, it follows that $a_1 \in \langle a_1, x \rangle \leq \langle x, y \rangle = K$, a contradiction. Hence $\langle a_1 \rangle \cap \langle x \rangle = \langle {a_1}^4 \rangle = \langle x^4 \rangle$. By $Cl(\langle a_1, x \rangle) \leq 2$, there exists a_3 of order 4 such that $\langle a_1, x \rangle = \langle a_3, x \rangle = \langle x \rangle \rtimes \langle a_3 \rangle$. Noticing $\langle a_3 \rangle^G \leq \Omega_2(\langle x, a_1 \rangle)$ and $\Omega_2(\langle x, a_1 \rangle) = \langle x^2 \rangle \times \langle a_3 \rangle$ is abelian, we assert $\langle a_3 \rangle \trianglelefteq G$. Otherwise, $\langle a_3 \rangle^G$ is abelian, a contradiction. Hence $\langle a_1, x \rangle$ is abelian. If $a_3 \in K$, then $a_1 \in \langle a_1, x \rangle = \langle a_3, x \rangle \leq K$, a contradiction. Thus $a_3 \notin K$. Let $T_2 = \langle y, a_3 \rangle$. Since ${T_2}^G = \langle a_3 \rangle^G \langle y \rangle^G = \langle a_3 \rangle \langle y \rangle^{T_2} = T_2$, $T_2 \trianglelefteq G$. Let $t \in T_2$, then $t = m_1 m_2$, where $m_1 \in \langle a_3 \rangle$ and $m_2 \in \langle y \rangle^G$. By Hall-Petrescu formula, we get $t^4 = (m_1 m_2)^4 = {m_1}^4 {m_2}^4 {c_2}^6 {c_3}^4 = 1$, where $c_i \in K_i(\langle m_1, m_2 \rangle)$, $i = 2, 3$. Hence $exp(T_2) = 4$, it is impossible by the same argument in the proof of (i). Therefore $|a| \neq 4$.

Subcase 3. Assume that $|a| \geq 8$. By $T = \langle y, a \rangle$, where $a \in G \setminus K$ and $|y| = 4$. Similarly, we have $T^G = \langle a \rangle \langle y \rangle^G = T \trianglelefteq G$. By the same reasoning as above, we have $\langle a \rangle \cap \langle y \rangle = 1$ and $Cl(\langle y, a \rangle) = 3$. Since $[a, y^2] = 1$, y induces an automorphism of order 2 of $\langle a \rangle$. We have $a^y = a^{1+2^{n-1}}$ or $a^y = a^{-1+k2^{n-1}}$. Since $Cl(\langle y, a \rangle) = 3$, then $a^y = a^{-1+k2^{n-1}}$ and $|a| = 8$. This case is subcase 2. Hence it is impossible if $|a| \geq 8$. Therefore, $G = K$.

If $Cl(K) = 2$, it follows that $[x, y^2] = [x^2, y] = [x, y]^2 = 1$ by $y^2 \in Z(G)$. Since $K' = \langle [x, y], \gamma_3(K) \rangle = \langle [x, y] \rangle$, we have $|K'| = 2$, hence K is a minimal non-abelian subgroup. Therefore $x \in K = \langle y \rangle^G$. But $\langle y \rangle^G$ is a minimal non-abelian subgroup, we get $exp(\langle y \rangle^G) = 4$, which contradicts $|x| = 8$. ∎

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References

[1] Herzog, M., Longobardi, P., Maj, M., Mann, A., *On generalized Dedekind groups and Tarski super Monsters*, Journal of Algebra, 226 (2000), 690-713.

[2] Lv, H., Zhou, W., Yu, D., *Some finite p-groups with bounded index of every cyclic subgroup in its normal closure*, Journal of Algebra, 338 (2011), 169-179.

[3] Berkovich, Y., Janko, Z., *Groups of Prime Order*, vol. 1, Walter de Gruyter, Berlin, 2008.

[4] Berkovich, Y., Janko, Z., *Groups of Prime Order*, vol. 2, Walter de Gruyter, Berlin, 2011.

[5] Berkovich, Y., Janko, Z., *Groups of Prime Order*, vol. 3, Walter de Gruyter, Berlin, 2008.

[6] Xu, Mingyao, *Finite p-groups* (Chinese), Peking University Press, Beijing, 2010.

[7] Heineken, H., *Engelsche elemente der lange drei* (German), Illinois Journal of Mathematics, 5 (1961), 681-707.

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