ON IMPROVED YOUNG TYPE INEQUALITIES FOR MATRICES

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Abstract. This paper aims to give improved Young type inequalities which are due to Hu [2]. Then we use these inequalities to establish corresponding Young type inequalities for matrices.

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1. Introduction

Let M_n be the space of $n \times n$ complex matrices. Let $\|\cdot\|$ denote any unitarily invariant norm on M_n. So, $\|UAV\| = \|A\|$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. For $A = (a_{ij}) \in M_n$, the Hilbert-Schmidt norm and the trace norm of A are defined by

$$\|A\|_2 = \sqrt{\sum_{j=1}^{n} s_j^2(A)}, \quad \|A\|_1 = \sum_{j=1}^{n} s_j(A), \text{ respectively,}$$

where $s_i(A)\,(i = 1, ..., n)$ are the singular values of A with $s_1(A) \geq \cdots \geq s_n(A)$, which are the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity.

The classical Young inequality says that if $a, b \geq 0$ and $0 \leq v \leq 1$, then

$$a^v b^{1-v} \leq va + (1-v)b \tag{1.1}$$

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with equality if and only if $a = b$.

The Kontorovich constant is defined as

$$K(t,2) = \frac{(t+1)^2}{4t}, \quad t > 0.$$

Zuo, Shi and Fujii [1] obtained an improvement of inequality (1.1) which can be stated as follows:

$$K(h,2)^r a^v b^{1-v} \leq va + (1-v)b, \tag{1.2}$$

where $h = \frac{a}{b}$ and $r = \min\{v, 1-v\}$.

In a recent work, Hu [2] gave the following Young type inequalities:

$$\left[(va)^v b^{1-v}\right]^2 + v^2(a-b)^2 \leq v^2a^2 + (1-v)^2 b^2, \quad 0 \leq v \leq \frac{1}{2}, \tag{1.3}$$

and

$$\{a^v[(1-v)b]^{1-v}\}^2 + (1-v)^2(a-b)^2 \leq v^2a^2 + (1-v)^2b^2, \ \frac{1}{2} \leq v \leq 1. \tag{1.4}$$

Based on the scalar Young type inequalities (1.3) and (1.4), Hu proved in [2] that if $A, B, X \in M_n$ such that A and B are positive semidefinite, then

$$\begin{aligned} \|vAX + (1-v)XB\|_2^2 &\geq v^2\|AX - XB\|_2^2 + v^{2v}\|A^vXB^{1-v}\|_2^2 \\ &\quad + 2v(1-v)\left\|A^{1/2}XB^{1/2}\right\|_2^2, \quad 0 \leq v \leq \frac{1}{2} \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} \|vAX + (1-v)XB\|_2^2 &\geq (1-v)^2\|AX - XB\|_2^2 \\ &\quad + (1-v)^{2-2v}\|A^vXB^{1-v}\|_2^2 \\ &\quad + 2v(1-v)\left\|A^{1/2}XB^{1/2}\right\|_2^2, \quad \frac{1}{2} \leq v \leq 1. \end{aligned} \tag{1.6}$$

These are the Hilbert-Schmidt norm versions of Young type inequalities.

At the same time, Hu [2] obtained that if $A, B \in M_n$ are positive definite, then

$$\begin{aligned} \det(vA + (1-v)B)^2 &\geq v^{2nv}\det(A^vB^{1-v})^2 + v^{2n}\det(A-B)^2 \\ &\quad + (2v(1-v))^n \det B^{1/2}AB^{1/2}, \quad 0 \leq v \leq \tfrac{1}{2} \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} \det(vA + (1-v)B)^2 &\geq (1-v)^{2n(1-v)}\det(A^vB^{1-v})^2 \\ &\quad + (1-v)^{2n}\det(A-B)^2 \\ &\quad + (2v(1-v))^n \det B^{1/2}AB^{1/2}, \quad \frac{1}{2} \leq v \leq 1. \end{aligned} \tag{1.8}$$

These are the determinant versions of Young type inequalities.

For more information on matrix versions of the Young inequality the reader is referred to [3]-[7]. In this paper, we present improvements of inequalities (1.5), (1.6), (1.7) and (1.8).

2. Young type inequalities for scalars

We begin this section with the Young type inequalities for scalars.

Theorem 1. *Let* $a, b \geq 0$. *If* $0 \leq v \leq \frac{1}{2}$, *then*

$$K(h,2)^r \left[(va)^v b^{1-v}\right]^2 + v^2 (a-b)^2 \leq v^2 a^2 + (1-v)^2 b^2, \tag{2.1}$$

where $h = \dfrac{va}{b}$, $r = \min\{2v, 1-2v\}$.

If $\dfrac{1}{2} \leq v \leq 1$, *then*

$$K(h,2)^r \left\{a^v [(1-v) b]^{1-v}\right\}^2 + (1-v)^2 (a-b)^2 \leq v^2 a^2 + (1-v)^2 b^2, \tag{2.2}$$

where $h = \dfrac{a}{(1-v) b}$, $r = \min\{2v-1, 2-2v\}$.

Proof. If $0 \leq v \leq \dfrac{1}{2}$. Then, by inequality (1.2), we have

$$\begin{aligned} v^2 a^2 + (1-v)^2 b^2 - v^2 (a-b)^2 &= b\left[2v(va) + (1-2v) b\right] \\ &\geq bK(h,2)^r (va)^{2v} b^{1-2v} \\ &= K(h,2)^r \left[(va)^v b^{1-v}\right]^2, \end{aligned}$$

and so

$$v^2 a^2 + (1-v)^2 b^2 \geq K(h,2)^r \left[(va)^v b^{1-v}\right]^2 + v^2 (a-b)^2.$$

If $\dfrac{1}{2} \leq v \leq 1$, then

$$\begin{aligned} v^2 a^2 + (1-v)^2 b^2 - (1-v)^2 (a-b)^2 &= a\left[(2v-1) a + 2(1-v)^2 b\right] \\ &\geq aK(h,2)^r a^{2v-1} [(1-v) b]^{2-2v} \\ &= K(h,2)^r \left\{a^v [(1-v) b]^{1-v}\right\}^2, \end{aligned}$$

and so

$$v^2 a^2 + (1-v)^2 b^2 \geq K(h,2)^r \left\{a^v [(1-v) b]^{1-v}\right\}^2 + (1-v)^2 (a-b)^2.$$

This completes the proof. ∎

Remark 1. Obviously, (2.1) and (2.2) are improvement of the scalar Young type inequalities (1.3) and (1.4).

3. Young type inequalities for matrices

Based on the scalar Young type inequalities (2.1) and (2.2), we obtain the Hilbert-Schmidt norm, the trace norm and the determinant versions of Young type inequalities.

Theorem 2. *Let $A, B, X \in M_n$ such that A and B are positive semidefinite. If $0 \le v \le \frac{1}{2}$, then*

$$\|vAX + (1-v)XB\|_2^2 \ge v^2 \|AX - XB\|_2^2 + K^r v^{2v} \|A^v XB^{1-v}\|_2^2 + 2v(1-v)\left\|A^{1/2}XB^{1/2}\right\|_2^2, \tag{3.1}$$

where $K = \min\left\{K\left(\frac{v\lambda_i}{\mu_j}, 2\right), i, j = 1, ..., n\right\}$, $r = \min\{2v, 1-2v\}$.

If $\frac{1}{2} \le v \le 1$, *then*

$$\|vAX + (1-v)XB\|_2^2 \ge (1-v)^2 \|AX - XB\|_2^2 + K^r (1-v)^{2-2v} \|A^v XB^{1-v}\|_2^2 + 2v(1-v)\left\|A^{1/2}XB^{1/2}\right\|_2^2, \tag{3.2}$$

where $K = \min\left\{K\left(\frac{\lambda_i}{(1-v)\mu_j}, 2\right), i, j = 1, ..., n\right\}$, $r = \min\{2v-1, 2-2v\}$.

Proof. Since every positive semidefinite matrix is unitarily diagonalizable, it follows that there are unitary matrices $U, V \in M_n$ such that $A = UDU^*$ and $B = VEV^*$, where

$$D = diag(\lambda_1, ..., \lambda_n), \quad E = diag(\mu_1, ..., \mu_n), \text{ and } \lambda_i, \mu_i \ge 0,\ i = 1, ..., n.$$

Let $Y = U^*XV = (y_{ij})$. Then

$$vAX + (1-v)XB = U(vDY + (1-v)YE)V^* = U((v\lambda_i + (1-v)\mu_j)y_{ij})V^*,$$
$$AX - XB = U((\lambda_i - \mu_j)y_{ij})V^*,\ A^{1/2}XB^{1/2} = U(\lambda_i^{1/2}\mu_j^{1/2}y_{ij})V^*$$

and

$$A^v XB^{1-v} = U\left(\lambda_i^v \mu_j^{1-v} y_{ij}\right)V^*.$$

If $0 \le v \le \frac{1}{2}$, by inequality (2.1), we have

$$\begin{aligned}
&\|vAX + (1-v)XB\|_2^2 = \sum_{i,j=1}^n (v\lambda_i + (1-v)\mu_j)^2 |y_{ij}|^2 \\
&= \sum_{i,j=1}^n \left(v^2\lambda_i^2 + (1-v)^2\mu_j^2 + 2v(1-v)\lambda_i\mu_j\right)|y_{ij}|^2 \\
&\ge v^2 \sum_{i,j=1}^n (\lambda_i - \mu_j)^2 |y_{ij}|^2 + K^r v^{2v} \sum_{i,j=1}^n \left(\lambda_i^v \mu_j^{1-v}\right)^2 |y_{ij}|^2 + \sum_{i,j=1}^n 2v(1-v)\lambda_i\mu_j |y_{ij}|^2 \\
&\ge v^2 \|AX - XB\|_2^2 + K^r v^{2v} \|A^v XB^{1-v}\|_2^2 + 2v(1-v)\left\|A^{1/2}XB^{1/2}\right\|_2^2.
\end{aligned}$$

If $\frac{1}{2} \le v \le 1$, then by the inequality (2.2) and the same method above, we have the inequality (3.2). This completes the proof. ∎

Remark 2. Obviously, (3.1) and (3.2) are improvement matrix Young type inequalities (1.5) and (1.6).

To obtain refinements of the trace norm versions of Young type inequalities, we need the following lemmas.

Lemma 1. (Cauchy-Schwarz Inequality) [8] *Let* $a_i \geq 0, b_i \geq 0$ *for* $i = 1, 2, ..., n$, *then*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} b_i^2\right)^{\frac{1}{2}}.$$

Lemma 2. [8] *Let* $A, B \in M_n$, *then*

$$\sum_{j=1}^{n} s_j(AB) \leq \sum_{j=1}^{n} s_j(A) s_j(B).$$

Theorem 3. *Let* $A, B \in M_n$ *be positive semidefinite. If* $0 \leq v \leq \frac{1}{2}$, *then*

$$K^r v^{2v} \|A^v B^{1-v}\|_1 \leq v^2 \|A\|_1 + (1-v)^2 \|B\|_1 - v^2 \Big(\sqrt{\|A\|_1} - \sqrt{\|B\|_1}\Big)^2, \tag{3.3}$$

where $K = \min\Big\{K\Big(\frac{v\sqrt{s_j(A)}}{\sqrt{s_j(B)}}, 2\Big),\ j = 1, ..., n\Big\}$, $r = \min\{2v, 1-2v\}$.

If $\frac{1}{2} \leq v \leq 1$, *then*

$$\begin{aligned} K^r (1-v)^{2(1-v)} \|A^v B^{1-v}\|_1 &\leq v^2 \|A\|_1 + (1-v)^2 \|B\|_1 \\ &\quad -(1-v)^2 (\sqrt{\|A\|_1} - \sqrt{\|B\|_1})^2, \end{aligned} \tag{3.4}$$

where $K = \min\Big\{K\Big(\frac{\sqrt{s_j(A)}}{(1-v)\sqrt{s_j(B)}}, 2\Big), j = 1, ..., n\Big\}$, $r = \min\{2v-1, 2-2v\}$.

Proof. If $0 \leq v \leq \frac{1}{2}$, then using Lemma 1, Lemma 2 and the inequality (2.1), we have

$$\begin{aligned} \operatorname{tr}\left(v^2 A + (1-v)^2 B\right) &= v^2 \operatorname{tr} A + (1-v)^2 \operatorname{tr} B \\ &= \sum_{j=1}^{n} \left(v^2 s_j(A) + (1-v)^2 s_j(B)\right) \\ &\geq \sum_{j=1}^{n} K\left(\frac{v\sqrt{s_j(A)}}{\sqrt{s_j(B)}}, 2\right)^r \left[\left(v\sqrt{s_j(A)}\right)^v \sqrt{s_j(B)}^{\,1-v}\right]^2 \\ &\quad + v^2 \left(\sum_{j=1}^{n} s_j(A) + \sum_{j=1}^{n} s_j(B) - 2\sum_{j=1}^{n} \sqrt{s_j(A) s_j(B)}\right) \end{aligned}$$

$$\geq K^r \sum_{j=1}^{n} \left[v^{2v} s_j\left(A^v\right) s_j\left(B^{1-v}\right)\right]$$
$$+v^2\left(\|A\|_1+\|B\|_1-2\left(\sum_{j=1}^{n} s_j(A)\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n} s_j(B)\right)^{\frac{1}{2}}\right)$$
$$=K^r v^{2v} \sum_{j=1}^{n}\left[s_j\left(A^v\right) s_j\left(B^{1-v}\right)\right]+v^2\left(\sqrt{\|A\|_1}-\sqrt{\|B\|_1}\right)^2.$$

Thus

$$v^2\|A\|_1+(1-v)^2\|B\|_1-v^2\left(\sqrt{\|A\|_1}-\sqrt{\|B\|_1}\right)^2 \geq K^r v^{2v} \sum_{j=1}^{n}\left[s_j\left(A^v\right) s_j\left(B^{1-v}\right)\right].$$

Then

$$K^r v^{2v}\left\|A^v B^{1-v}\right\|_1 \leq v^2\|A\|_1+(1-v)^2\|B\|_1-v^2\left(\sqrt{\|A\|_1}-\sqrt{\|B\|_1}\right)^2.$$

If $\frac{1}{2} \leq v \leq 1$, then by the inequality (2.2) and the same method above, we have the inequality (3.4). This completes the proof. ∎

Theorem 4. *Let $A, B \in M_n$ be positive definite. If $0 \leq v \leq \frac{1}{2}$, then*

$$\det(vA+(1-v)B)^2 \geq K^{nr} v^{2nv} \det\left(A^v B^{1-v}\right)^2+v^{2n} \det(A-B)^2 + (2v(1-v))^n \det B^{1/2} A B^{1/2}, \tag{3.5}$$

where $K=\min\left\{K\left(v s_j\left(B^{-1/2} A B^{-1/2}\right), 2\right), j=1, \ldots, n\right\}$, $r=\min\{2v, 1-2v\}$.

If $\frac{1}{2} \leq v \leq 1$, *then*

$$\det(vA+(1-v)B)^2 \geq K^{nr}(1-v)^{2n(1-v)} \det\left(A^v B^{1-v}\right)^2 +(1-v)^{2n} \det(A-B)^2 +(2v(1-v))^n \det B^{1/2} A B^{1/2}, \tag{3.6}$$

where

$$K=\min\left\{K\left(\frac{s_j\left(B^{-1/2} A B^{-1/2}\right)}{1-v}, 2\right), j=1, \ldots, n\right\},\ r=\min\{2v-1, 2-2v\}.$$

Proof. If $0 \leq v \leq \frac{1}{2}$, then

$$\det\left(vB^{-1/2}AB^{-1/2} + (1-v)\,I\right)^2 = \prod_{j=1}^{n}\left(vs_j\left(B^{-1/2}AB^{-1/2}\right) + 1 - v\right)^2$$

$$= \prod_{j=1}^{n}\left(v^2 s_j^2\left(B^{-1/2}AB^{-1/2}\right) + (1-v)^2 + 2v\,(1-v)\,s_j\left(B^{-1/2}AB^{-1/2}\right)\right)$$

$$\geq \prod_{j=1}^{n}(K^r v^{2v} s_j^{2v}\left(B^{-1/2}AB^{-1/2}\right) + v^2\left(s_j\left(B^{-1/2}AB^{-1/2}\right) - 1\right)^2$$

$$+2v\,(1-v)\,s_j\left(B^{-1/2}AB^{-1/2}\right))$$

$$\geq K^{nr}v^{2nv}\prod_{j=1}^{n} s_j^{2v}\left(B^{-1/2}AB^{-1/2}\right) + v^{2n}\prod_{j=1}^{n}\left(s_j\left(B^{-1/2}AB^{-1/2}\right) - 1\right)^2$$

$$+\left(2v\,(1-v)\right)^n \prod_{j=1}^{n} s_j\left(B^{-1/2}AB^{-1/2}\right)$$

$$= K^{nr}v^{2nv}\det\left(B^{-1/2}AB^{-1/2}\right)^{2v} + v^{2n}\det\left(B^{-1/2}AB^{-1/2} - I\right)^2$$

$$+\left(2v\,(1-v)\right)^n \det B^{-1/2}AB^{-1/2}.$$

Thus, we have

$$\det\left(vA + (1-v)\,B\right)^2 \geq K^{nr}v^{2nv}\det\left(A^v B^{1-v}\right)^2 + v^{2n}\det\left(A - B\right)^2$$

$$+\left(2v\,(1-v)\right)^n \det B^{1/2}AB^{1/2}.$$

If $\frac{1}{2} \leq v \leq 1$, then by the inequality (2.2) and the same method above, we have the inequality(3.6). This completes the proof. ∎

Remark 3. Obviously, (3.5) and (3.6) are improvement determinant versions of Young type inequalities (1.7) and (1.8).

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