INTUITIONISTIC FUZZY SETS IN UP-ALGEBRAS[1]

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Abstract. The concept of intuitionistic fuzzy sets was first introduced by Atanassov, which is a generalization of the concept of fuzzy sets. In this paper, we apply the concept of intuitionistic fuzzy sets to UP-algebras. The notions of intuitionistic fuzzy UP-ideals and intuitionistic fuzzy UP-subalgebras of UP-algebras are introduced and their basic properties are investigated. Upper t-(strong) level subsets and lower t-(strong) level subsets are derived from some intuitionistic fuzzy sets.

Keywords: UP-algebra, intuitionistic fuzzy set, intuitionistic fuzzy UP-ideal, intuitionistic fuzzy UP-subalgebra, upper t-(strong) level subset, lower t-(strong) level subset.

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1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [6], BCI-algebras [7], BCH-algebras [4], KU-algebras [18], SU-algebras [9] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [7] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [6], [7] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

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The fundamental concept of fuzzy sets in a set was first introduced by Zadeh [27] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere. The concept of intuitionistic fuzzy sets was first published by Atanassov in his pioneer papers [2], [3], as generalization of the notion of fuzzy sets. Several researches were conducted on the generalizations of the notion of intuitionistic fuzzy sets and application to many logical algebras such as: In 2000, Jun and Kim [8] introduced the notion of equivalence relations on the family of all intuitionistic fuzzy ideals of BCK-algebras. In 2004, Zhan and Z. Tan [30] introduced the notion of intuitionistic fuzzy α-ideals of BCI-algebras. In 2005, Kim and Jeong [12] introduced the notion of intuitionistic fuzzy o-subalgebra of BCK-algebras with condition (S). Xueling and Jianming [26] introduced the notion of intuitionistic Ω-fuzzy ideals of BCK-algebras. Zahedi and Torkzadeh [28] introduced the notions of intuitionistic fuzzy dual positive implicative hyper K-ideals of types 1,2,3,4 and intuitionistic fuzzy dual hyper K-ideals. In 2006, Kim and Jeong [10] introduced the notion of intuitionistic fuzzy subalgebras of B-algebras which is related to several classes of algebras such as BCI/BCK-algebras. In 2007, Kim [11] introduced the notion of intuitionistic (T, S)-normed fuzzy subalgebras in BCK/BCI-algebras. Zarandi and A. B. Saeid [29] studied the intuitionistic fuzzification of the concept of subalgebras and ideals of BG-algebras. In 2008, Akram, Dar, Meng and Shum [1] introduced the notion of interval-valued intuitionistic fuzzy ideals of K-algebras. In 2011, Mostafa, Naby and Elgendy [14] introduced the intuitionistic fuzzification of the concept of KU-ideals and the image (preimage) of KU-ideals in KU-algebras. Satyanarayana and Prasad [21] studied the intuitionistic fuzzy implicative ideals, intuitionistic fuzzy positive implicative ideals and intuitionistic fuzzy commutative ideals in BCK-algebras. In 2012, Malik and Touqeer [13] introduced the intuitionistic fuzzification of the concept of BCI-commutative ideals of BCI-algebras. Palaniappan, Veerappan and Devi [17] introduced the notion of interval valued intuitionistic fuzzy H-ideals of BCI-algebras. Senapati, Bhowmik and Pal [22] introduced the notion of interval-valued intuitionistic fuzzy closed ideals of BG-algebras. In 2013, Nezhad, Rayeni and Rezaei [15] introduced the notion of intuitionistic fuzzy soft subalgebras (filters) of BE-algebras. Palaniappan, Devi and Veerappan [16] introduced the notion of intuitionistic fuzzy n-fold positive implicative ideals of BCI-algebras. In 2014, Ragavan, Solairaju and Balamurugan [19] introduced the notion of interval valued Intuitionistic Fuzzy R-ideals of BCI-algebras. Satyanarayana, Krishna and Prasad [20] introduced the notions of intuitionistic fuzzy (weak) implicative hyper BCK-ideals of hyper BCK-algebras. Senapati, Bhowmik and Pal [23] introduced the notions of fuzzy dot subalgebras, fuzzy normal dot subalgebras and fuzzy dot ideals of B-algebras. Sun and Li [25] introduced the notions of intuitionistic fuzzy subalgebras with thresholds (λ, μ) and intuitionistic fuzzy ideals with thresholds (λ, μ) of BCI-algebras.

Iampan [5] now introduced a new algebraic structure, called a UP-algebra and a concept of UP-ideals and UP-subalgebras of UP-algebras. The notions of intuitionistic fuzzy UP-ideals and intuitionistic fuzzy UP-subalgebras play an important role in studying the many logical algebras. In this paper, we introduce the notions of intuitionistic fuzzy UP-ideals and intuitionistic fuzzy UP-subalgebras of UP-algebras, and their properties are investigated.

Before we begin our study, we will introduce to the definition of a UP-algebra.

Definition 1.1. [5] An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = y \cdot x = 0$ implies $x = y$.

Example 1.2. [5] Let X be a set. Define a binary operation $\cdot$ on the power set of X by putting $A \cdot B = B \cap A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); \cdot, \emptyset)$ is a UP-algebra.

We can easily show the following example.

Example 1.3. [5] Let $A = \{0, a, b, c\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

(1.1)

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Then $(A; \cdot, 0)$ is a UP-algebra.

In what follows, let A denote a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.4. [5] *In a UP-algebra A, the following properties hold: for any $x, y \in A$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

On a UP-algebra $A = (A; \cdot, 0)$, we define a binary relation $\leq$ on A as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0. \tag{1.2}$$

Proposition 1.5 obviously follows from Proposition 1.4.

Proposition 1.5. *[5] In a UP-algebra A, the following properties hold: for any $x, y \in A$,*

(1) $x \leq x$,

(2) $x \leq y$ *and* $y \leq x$ *imply* $x = y$,

(3) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$,

(4) $x \leq y$ *implies* $z \cdot x \leq z \cdot y$,

(5) $x \leq y$ *implies* $y \cdot z \leq x \cdot z$,

(6) $x \leq y \cdot x$, *and*

(7) $x \leq y \cdot y$.

From Proposition 1.5 and UP-3, we have Proposition 1.6.

Proposition 1.6. [5] *Let A be a UP-algebra with a binary relation $\leq$ defined by* (1.2). *Then $(A, \leq)$ is a partially ordered set with* 0 *as the greatest element.*

We often call the partial ordering $\leq$ defined by (1.2) the *UP-ordering* on A. From now on, the symbol $\leq$ will be used to denote the UP-ordering, unless specified otherwise.

Definition 1.7. [5] A nonempty subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and

(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Clearly, A and $\{0\}$ are UP-ideals of A.

Theorem 1.8. [5] *Let A be a UP-algebra and $\{B_i\}_{i\in I}$ a family of UP-ideals of A. Then $\bigcap_{i\in I} B_i$ is a UP-ideal of A.*

Definition 1.9. [5] A subset S of A is called a *UP-subalgebra* of A if it constant 0 of A is in S, and $(S; \cdot, 0)$ itself forms a UP-algebra. Clearly, A and $\{0\}$ are UP-subalgebras of A.

Applying Proposition 1.4 1.4, we can then easily prove the following proposition.

Proposition 1.10. [5] *A nonempty subset S of a UP-algebra $A = (A; \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Theorem 1.11. [5] *Let A be a UP-algebra and $\{B_i\}_{i\in I}$ a family of UP-subalgebras of A. Then $\bigcap\limits_{i\in I} B_i$ is a UP-subalgebra of A.*

Theorem 1.12. [5] *Let A be a UP-algebra and B a UP-ideal of A. Then $A \cdot B \subseteq B$. In particular, B is a UP-subalgebra of A.*

We can easily show the following example.

Example 1.13. [5] Let $A = \{0, a, b, c, d\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

(1.3)

$\cdot$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

Using the following program in the software "MATLAB", we know that $(A; \cdot, 0)$ is a UP-algebra, where we use numbers $1, 2, 3, 4$ and 5 instead of $0, a, b, c$ and d, respectively.

Program for test UP-1

```
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
if n == 4
    b = [   1    2    3    4;
            1    1    1    1;
            1    2    1    4;
            1    2    3    1   ];
else
     b = [  1    2    3    4    5;
            1    1    3    4    5;
            1    1    1    4    5;
            1    1    3    1    5;
            1    1    1    1    1   ];
end
tc = 0;
cp = 0;
np = 0;
for i = 1:n
    for j = 1:n
```

```
        for k = 1:n
            tc = tc + 1;
            rc = b(b(j,k),b(b(i,j),b(i,k)));
            if rc == 1
                cp = cp + 1;
            else
                np = np + 1;
             end
        end
    end
end
```

We can check condition 1.7 in Definition 1.7 that the set $\{0, a, c\}$ is a UP-ideal of A by using the following program.

Program for test Definition 1.7 1.7

```
clc,clear
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
if n == 4
    b = [   1   2   3   4;
            1   1   1   1;
            1   2   1   4;
            1   2   3   1   ];
else
     b = [  1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
end
tc  = 0;
cp  = 0;
scp = 0;
ncp = 0;
np  = 0;
for i = 1:n
    for j = 1:4
        for k = 1:n
            rc = b(i,b(j,k));
            if (rc <= 2) | (rc == 4)
                tc = tc + 1;
                   if j ~= 3
                    cp = cp + 1;
```

```
                    src = b(i,k);
                    if (src <= 2) | (src == 4)
                        scp = scp + 1;
                    else
                        ncp = ncp + 1;
                    end
                end
            end
            if ((rc == 3) | (rc ==5)) & (j == 3)
                np = np + 1;
            end
        end
    end
end
```

We can check that the set $\{0, a, b\}$ is a UP-ideal of A.

By Proposition 1.10, we can check that the set $\{0, a, b, c\}$ is a UP-subalgebra of A.

2. Main results

In this section, firstly, we recall the definition of a fuzzy set in a nonempty set and the definitions of a fuzzy UP-ideal and a fuzzy UP-subalgebra of a UP-algebra. Secondly, we introduce the notions of a intuitionistic fuzzy UP-ideal and a intuitionistic fuzzy UP-subalgebra of a UP-algebra and study some of their basic properties.

Definition 2.1. [27] A *fuzzy set* in a nonempty set X (or a fuzzy subset of X) is an arbitrary function $f\colon X \to [0, 1]$ where $[0, 1]$ is the unit segment of the real line. If $A \subseteq X$, the *characteristic function* f_A of X is a function of X into $\{0, 1\}$ defined as follows:

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

By the definition of the characteristic function, f_A is a function of X into $\{0, 1\} \subset [0, 1]$. Hence, f_A is a fuzzy set in X.

Definition 2.2. Let f be a fuzzy set in A. The fuzzy set $\overline{f}$ defined by $\overline{f}(x) = 1 - f(x)$ for all $x \in A$ is called the *complement* of f in A.

Definition 2.3. [24] A fuzzy set f in A is called a *fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $f(0) \geq f(x)$, and

(2) $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\}$.

Example 2.4. By Example 1.13, we get $\{0, a, b\}$ is a UP-ideal of A. Then

$$f(x) = \begin{cases} 1 & \text{if } x \in \{0, a, b\}, \\ 0 & \text{if } x \in \{c, d\} \end{cases}$$

is a fuzzy UP-ideal of A by using the following program.

```
clc,clear
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
f = zeros(n,n);
if n == 4
      b = [ 1   2   3   4;
            1   1   1   1;
            1   2   1   4;
            1   2   3   1   ];
     f = [  1   1   0.3 0.4;
            1   1   1   1;
            1   1   1   0.4;
            1   1   0.3 1  ];
else
    b = [   1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
    f = [   1   1   1   0   0;
            1   1   1   0   0;
            1   1   1   0   0;
            1   1   1   1   0;
            1   1   1   1   1   ];
end
tc  = 0;
cp  = 0;
ncp = 0;
az  = 1;
bz  = 1;
cz  = 1;
dz  = 0;
ez  = 0;
for i = 1:n
    for j = 1:n
        for k = 1:n
            re = b(j,k);
            rc = f(i,re);
```

```
            rm = b(i,k);
            rd = f(i,k);
            if(j==1)
                tc = tc + 1;
              if(rd >= min(rc,az))
                cp=cp+1;
              else
                 ncp=ncp+1;
              end
            end
            if(j==2)
                tc = tc + 1;
              if(rd >= min(rc,bz))
                  cp=cp+1;
              else
                  ncp=ncp+1;
               end
            end
            if(j==3)
                tc = tc + 1;
              if(rd >= min(rc,cz))
                  cp=cp+1;
              else
                  ncp=ncp+1;
              end
            end
            if(j==4)
                tc = tc + 1;
              if(rd >= min(rc,dz))
                  cp=cp+1;
              else
                  ncp=ncp+1;
              end
            end
            if(j==5)
                tc = tc + 1;
              if(rd >= min(rc,ez))
                  cp=cp+1;
              else
                  ncp=ncp+1;
              end
            end
        end
    end
end
```

Definition 2.5. [24] A fuzzy set f in A is called a *fuzzy UP-subalgebra* in A if for any $x, y \in A$,

$$f(x \cdot y) \geq \min\{f(x), f(y)\}. \tag{2.1}$$

Example 2.6. By Example 1.13, we get $\{0, a, b, c\}$ is a UP-subalgebra of A. Then

$$f(x) = \begin{cases} 1 & \text{if } x \in \{0, a, b, c\}, \\ 0 & \text{if } x \in \{d\} \end{cases}$$

is a fuzzy UP-subalgebra of A by using the following program.

```
clc,clear
display(['Input n = 4 or n = 5']);
n = input('n = ');
g = zeros(n,n);
b = zeros(n,n);
f = zeros(n,n);
if n == 4
    b = [   0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.7;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.7 ];
    f = [   0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3 ];
else
    g = [   1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
    b = [   1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   1   ];
    f = [   1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   0;
            1   1   1   1   0   ];
end
tc  = 0;
cp  = 0;
```

```
ncp = 0;
az  = 0.7;
bz  = 0.7;
cz  = 0.7;
dz  = 0.3;
ez  = 0.2;
for i = 1:n
    for j = 1:n
             rc = b(i,j);
             rd = f(i,j);
             if(i==1)
                 tc = tc + 1;
              if(rc >= min(az,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==2)
                 tc = tc + 1;
              if(rc >= min(bz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==3)
                 tc = tc + 1;
              if(rc >= min(cz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==4)
                 tc = tc + 1;
              if(rc >= min(dz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==5)
                 tc = tc + 1;
              if(rc >= min(ez,rd))
```

```
                    cp = cp + 1;
                else
                    ncp = ncp + 1;
                end
            end
        end
    end
```

Definition 2.7. [2], [3] An *intuitionistic fuzzy set* (briefly, IFS) in a nonempty set X is an object F having the form

$$F = \{(x, \mu_F(x), \gamma_F(x)) \mid x \in X\} \tag{2.2}$$

where the fuzzy sets $\mu_F\colon X \to [0,1]$ and $\gamma_F\colon X \to [0,1]$ denote the degree of membership and the degree of nonmembership, respectively, and for all $x \in X$,

$$0 \leq \mu_F(x) + \gamma_F(x) \leq 1. \tag{2.3}$$

An intuitionistic fuzzy set $F = \{(x, \mu_F(x), \gamma_F(x)) \mid x \in X\}$ in X can be identified to an ordered pair (μ_F, γ_F) in $[0,1]^X \times [0,1]^X$. For the sake of simplicity, we shall use the symbol $F = (\mu_F, \gamma_F)$ for the IFS $F = \{(x, \mu_F(x), \gamma_F(x)) \mid x \in X\}$.

Definition 2.8. An IFS $F = (\mu_F, \gamma_F)$ in A is called an *intuitionistic fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $\mu_F(0) \geq \mu_F(x)$,

(2) $\gamma_F(0) \leq \gamma_F(x)$,

(3) $\mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$, and

(4) $\gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}$.

Definition 2.9. An IFS $F = (\mu_F, \gamma_F)$ in A is called an *intuitionistic fuzzy UP-subalgebra* of A if it satisfies the following properties: for any $x, y \in A$,

(1) $\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\}$, and

(2) $\gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}$.

Example 2.10. Consider a UP-algebra $A = \{0, a, b, c\}$ with the following Cayley table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Let $F = (\mu_F, \gamma_F)$ be an IFS in A defined by

$$\mu_F(x) = \begin{cases} 0.3 & \text{if } x = c, \\ 0.7 & \text{if } x \neq c \end{cases}$$

and

$$\gamma_F(x) = \begin{cases} 0.5 & \text{if } x = c, \\ 0.2 & \text{if } x \neq c. \end{cases}$$

Then $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A by using the following programs.

Program for test μ_F

```
clc,clear
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
f = zeros(n,n);
if n == 4
    b = [   0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.7;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.7 ];
    f = [   0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3;
            0.7 0.7 0.7 0.3 ];
else
    b = [   1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
end
tc  = 0;
cp  = 0;
ncp = 0;
az  = 0.7;
bz  = 0.7;
cz  = 0.7;
dz  = 0.3;
for i = 1:n
    for j = 1:n
             rc = b(i,j);
             rd = f(i,j);
             if(i==1)
                 tc = tc + 1;
              if(rc >= min(az,rd))
                  cp = cp + 1;
              else
```

```
                ncp = ncp + 1;
            end
          end
          if(i==2)
              tc = tc + 1;
            if(rc >= min(bz,rd))
                cp = cp + 1;
            else
                ncp = ncp + 1;
            end
          end
          if(i==3)
              tc = tc + 1;
            if(rc >= min(cz,rd))
                cp = cp + 1;
            else
                ncp = ncp + 1;
            end
          end
          if(i==4)
              tc = tc + 1;
            if(rc >= min(dz,rd))
                cp = cp + 1;
            else
                ncp = ncp + 1;
            end
        end
     end
end
```

Program for test γ_F

```
clc,clear
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
f = zeros(n,n);
if n == 4
    b = [   0.2 0.2 0.2 0.5;
            0.2 0.2 0.2 0.2;
            0.2 0.2 0.2 0.5;
            0.2 0.2 0.2 0.2 ];
    f = [   0.2 0.2 0.2 0.5;
            0.2 0.2 0.2 0.5;
            0.2 0.2 0.2 0.5;
            0.2 0.2 0.2 0.5 ];
```

```
else
    b = [   1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
end
tc  = 0;
cp  = 0;
ncp = 0;
az  = 0.2;
bz  = 0.2;
cz  = 0.2;
dz  = 0.5;
for i = 1:n
    for j = 1:n
             rc = b(i,j);
             rd = f(i,j);
             if(i==1)
                 tc = tc + 1;
              if(rc <= max(az,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==2)
                 tc = tc + 1;
              if(rc <= max(bz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==3)
                 tc = tc + 1;
              if(rc <= max(cz,rd))
                  cp = cp + 1;
              else
                  ncp = ncp + 1;
              end
             end
             if(i==4)
                 tc = tc + 1;
              if(rc <= max(dz,rd))
```

```
                    cp = cp + 1;
                else
                    ncp = ncp + 1;
                end
          end
      end
end
```

Lemma 2.11. *Every intuitionistic fuzzy UP-subalgebra $F = (\mu_F, \gamma_F)$ of A satisfies the inequalities: for all $x \in A$,*

(1) $\mu_F(0) \geq \mu_F(x)$, *and*

(2) $\gamma_F(0) \leq \gamma_F(x)$.

Proof. Let $x \in A$. Then

(By Proposition 1.4 1.4)
$$\begin{aligned}\mu_F(0) &= \mu_F(x \cdot x)\\ &\geq \min\{\mu_F(x), \mu_F(x)\}\\ &= \min\{\mu_F(x)\}\\ &= \mu_F(x)\end{aligned}$$

and

(By Proposition 1.4 1.4)
$$\begin{aligned}\gamma_F(0) &= \gamma_F(x \cdot x)\\ &\leq \max\{\gamma_F(x), \gamma_F(x)\}\\ &= \max\{\gamma_F(x)\}\\ &= \gamma_F(x).\blacksquare\end{aligned}$$

Lemma 2.12. *Let an IFS $F = (\mu_F, \gamma_F)$ in A be an intuitionistic fuzzy UP-ideal of A. If $x, y \in A$ is such that $y \leq x$ in A, then*

(1) $\mu_F(y) \leq \mu_F(x)$, *and*

(2) $\gamma_F(y) \geq \gamma_F(x)$.

Proof. Let $x, y \in A$ be such that $y \leq x$ in A. Then $y \cdot x = 0$. Thus

$$\begin{aligned}&\text{(By UP-2)} & \mu_F(x) &= \mu_F(0 \cdot x)\\ && &\geq \min\{\mu_F(0 \cdot (y \cdot x)), \mu_F(y)\}\\ &\text{(By UP-2)} & &= \min\{\mu_F(y \cdot x), \mu_F(y)\}\\ && &= \min\{\mu_F(0), \mu_F(y)\}\\ && &= \mu_F(y)\end{aligned}$$

and

(By UP-2) $\gamma_F(x) = \gamma_F(0 \cdot x)$
$\leq \max\{\gamma_F(0 \cdot (y \cdot x)), \gamma_F(y)\}$
(By UP-2) $= \max\{\gamma_F(y \cdot x), \gamma_F(y)\}$
$= \max\{\gamma_F(0), \gamma_F(y)\}$
$= \gamma_F(y)$.

Hence, μ_F is an order preserving fuzzy set and γ_F is an anti order preserving fuzzy set in A. ■

Lemma 2.13. *Let an IFS $F = (\mu_F, \gamma_F)$ in A be an intuitionistic fuzzy UP-ideal of A. If $w, x, y, z \in A$ is such that $x \leq w \cdot (y \cdot z)$ in A, then*

(1) $\mu_F(x \cdot z) \geq \min\{\mu_F(w), \mu_F(y)\}$, *and*

(2) $\gamma_F(x \cdot z) \leq \max\{\gamma_F(w), \gamma_F(y)\}$.

Proof. Let $w, x, y, z \in A$ be such that $x \leq w \cdot (y \cdot z)$ in A. Then $x \cdot (w \cdot (y \cdot z)) = 0$. Hence,

(By Definition 2.8 2.8)
$\mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$
(By Definition 2.8 2.8) $\geq \min\{\min\{\mu_F(x \cdot (w \cdot (y \cdot z))), \mu_F(w)\}, \mu_F(y)\}$
$= \min\{\min\{\mu_F(0), \mu_F(w)\}, \mu_F(y)\}$
(By Definition 2.8 2.8) $= \min\{\mu_F(w), \mu_F(y)\}$

and

(By Definition 2.8 2.8)
$\gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}$
(By Definition 2.8 2.8) $\leq \max\{\max\{\gamma_F(x \cdot (w \cdot (y \cdot z))), \gamma_F(w)\}, \gamma_F(y)\}$
$= \max\{\max\{\gamma_F(0), \gamma_F(w)\}, \gamma_F(y)\}$
(By Definition 2.8 2.8) $= \max\{\gamma_F(w), \gamma_F(y)\}$.. ■

Corollary 2.14. *Let an IFS $F = (\mu_F, \gamma_F)$ in A be an intuitionistic fuzzy UP-ideal of A. If $x, y, z \in A$ is such that $x \leq y \cdot z$ in A, then*

(1) $\mu_F(x \cdot z) \geq \mu_F(y)$, *and*

(2) $\gamma_F(x \cdot z) \leq \gamma_F(y)$.

Proof. Let $x, y, z \in A$ be such that $x \leq y \cdot z$ in A. By Lemma 2.13, put $w = 0$. By UP-2, we have that $x \leq 0 \cdot (y \cdot z)$. Hence,

$$\mu_F(x \cdot z) \geq \min\{\mu_F(0), \mu_F(y)\} = \mu_F(y)$$

and

$$\gamma_F(x \cdot z) \leq \max\{\gamma_F(0), \gamma_F(y)\} = \gamma_F(y).$$ ■

Theorem 2.15. *Every intuitionistic fuzzy UP-ideal of A is an intuitionistic fuzzy UP-subalgebra of A.*

Proof. Let $F = (\mu_F, \gamma_F)$ be an intuitionistic fuzzy UP-ideal of A and let $x, y \in A$. By Proposition 1.5 1.5, we have $x \leq y \cdot x$. It follows from Lemma 2.12 that

$$\mu_F(y \cdot x) \geq \mu_F(x) \geq \min\{\mu_F(y), \mu_F(x)\}$$

and

$$\gamma_F(y \cdot x) \leq \gamma_F(x) \leq \max\{\gamma_F(y), \gamma_F(x)\}.$$

Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. ∎

The converse of Theorem 2.15 may not be true. For example, the intuitionistic fuzzy UP-subalgebra $F = (\mu_F, \gamma_F)$ in Example 2.10 is not an intuitionistic fuzzy UP-ideal of A since

$$\gamma_F(b \cdot c) = 0.5 > 0.2 = \max\{\gamma_F(b \cdot (a \cdot c)), \gamma_F(a)\}. \tag{2.4}$$

Lemma 2.16. *Let f be a fuzzy set in A. Then the following statements hold: for any $x, y \in A$,*

(1) $1 - \max\{f(x), f(y)\} = \min\{1 - f(x), 1 - f(y)\}$, *and*

(2) $1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\}$.

Proof. 2.16 If $\max\{f(x), f(y)\} = f(x)$, then $f(y) \leq f(x)$. Thus $1 - f(y) \geq 1 - f(x)$, so $\min\{1 - f(x), 1 - f(y)\} = 1 - f(x) = 1 - \max\{f(x), f(y)\}$. Similarly, if $\max\{f(x), f(y)\} = f(y)$, then

$$\min\{1 - f(x), 1 - f(y)\} = 1 - f(y) = 1 - \max\{f(x), f(y)\}.$$

2.16 If $\min\{f(x), f(y)\} = f(x)$, then $f(x) \leq f(y)$. Thus $1 - f(x) \geq 1 - f(y)$, so $\max\{1 - f(x), 1 - f(y)\} = 1 - f(x) = 1 - \min\{f(x), f(y)\}$. Similarly, if $\min\{f(x), f(y)\} = f(y)$, then

$$\max\{1 - f(x), 1 - f(y)\} = 1 - f(y) = 1 - \min\{f(x), f(y)\}.$$ ∎

Theorem 2.17. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A if and only if the fuzzy sets μ_F and $\overline{\gamma}_F$ are fuzzy UP-ideals of A.*

Proof. Assume that an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. Then for any $x, y, z \in A$, we have

$$\mu_F(0) \geq \mu_F(x) \text{ and } \mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}.$$

Hence, μ_F is a fuzzy UP-ideal of A. Now, for any $x, y, z \in A$, we have

$$\gamma_F(0) \leq \gamma_F(x) \text{ and } \gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}.$$

Thus $\overline{\gamma}_F(0) = 1 - \gamma_F(0) \geq 1 - \gamma_F(x) = \overline{\gamma}_F(x)$ and

$$\begin{aligned}\overline{\gamma}_F(x \cdot z) &= 1 - \gamma_F(x \cdot z)\\ &\geq 1 - \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}\\ \text{(By Lemma 2.16 2.16)} \qquad &= \min\{1 - \gamma_F(x \cdot (y \cdot z)), 1 - \gamma_F(y)\}\\ &= \min\{\overline{\gamma}_F(x \cdot (y \cdot z)), \overline{\gamma}_F(y)\}.\end{aligned}$$

Hence, $\overline{\gamma}_F$ is a fuzzy UP-ideal of A.

Conversely, assume that μ_F and $\overline{\gamma}_F$ are fuzzy UP-ideals of A. Then for any $x, y, z \in A$, we have

$$\mu_F(0) \geq \mu_F(x) \text{ and } \mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}.$$

Now, for any $x, y, z \in A$, we have

$$\overline{\gamma}_F(0) \geq \overline{\gamma}_F(x) \text{ and } \overline{\gamma}_F(x \cdot z) \geq \min\{\overline{\gamma}_F(x \cdot (y \cdot z)), \overline{\gamma}_F(y)\}.$$

Thus $1 - \gamma_F(0) \geq 1 - \gamma_F(x)$, so $\gamma_F(0) \leq \gamma_F(x)$. Now,

$$\begin{aligned}1 - \gamma_F(x \cdot z) &\geq \min\{1 - \gamma_F(x \cdot (y \cdot z)), 1 - \gamma_F(y)\}\\ \text{(By Lemma 2.16 2.16)} \qquad &= 1 - \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\},\end{aligned}$$

so $\gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}$. Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. ∎

Theorem 2.18. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A if and only if the fuzzy sets μ_F and $\overline{\gamma}_F$ are fuzzy UP-subalgebras of A.*

Proof. Assume that an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. Then for any $x, y \in A$, we have

$$\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\}.$$

Hence, μ_F is a fuzzy UP-subalgebra of A. Now, for any $x, y \in A$, we have

$$\gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}.$$

Thus

$$\begin{aligned}\overline{\gamma}_F(x \cdot y) &= 1 - \gamma_F(x \cdot y)\\ &\geq 1 - \max\{\gamma_F(x), \gamma_F(y)\}\\ \text{(By Lemma 2.16 2.16)} \qquad &= \min\{1 - \gamma_F(x), 1 - \gamma_F(y)\}\\ &= \min\{\overline{\gamma}_F(x), \overline{\gamma}_F(y)\}.\end{aligned}$$

Hence, $\overline{\gamma}_F$ is a fuzzy UP-subalgebra of A.

Conversely, assume that μ_F and $\overline{\gamma}_F$ are fuzzy UP-subalgebras of A. Then for any $x, y \in A$, we have

$$\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\}.$$

Now, for any $x, y \in A$, we have

$$\overline{\gamma}_F(x \cdot y) \geq \min\{\overline{\gamma}_F(x), \overline{\gamma}_F(y)\}.$$

Thus

$$\begin{aligned} 1 - \gamma_F(x \cdot y) &\geq \min\{1 - \gamma_F(x), 1 - \gamma_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \qquad &= 1 - \max\{\gamma_F(x), \gamma_F(y)\}, \end{aligned}$$

so $\gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}$. Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. ∎

Theorem 2.19. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A if and only if the IFSs $\Box F = (\mu_F, \overline{\mu}_F)$ and $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ are intuitionistic fuzzy UP-ideals of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. Then for any $x, y, z \in A$, we have

$$\mu_F(0) \geq \mu_F(x) \text{ and } \mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}.$$

Thus for any $x, y, z \in A$, we have $\overline{\mu}_F(0) = 1 - \mu_F(0) \leq 1 - \mu_F(x) = \overline{\mu}_F(x)$ and

$$\begin{aligned} \overline{\mu}_F(x \cdot z) &= 1 - \mu_F(x \cdot z) \\ &\leq 1 - \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \qquad &= \max\{1 - \mu_F(x \cdot (y \cdot z)), 1 - \mu_F(y)\} \\ &= \max\{\overline{\mu}_F(x \cdot (y \cdot z)), \overline{\mu}_F(y)\}. \end{aligned}$$

Hence, $\Box F = (\mu_F, \overline{\mu}_F)$ is an intuitionistic fuzzy UP-ideal of A. Now, for any $x, y, z \in A$, we have

$$\gamma_F(0) \leq \gamma_F(x) \text{ and } \gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}.$$

Thus for any $x, y, z \in A$, we have $\overline{\gamma}_F(0) = 1 - \gamma_F(0) \geq 1 - \gamma_F(x) = \overline{\gamma}_F(x)$ and

$$\begin{aligned} \overline{\gamma}_F(x \cdot z) &= 1 - \gamma_F(x \cdot z) \\ &\geq 1 - \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \qquad &= \min\{1 - \gamma_F(x \cdot (y \cdot z)), 1 - \gamma_F(y)\} \\ &= \min\{\overline{\gamma}_F(x \cdot (y \cdot z)), \overline{\gamma}_F(y)\}. \end{aligned}$$

Hence, $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A.

Conversely, assume that $\Box F = (\mu_F, \overline{\mu}_F)$ and $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ are intuitionistic fuzzy UP-ideals of A. Then for any $x, y, z \in A$, we have

$$\mu_F(0) \geq \mu_F(x) \text{ and } \mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\},$$

and

$$\gamma_F(0) \leq \gamma_F(x) \text{ and } \gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\}.$$

Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. ∎

Theorem 2.20. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A if and only if the IFSs $\Box F = (\mu_F, \overline{\mu}_F)$ and $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ are intuitionistic fuzzy UP-subalgebras of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. Then for any $x, y \in A$, we have

$$\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\}.$$

Thus for any $x, y \in A$, we have

$$\begin{aligned} \overline{\mu}_F(x \cdot y) &= 1 - \mu_F(x \cdot y) \\ &\leq 1 - \min\{\mu_F(x), \mu_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \quad &= \max\{1 - \mu_F(x), 1 - \mu_F(y)\} \\ &= \max\{\overline{\mu}_F(x), \overline{\mu}_F(y)\}. \end{aligned}$$

Hence, $\Box F = (\mu_F, \overline{\mu}_F)$ is an intuitionistic fuzzy UP-subalgebra of A. Now, for any $x, y \in A$, we have

$$\gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}.$$

Thus for any $x, y \in A$, we have

$$\begin{aligned} \overline{\gamma}_F(x \cdot y) &= 1 - \gamma_F(x \cdot y) \\ &\geq 1 - \max\{\gamma_F(x), \gamma_F(y)\} \\ \text{(By Lemma 2.16 2.16)} \quad &= \min\{1 - \gamma_F(x), 1 - \gamma_F(y)\} \\ &= \min\{\overline{\gamma}_F(x), \overline{\gamma}_F(y)\}. \end{aligned}$$

Hence, $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ in an intuitionistic fuzzy UP-subalgebra of A.

Conversely, assume that $\Box F = (\mu_F, \overline{\mu}_F)$ and $\Diamond F = (\overline{\gamma}_F, \gamma_F)$ are intuitionistic fuzzy UP-subalgebra of A. Then for any $x, y \in A$, we have

$$\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\} \text{ and } \gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}.$$

Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. ∎

Definition 2.21. Let f be a fuzzy set in A. For any $t \in [0, 1]$, the set

$$U(f; t) = \{x \in A \mid f(x) \geq t\} \text{ and } U^+(f; t) = \{x \in A \mid f(x) > t\}$$

are called an *upper t-level subset* and an *upper t-strong level subset* of f, respectively. The set

$$L(f; t) = \{x \in A \mid f(x) \leq t\} \text{ and } L^-(f; t) = \{x \in A \mid f(x) < t\}$$

are called a *lower t-level subset* and a *lower t-strong level subset* of f, respectively.

Theorem 2.22. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A if and only if for all $s, t \in [0, 1]$, the sets $U(\mu_F; t)$ and $L(\gamma_F; s)$ are either empty or UP-ideals of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. Let $U(\mu_F; t)$ and $L(\gamma_F; s)$ be nonempty subsets of A for all $s, t \in [0, 1]$. Then there exist $a \in U(\mu_F; t)$ and $b \in L(\gamma_F; s)$, that is, $\mu_F(a) \geq t$ and $\gamma_F(b) \leq s$. Since $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A, we have $\mu_F(0) \geq \mu_F(x)$ and $\gamma_F(0) \leq \gamma_F(x)$ for all $x \in A$. Thus $\mu_F(0) \geq \mu_F(a) \geq t$ and $\gamma_F(0) \leq \gamma_F(b) \leq s$, so $0 \in U(\mu_F; t)$ and $0 \in L(\gamma_F; s)$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(\mu_F; t)$ and $y \in U(\mu_F; t)$. Then $\mu_F(x \cdot (y \cdot z)) \geq t$ and $\mu_F(y) \geq t$. Thus

$$\begin{aligned} \text{(By Definition 2.8 2.8)} \qquad \mu_F(x \cdot z) &\geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\} \\ &\geq \min\{t, t\} \\ &= t, \end{aligned}$$

so $x \cdot z \in U(\mu_F; t)$. Hence, $U(\mu_F; t)$ is a UP-ideal of A. Finally, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L(\gamma_F; s)$ and $y \in L(\gamma_F; s)$. Then $\gamma_F(x \cdot (y \cdot z)) \leq s$ and $\gamma_F(y) \leq s$. Thus

$$\begin{aligned} \text{(By Definition 2.8 2.8)} \qquad \gamma_F(x \cdot z) &\leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\} \\ &\leq \max\{s, s\} \\ &= s, \end{aligned}$$

so $x \cdot z \in L(\gamma_F; s)$. Hence, $L(\gamma_F; s)$ is a UP-ideal of A.

Conversely, assume that for any $s, t \in [0, 1]$, the sets $U(\mu_F; t)$ and $L(\gamma_F; s)$ are either empty or UP-ideals of A. For any $x \in A$, let $\mu_F(x) = t$ and $\gamma_F(x) = s$. Then $x \in U(\mu_F; t) \neq \emptyset$ and $x \in L(\gamma_F; s) \neq \emptyset$. By assumption, we have $U(\mu_F; t)$ and $L(\gamma_F; s)$ are UP-ideals of A. Thus $0 \in U(\mu_F; t)$ and $0 \in L(\gamma_F; s)$, so $\mu_F(0) \geq t = \mu_F(x)$ and $\gamma_F(0) \leq s = \gamma_F(x)$ for all $x \in A$. Suppose that there exist $x, y, z \in A$ such that $\mu_F(x \cdot z) < \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$. Put

$$t_0 = \frac{1}{2}[\mu_F(x \cdot z) + \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}].$$

Thus $t_0 \in [0, 1]$ and $\mu_F(x \cdot z) < t_0 < \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$. This implies that $x \cdot z \notin U(\mu_F; t_0), x \cdot (y \cdot z) \in U(\mu_F; t_0)$ and $y \in U(\mu_F; t_0)$. Thus $U(\mu_F; t_0)$ is not a UP-ideal of A. Now, suppose that there exist $a, b, c \in A$ such that $\gamma_F(a \cdot c) > \max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\}$. Put

$$s_0 = \frac{1}{2}[\gamma_F(a \cdot c) + \max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\}].$$

Thus $s_0 \in [0, 1]$ and $\max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\} < s_0 < \gamma_F(a \cdot c)$. This implies that $a \cdot c \notin L(\gamma_F; s_0), a \cdot (b \cdot c) \in L(\gamma_F; s_0)$ and $b \in L(\gamma_F; s_0)$. Thus $L(\gamma_F; s_0)$ is not a UP-ideal of A. By assumption, we have $U(\mu_F; t_0)$ and $L(\gamma_F; s_0)$ are empty. This is a contradiction to the fact that $y \in U(\mu_F; t_0) \neq \emptyset$ and $b \in L(\gamma_F; s_0) \neq \emptyset$. Hence, $\mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$ and $\gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(b)\}$ for all $x, y, z \in A$. Therefore, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. ∎

Theorem 2.23. *An IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A if and only if for all $s, t \in [0, 1]$, the sets $U(\mu_F; t)$ and $L(\gamma_F; s)$ are either empty or UP-subalgebras of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. Let $U(\mu_F; t)$ and $L(\gamma_F; s)$ be nonempty subsets of A for all $s, t \in [0, 1]$. Let $x, y \in U(\mu_F; t)$. Then $\mu_F(x) \geq t$. Thus

(By Definition 2.9 2.9)
$$\begin{aligned} \mu_F(x \cdot y) &\geq \min\{\mu_F(x), \mu_F(y)\} \\ &\geq \min\{t, t\} \\ &= t, \end{aligned}$$

so $x \cdot y \in U(\mu_F; t)$. It follows from Proposition 1.10 that $U(\mu_F; t)$ is a UP-subalgebra of A. Finally, let $x, y \in L(\mu_F; t)$. Then $\mu_F(y) \geq t$ and

(By Definition 2.9 2.9)
$$\begin{aligned} \gamma_F(x \cdot y) &\leq \max\{\gamma_F(x), \gamma_F(y)\} \\ &\leq \max\{s, s\} \\ &= s, \end{aligned}$$

so $x \cdot y \in L(\gamma_F; s)$. It follows from Proposition 1.10 that $L(\gamma_F; s)$ is a UP-subalgebra of A. Conversely, assume that for any $s, t \in [0, 1]$, the set $U(\mu_F; t)$ and $L(\gamma_F; s)$ are either empty or UP-subalgebras of A. For any $x, y \in A$, let $\min\{\mu_F(x), \mu_F(y)\} = t$ and $\max\{\gamma_F(x), \gamma_F(y)\} = s$. Then $x, y \in U(\mu_F; t) \neq \emptyset$ and $x, y \in L(\gamma_F; s) \neq \emptyset$. By assumption, we have $U(\mu_F; t)$ and $L(\gamma_F; s)$ are UP-subalgebras of A and so $x \cdot y \in U(\mu_F; t)$ and $x \cdot y \in L(\gamma_F; s)$. It follows that $\mu_F(x \cdot y) \geq t = \min\{\mu_F(x), \mu_F(y)\}$ and $\gamma_F(x \cdot y) \leq s = \max\{\gamma_F(x), \gamma_F(y)\}$. Hence, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. ■

Theorem 2.24. *If an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A, then for all $s, t \in [0, 1]$, the sets $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are either empty or UP-ideals of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. Let $s, t \in [0, 1]$ be such that $U^+(\mu_F; t)$ and $L^-(\gamma_F; s)$ are nonempty subsets of A. Then there exist $a \in U^+(\mu_F; t)$ and $b \in L^-(\gamma_F; s)$, that is, $\mu_F(a) > t$ and $\gamma_F(b) < s$. Since $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A, we have $\mu_F(0) \geq \mu_F(x)$ and $\gamma_F(0) \leq \gamma_F(x)$ for all $x \in A$. Thus $\mu_F(0) \geq \mu_F(a) > t$ and $\gamma_F(0) \leq \gamma_F(b) < s$, so $0 \in U^+(\mu_F; t)$ and $0 \in L^-(\gamma_F; s)$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U^+(\mu_F; t)$ and $y \in U^+(\mu_F; t)$. Then $\mu_F(x \cdot (y \cdot z)) > t$ and $\mu_F(y) > t$. Thus

(By Definition 2.8 2.8)
$$\begin{aligned} \mu_F(x \cdot z) &\geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\} \\ &> \min\{t, t\} \\ &= t, \end{aligned}$$

so $x \cdot z \in U^+(\mu_F; t)$. Hence, $U^+(\mu_F; t)$ is a UP-ideal of A. Finally, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L^-(\gamma_F; s)$ and $y \in L^-(\gamma_F; s)$. Then $\gamma_F(x \cdot (y \cdot z)) < s$ and $\gamma_F(y) < s$. Thus

(By Definition 2.8 2.8)
$$\begin{aligned} \gamma_F(x \cdot z) &\leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(y)\} \\ &< \max\{s, s\} \\ &= s, \end{aligned}$$

so $x \cdot z \in L^-(\gamma_F; s)$. Hence, $L^-(\gamma_F; s)$ is a UP-ideal of A. ■

Theorem 2.25. *If for all $s, t \in [0,1]$, the sets $U^{+}(\mu_F; t)$ and $L^{-}(\gamma_F; s)$ are UP-ideals of A, then an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A.*

Proof. Assume that for all $s, t \in [0,1]$, the sets $U^{+}(\mu_F; t)$ and $L^{-}(\gamma_F; s)$ are UP-ideals of A. For any $x \in A$, we have $\mu_F(x) \in [0,1]$ and $\gamma_F(x) \in [0,1]$. By assumption, we have $U^{+}(\mu_F; \mu_F(x))$ and $L^{-}(\gamma_F; \gamma_F(x))$ are UP-ideals of A. Thus $0 \in U^{+}(\mu_F; \mu_F(x))$ and $0 \in L^{-}(\gamma_F; \gamma_F(x))$, that is, $\mu_F(0) > \mu_F(x)$ and $\gamma_F(0) < \gamma_F(x)$. Suppose that there exist $x, y, z \in A$ such that $\mu_F(x \cdot z) < \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$. Put $t_0 = \frac{1}{2}[\mu_F(x \cdot z) + \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}]$. Thus $t_0 \in [0,1]$ and $\mu_F(x \cdot z) < t_0 < \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$. This implies that $x \cdot z \notin U^{+}(\mu_F; t_0), x \cdot (y \cdot z) \in U^{+}(\mu_F; t_0)$ and $y \in U^{+}(\mu_F; t_0)$. Thus $U^{+}(\mu_F; t_0)$ is not a UP-ideal of A. Now, suppose that there exist $a, b, c \in A$ such that $\gamma_F(a \cdot c) > \max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\}$. Put $s_0 = \frac{1}{2}[\gamma_F(a \cdot c) + \max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\}]$. Thus $s_0 \in [0,1]$ and $\max\{\gamma_F(a \cdot (b \cdot c)), \gamma_F(b)\} < s_0 < \gamma_F(a \cdot c)$. This implies that $a \cdot c \notin L^{-}(\gamma_F; s_0), a \cdot (b \cdot c) \in L^{-}(\gamma_F; s_0)$ and $b \in L^{-}(\gamma_F; s_0)$. Thus $L^{-}(\gamma_F; s_0)$ is not a UP-ideal of A. This is a contradiction to the fact that for all $s, t \in [0,1]$, the sets $U^{+}(\mu_F; t)$ and $L^{-}(\gamma_F; s)$ are UP-ideals of A. Hence, $\mu_F(x \cdot z) \geq \min\{\mu_F(x \cdot (y \cdot z)), \mu_F(y)\}$ and $\gamma_F(x \cdot z) \leq \max\{\gamma_F(x \cdot (y \cdot z)), \gamma_F(b)\}$ for all $x, y, z \in A$. Therefore, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-ideal of A. ∎

Theorem 2.26. *If an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A, then for all $s, t \in [0,1]$, the sets $U^{+}(\mu_F; t)$ and $L^{-}(\gamma_F; s)$ are either empty or UP-subalgebras of A.*

Proof. Assume that $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. Let $s, t \in [0,1]$ be such that $U^{+}(\mu_F; t)$ and $L^{-}(\gamma_F; s)$ are nonempty subsets of A. Let $x, y \in U^{+}(\mu_F; t)$. Then $\mu_F(x) > t$ and $\mu_F(y) > t$. Thus

(By Definition 2.9 2.9)
$$\begin{aligned} \mu_F(x \cdot y) &\geq \min\{\mu_F(x), \mu_F(y)\} \\ &> \min\{t, t\} \\ &= t, \end{aligned}$$

so $x \cdot y \in U^{+}(\mu_F; t)$. It follows from Proposition 1.10 that $U^{+}(\mu_F; t)$ is a UP-subalgebra of A. Finally, let $x, y \in L^{-}(\gamma_F; s)$. Then $\gamma_F(x) < s$ and $\gamma_F(y) < s$. Thus

(By Definition 2.9 2.9)
$$\begin{aligned} \gamma_F(x \cdot y) &\leq \max\{\gamma_F(x), \gamma_F(y)\} \\ &< \max\{s, s\} \\ &= s, \end{aligned}$$

so $x \cdot y \in L^{-}(\gamma_F; s)$. It follows from Proposition 1.10 that $L^{-}(\gamma_F; s)$ is a UP-subalgebra of A. ∎

Theorem 2.27. *If for all $s, t \in [0,1]$, the sets $U^{+}(\mu_F; t)$ and $L^{-}(\gamma_F; s)$ are UP-subalgebras of A, then an IFS $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A.*

Proof. Assume that for all $s, t \in [0,1]$, the sets $U^{+}(\mu_F; t)$ and $L^{-}(\gamma_F; s)$ are UP-subalgebras of A. Suppose that there exist $x, y \in A$ such that $\mu_F(x \cdot y) < \min\{\mu_F(x), \mu_F(y)\}$. Put $t_0 = \frac{1}{2}[\mu_F(x \cdot y) + \min\{\mu_F(x), \mu_F(y)\}]$. Thus $t_0 \in [0,1]$ and $\mu_F(x \cdot y) < t_0 < \min\{\mu_F(x), \mu_F(y)\}$. This implies that $x \cdot y \notin U^{+}(\mu_F; t_0)$, $x \in U^{+}(\mu_F; t_0)$ and $y \in U^{+}(\mu_F; t_0)$. Thus $U^{+}(\mu_F; t_0)$ is not a UP-subalgebra of A. Now, suppose that there exist $a, b \in A$ such that $\gamma_F(a \cdot b) > \max\{\gamma_F(a), \gamma_F(b)\}$. Put $s_0 = \frac{1}{2}[\gamma_F(a \cdot b) + \max\{\gamma_F(a), \gamma_F(b)\}]$. Thus $s_0 \in [0,1]$ and $\max\{\gamma_F(a), \gamma_F(b)\} < s_0 < \gamma_F(a \cdot b)$. This implies that $a \cdot b \notin L^{-}(\gamma_F; s_0), a \in L^{-}(\gamma_F; s_0)$ and $b \in L^{-}(\gamma_F; s_0)$. Thus $L^{-}(\gamma_F; s_0)$ is not a UP-subalgebra of A. This is a contradiction to the fact that for all $s, t \in [0,1]$, the sets $U^{+}(\mu_F; t)$ and $L^{-}(\gamma_F; s)$ are UP-subalgebras of A. Hence, $\mu_F(x \cdot y) \geq \min\{\mu_F(x), \mu_F(y)\}$ and $\gamma_F(x \cdot y) \leq \max\{\gamma_F(x), \gamma_F(y)\}$ for all $x, y \in A$. Therefore, $F = (\mu_F, \gamma_F)$ is an intuitionistic fuzzy UP-subalgebra of A. ∎

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