CHARACTERIZATION OF BI Γ-TERNARY SEMIGROUPS BY THEIR IDEALS

Muhammad Akram
Jacob Kavikumar
Azme Khamis

Department of Mathematics and Statistics
Faculty of Science, Technology and Human Development
Universiti Tun Hussein Onn Malaysia
Batu Pahat
Malaysia
emails: makram_69@yahoo.com
kavi@uthm.edu.my
azme@uthm.edu.my

Abstract. In this paper, the concept of biΓ-ternary semigroup has been introduced. The notion of biΓ-ternary subsemigroup, biΓ left (right, lateral) ideals, biΓ-quasi and biΓ-bi-ideals of this newly defined structure has been introduced. Also the regular biΓ-ternary semigroups have been studied in terms of biΓ-ideals.

Keywords: ternary semigroup, Γ-semigroup, biΓ-ternary smigroup, biΓ-ideal, regular biΓ-ternary smigroup.

2010 Mathematical Subject Classification: 20N10, 20N99, 20M12.

1. Introduction

The concept of a semigroup is very simple but it plays a key role in the development of Mathematics. The formal study of semigroups began in the early 20^{th} century. The semigroups are significantly important in many areas of mathematics because they are the abstract algebraic underpinning of "memoryless" systems: time-dependent systems that start from scratch at each iteration. In applied mathematics, semigroups are fundamental models for linear time-invariant systems. In partial differential equations, a semigroup is associated to any equation whose spatial evolution is independent of time. The theory of finite semigroups has been of particular importance in theoretical computer science since 1950s because of the natural link between finite semigroups and finite automata. In probability theory, semigroups are associated with Markov process.

The algebraic theory of semigroups was widely studied by Clifford and Preston [1], [2], Petrich [15], [16], [17] and Ljapin [14]. They all discussed the notion of an ideal in semigroups. Good and Hughes [6] and Lajos [11] presented the idea of bi-ideals in the semigroup. Lajos [12] and Szasz [26], [27] gave the notion of interior ideals in the semigroup. Steinfeld [25] introduced the notion of quasi-ideals in the semigroups.

Lehmer [13], gave the formal definition of a ternary semigroup in 1932 but Kasner and Prüfer [10], [18] studied such structures earlier. Sioson [24] developed the ideal theory of ternary semigroups. Dixit and Dewan [5] enhanced the theory of quasi-ideal and bi-ideal of the ternary semigroups. Santiago [21], worked on the theory of ternary semigroups and semiheaps. Dutta et al. [4] studied regular ternary semigroups.

As a generalization of semigroup and ternary semigroup, Sen [22] introduced the notion of Γ-semigroup in 1981 and developed a theory on Γ-semigroups [23]. Many classical notions of semigroups have been extended to Γ-semigroups by Saha and Sen in [19, 20, 23]. The notion of bi-ideal in Γ-semigroup was introduced by Chinram and Jirojkul [3, 9]. Iampan [7] and Islam [8] extended the work on bi-ideals in Γ-semigroups.

In this paper we inspired from the concept of ternary semigroup and Γ-semigroup and obtain a new algebraic structure called biΓ-ternary semigroup. The word biΓ is used due to the double appearance of the nonempty set Γ in the structure. Here the notions of biΓ-ternary subsemigroup, biΓ-left (right, lateral) ideal, biΓ-quasi ideal and biΓ-bi-ideal have been presented with the characterization of regular biΓ-ternary semigroup by these ideals.

2. Preliminaries

2.1. Semigroup

A semigroup is a set S along with a binary operation " $*$ " (that is, a function $* : S \times S \to S$) that satisfies the associative property. For all $a, b, c \in S$, the equation $(a * b) * c = a * (b * c)$ holds. Generally, we write this as $(ab)c = a(bc)$. The semigroup operation induces an operation on the collection of its subsets: given subsets A and B of a semigroup S, their product $A * B$, written commonly as AB, is the set $\{ab \mid a \in A \text{ and } b \in B\}$. In terms of this operations, a subset A of S is called a subsemigroup of S if $AA \subseteq A$, a right ideal if $AS \subseteq A$, and a left ideal if $SA \subseteq A$. If A is both a left ideal and a right ideal then it is called an ideal (or a two-sided ideal). A subsemigroup A of S is called a bi-ideal of S if $ASA \subseteq A$. A nonempty subset A of S is called an interior ideal of S if $SAS \subseteq A$.

2.2. Ternary semigroup

A ternary semigroup T is a nonempty set whose elements are closed under the ternary operation of multiplication and satisfies the associative law defined as

$$[[abc]\, de] = [a\, [bcd]\, e] = [ab\, [cde]]\,, \quad \text{for all } a, b, c, d, e \in T.$$

For simplicity we shall write $[abc]$ as abc. A nonempty subset A of a ternary semigroup T is called a ternary subsemigroup of T if $AAA \subseteq A$ and is called an idempotent if $AAA = A^3 = A$. A left (right, lateral) ideal of a ternary semigroup T is a nonempty subset A of T such that $TTA \subseteq A$ ($ATT \subseteq A$, $TAT \subseteq A$). A nonempty subset of T is called an ideal if it is a left, a right and a lateral ideal of T. A subsemigroup B of a ternary semigroup T is called a bi-ideal of T if $BTBTB \subseteq B$. A nonempty subset A of T is called an interior ideal of T if $TTATT \subseteq A$.

2.3. Γ-Semigroup

Let $S = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two nonempty sets. Then S is called a Γ-semigroup if it satisfies,

(i) $x\gamma y \in S$

(ii) $(x\beta y)\gamma z = x\beta(y\gamma z)$, for all $x, y, z \in S$ and $\beta, \gamma \in \Gamma$.

A nonempty subset 'A' of a Γ-semigroup S is called Γ-subsemigroup of S if $A\Gamma A \subseteq A$. By a left (right) Γ-ideal of a Γ-semigroup S we mean a nonempty subset A of S such that $S\Gamma A \subseteq A$ ($A\Gamma S \subseteq A$) and a two sided Γ-ideal or simply a Γ-ideal is that which is both a left and a right Γ-ideal of S. A Γ-subsemigroup B of a Γ-semigroup S is called a Γ-bi-ideal of S if $B\Gamma S\Gamma B \subseteq B$. A nonempty subset A of T is called an interior ideal of T if $TTATT \subseteq A$.

3. BiΓ-ternary semigroup

3.1. Basic concepts

Here, we define the basic concepts of BiΓ-ternary semigroup.

Definition 3.1.1 Let $T = \{x, y, z, ...\}$ and $\Gamma = \{\alpha, \beta, \gamma, ...\}$ be two nonempty sets. Then we call T as a BiΓ-ternary semigroup if it satisfies,

(i) $(x\alpha y)\beta z \in T$

(ii) $((v\alpha w\beta x)\gamma y)\delta z = (v\alpha(w\beta x\gamma y))\delta z = v\alpha(w\beta(x\gamma y\delta z)$,

for all $x, y, z, v, w \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$.

Example 3.1.2 Let $T = \{4n + 3, n \in N\}$ and $\Gamma = \{4n + 1, n \in N\}$. Define the mapping $T \times \Gamma \times T \times \Gamma \times T \to T$ as $(x\gamma y)\delta z = x + \gamma + y + \delta + z$. Let $x, y, z \in T$ and $\gamma, \delta \in \Gamma$, then

$$\begin{aligned}
(x\alpha y)\beta z &= x + \alpha + y + \beta + z \\
&= 4n_1 + 3 + 4n' + 1 + 4n_2 + 3 + 4n'' + 1 + 4n_3 + 3 \\
&= 4(n_1 + n' + n_2 + n'' + n_3 + 2) + 3 \\
&= 4n + 3, \\
(\text{where, } n &= n_1 + n' + n_2 + n'' + n_3 + 2 \in N, \text{ for } n_1, n', n_2, n'', n_3 \in N\ )
\end{aligned}$$

Also it is clear that $((v\alpha w\beta x)\gamma y)\delta z = (v\alpha(w\beta x\gamma y))\delta z = v\alpha(w\beta(x\gamma y\delta z)$, for all $x, y, z, v, w \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Hence T is a biΓ-ternary semigroup.

Example 3.1.3 Let $T = \{2n, n \in N\}, \Gamma = \{\alpha, \beta, \gamma ...\}$. Define $(x\gamma y)\delta z = x + y + z$, for all, $x, y, z \in T$ and $\gamma, \delta \in \Gamma$. Then T is a biΓ-ternary semigroup.

Example 3.1.4 Let $S = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta\}$, consider the operation defined bellow

α	0	a	b	c
0	0	0	0	0
a	0	b	0	a
b	0	b	0	c
c	0	0	0	b

and

β	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	0
c	a	a	a	c

Then S is a neither a Γ-semigroup nor a biΓ-ternary semigroup, as we can see,

$$\begin{aligned}
(a\alpha c)\alpha c &= a \neq 0 = a\alpha(c\alpha c) \text{ and} \\
((a\alpha c)\beta b)\alpha a &= (a\beta b)\alpha a = a\alpha a = b \\
(a\alpha(c\beta b))\alpha a &= b \\
a\alpha((c\beta b)\alpha a) &= 0 \neq b \\
a\alpha(c\beta(b\alpha a) &= b \\
0 &\neq b.
\end{aligned}$$

which implies that S is not a biΓ-ternary semigroup.

Remark 3.1.5 Every Γ-semigroup is a biΓ-ternary semigroup but the converse is not true.

Example 3.1.6 Let $T = Z^-$ and $\Gamma \subseteq Z^+$. Define $(x\gamma y)\delta z$, for $x, y, z \in T$ and $\gamma, \delta \in \Gamma$ as the usual multiplication of integers. Then for $x, y, z \in T$ and $\gamma, \delta \in \Gamma, (x\gamma y)\delta z \in T$ and $((v\alpha w\beta x)\gamma y)\delta z = (v\alpha(w\beta x\gamma y))\delta z = v\alpha(w\beta(x\gamma y\delta z)$, for all $x, y, z, v, w \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Hence T is a biΓ-ternary semigroup.

Now for $x, y \in T = Z^-$ and $\alpha \in \Gamma = Z^+$, $x\alpha y \notin T = Z^-$. Which shows that $T = Z^-$ is not a Γ-semigroup.

Example 3.1.7 Let $T = iR$,where, $i = \sqrt{-1}$ and R is the set of real numbers. If $\Gamma \subseteq R$ and $(x\alpha y)\beta z$ is defined as the usual multiplication of complex numbers. Then, for $x, y, z \in T$ there exist $a, b, c \in R$ so that $x = ai$, $y = bi$ and $z = ci$. For, $\alpha, \beta \in \Gamma$,

$$(x\alpha y)\beta z = (ai\alpha bi)\beta ci = abc\alpha\beta i^3 = -abc\alpha\beta i = ri, \text{ where } r = -abc\alpha\beta \in R.$$

Also,

$$((v\alpha w\beta x)\gamma y)\delta z = (v\alpha(w\beta x\gamma y))\delta z = v\alpha(w\beta(x\gamma y\delta z),$$

for all $x, y, z, v, w \in S$ and $\alpha, \beta, \gamma, \delta \in \Gamma$. Hence T is a biΓ-ternary semigroup. But, for $x = ai, y = bi \in T = iR$ and $\alpha \in \Gamma$,

$$x\alpha y = ai\alpha bi = ab\alpha i^2 = -ab\alpha \notin T = iR,$$

which shows that T is not a Γ-semigroup.

Definition 3.1.8 Let T be a biΓ-ternary semigroup and A be a nonempty subset of T. Then A is called a biΓ-ternary subsemigroup of T if,

$$A\Gamma A\Gamma A \subseteq A.$$

Example 3.1.9 let $T = N = \{1, 2, 3, ...\}$ and $\Gamma = \{4n + 2, n \in N\}$. Define $(x\alpha y)\beta z = x + \alpha + y + \beta + z$. Under this operation T is a biΓ-ternary semigroup.

Let $A = \{4n, n \in N\}$ be a nonempty subset of T. For $x, y, z \in A$ and $\alpha, \beta \in \Gamma$,

$$\begin{aligned} (x\alpha y)\beta z &= (x + \alpha + y) + \beta + z \\ &= (4n_1 + 4n' + 2 + 4n_2) + 4n'' + 2 + 4n_3 \\ &= 4(n_1 + n' + n_2 + n'' + n_3 + 1) \\ &= 4n \in A \\ \text{Where, } n &= n_1 + n' + n_2 + n'' + n_3 + 1 \in N, \text{ for } n_1, n', n_2, n'', n_3 \in N. \end{aligned}$$

which implies that $A\Gamma A\Gamma A \subseteq A$. Hence A is a biΓ-ternary subsemigroup.

Definition 3.1.10 Let T be a biΓ-ternary semigroup and A a nonempty subset of T. Then A is called a biΓ-left (right, lateral) ideal of T if

$$T\Gamma T\Gamma A \subseteq A \ (A\Gamma T\Gamma T \subseteq A, T\Gamma A\Gamma T \subseteq A)$$

A is called a biΓ-ideal of T if it is a biΓ-left, a biΓ-right and a biΓ-lateral ideal of T.

Example 3.1.11 Let $T = \{2n, n \in N\}, \Gamma = \{\alpha, \beta, \gamma, ...\}$ and $A = \{4n, n \in N\}$. Define, $(x\gamma y)\delta z = (2x + 2y) + z$, for $x, y, z \in T$ and $\gamma, \delta \in \Gamma$. Then T is a biΓ-ternary semigroup. For $x, y \in T$, $a \in A$ and $\alpha, \beta \in \Gamma$, we have

$$\begin{aligned} (x\gamma y)\delta a &= (2x + 2y) + a \\ &= 2(2n_1 + 2n_2) + 4n', x = 2n_1, y = 2n_2 \text{ and } a = 4n' \\ &= 4(n_1 + n_2 + n') \\ &= 4n \in A \text{ (where, } n = n_1 + n_2 + n' \in N, \text{ for } n_1, n_2, n' \in N. \end{aligned}$$

which implies that $T\Gamma T\Gamma A \subseteq A$. Hence A is a biΓ-left ideal of T.
Now, consider

$$\begin{aligned} (a\gamma x)\delta y &= (2a + 2x) + y \\ &= (8n' + 4n_1) + 2n_2, x = 2n_1, y = 2n_2 \text{ and } a = 4n' \\ &= 4(2n' + n_1) + 2n_2. \\ \text{Taking } n' &= n_1 = n_2 = 1, \Rightarrow (a\gamma x)\delta y = 4(2.1 + 1) + 2.1 = 14 \notin A. \end{aligned}$$

which implies that $A\Gamma T\Gamma T \notin A$. Similarly we can show that $T\Gamma A\Gamma T \notin A$. Hence A is neither a biΓ-right nor a biΓ-lateral ideal of T.

Remark 3.1.12 If we define, $(x\gamma y)\delta z = (x+2y)+2z$ and $(x\gamma y)\delta z = (2x+y)+2z$ respectively, then A is a biΓ-right and a biΓ-lateral ideal of T.

Example 3.1.13 In the above example if we define, $(x\gamma y)\delta z = (2x+2y)+2z$, then A is a biΓ-left, a biΓ-right and a biΓ-lateral ideal of T. Hence A is a bi Γ-ideal of T.

3.2. Main results

In what follows, let T denotes a biΓ-ternary semigroup, unless otherwise it is stated. In short, we shall use $B\Gamma TS(s)$ for biΓ-ternary semigroup(s), $B\Gamma TSS(s)$ for biΓ-ternary subsemigroup(s), $B\Gamma LI(s), B\Gamma RI(s), B\Gamma MI(s)$ and $B\Gamma I(s)$ for biΓ-left ideal(s), biΓ-right ideal(s), biΓ-lateral ideal(s) and biΓ-ideal(s) of a biΓ-ternary semigroup.

Proposition 3.2.1 *Let T be a $B\Gamma TS$ and $\phi \neq X \subseteq T$, then*

(i) $T\Gamma T\Gamma X$ *be a* $B\Gamma LI$ *of* T.

(ii) $X\Gamma T\Gamma T$ *be a* $B\Gamma RI$ *of* T.

(iii) $T\Gamma X\Gamma T \cup T\Gamma T\Gamma X\Gamma T\ \Gamma T$ *be a* $B\Gamma MI$ *of* T.

Proof. It follows directly from the definitions of $B\Gamma LI, B\Gamma RI$ and $B\Gamma MI$. ∎

Lemma 3.2.2 *Let T be a $B\Gamma TS$, for any $t \in T$, define,*

(i) $(t)_l = t \cup T\Gamma T\Gamma t$

(ii) $(t)_r = t \cup t\Gamma T\Gamma T$

(iii) $(t)_m = t \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T$

(iv) $(t) = t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T$.

Then $(t)_l, (t)_r, (t)_m$ and (t) are $B\Gamma LI$, $B\Gamma RI$, $B\Gamma MI$ and $B\Gamma I$ of T respectively.

Proof. (i) Since $(t)_l = t \cup T\Gamma T\Gamma t$, then

$$\begin{aligned}
T\Gamma T\Gamma (t)_l &= T\Gamma T\Gamma (t \cup T\Gamma T\Gamma t) \\
&= T\Gamma T\Gamma t \cup T\Gamma T\Gamma T\Gamma T\Gamma t \\
&\subseteq T\Gamma T\Gamma t \cup T\Gamma T\Gamma t, \quad \text{since } T\Gamma T\Gamma T \subseteq T. \\
&= T\Gamma T\Gamma t \subseteq t \cup T\Gamma T\Gamma t = (t)_l \\
T\Gamma T\Gamma (t)_l &\subseteq (t)_l, \text{ implies that } (t)_l \text{ is bi}\Gamma\text{-left ideal of } T.
\end{aligned}$$

(ii) and (iii). Proof is similar as (i).

(iv) $(t) = t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T$. As,

$$\begin{aligned}
T\Gamma T\Gamma(t) &= T\Gamma T\Gamma(t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T) \\
&= T\Gamma T\Gamma t \cup T\Gamma T\Gamma T\Gamma T\Gamma t \cup T\Gamma T\Gamma t\Gamma T\Gamma T \cup T\Gamma T\Gamma T\Gamma t\Gamma T \\
&\quad \cup T\Gamma T\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T \\
&\subseteq T\Gamma T\Gamma t \cup T\Gamma T\Gamma t \cup T\Gamma T\Gamma t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
&= T\Gamma T\Gamma t \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
&\subseteq t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T = (t) \\
T\Gamma T\Gamma(t) &\subseteq (t),
\end{aligned}$$

implies that (t) is biΓ-left ideal. Similarly, we can show that it is biΓ-right ideal. Now consider,

$$\begin{aligned}
T\Gamma(t)\Gamma T &= T\Gamma(t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T)\Gamma T \\
&= T\Gamma t\Gamma T \cup T\Gamma T\Gamma T\Gamma t\Gamma T \cup T\Gamma t\Gamma T\Gamma T\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
&\quad \cup T\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T\Gamma T \\
&\subseteq T\Gamma t\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \\
&= T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T \\
&\subseteq t \cup T\Gamma T\Gamma t \cup t\Gamma T\Gamma T \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T = (t) \\
T\Gamma(t)\Gamma T &\subseteq (t), \text{ implies that } (t) \text{ is bi}\Gamma\text{-lateral ideal.}
\end{aligned}$$

Hence (t) is biΓ-ideal of T. ∎

Remark 3.2.3 The ideals $(t)_l, (t)_m, (t)_r, (t)$ are called principal biΓ-left, biΓ-right, biΓ-lateral and biΓ-ideal of T generated by t. Note that for any $a \in A \subseteq T$, $\underset{a\in A}{\cup}(a)_l = (A)_l$, $\underset{a\in A}{\cup}(a)_m = (A)_m$, $\underset{a\in A}{\cup}(a)_r = (A)_r$ and $\underset{a\in A}{\cup}(a) = (A)$ are biΓ-left ideal, biΓ-right ideal, biΓ-lateral ideal and biΓ-ideal of T generated by A.

Lemma 3.2.4 *Let T be a $B\Gamma TS$. Then*

(i) *The orbitrary intersection of $B\Gamma TSS(s)$ of T is again a $B\Gamma TSS$ of T.*

(ii) *The orbitrary intersection of $B\Gamma LI(s)$ ($B\Gamma RI(s)$, $B\Gamma MI(s)$, $B\Gamma I(s)$) of T is a $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$, $B\Gamma I$) of T.*

Proof. (i) Let $\{A_i, i \in I\}$ be a collection of biΓ-ternary subsemigroups of T, then $A_i\Gamma A_i\Gamma A_i \subseteq A_i$, for all $i \in I$. Also $\underset{i\in I}{\cap} A_i \subseteq A_i$ for all $i \in I$ then,

$$(\underset{i\in I}{\cap} A_i)\Gamma(\underset{i\in I}{\cap} A_i)\Gamma(\underset{i\in I}{\cap} A_i) \subseteq A_i\Gamma A_i\Gamma A_i \subseteq A_i, \text{ for all } i \in I.$$

implies that

$$(\underset{i\in I}{\cap} A_i)\Gamma(\underset{i\in I}{\cap} A_i)\Gamma(\underset{i\in I}{\cap} A_i) \subseteq \underset{i\in I}{\cap} A_{i.}$$

Hence $\underset{i\in I}{\cap} A_i$ is a biΓ-ternary subsemigroup of T.

(ii) Let $\{L_i, i \in I\}$ be a collection of biΓ-left ideals of Tthen $T\Gamma T\Gamma L_i \subseteq L_i$, for all $i \in I$. Also $\underset{i\in I}{\cap} L_i \subseteq L_i$ for all $i \in I$ then,

$$\begin{aligned} T\Gamma T\Gamma(\underset{i\in I}{\cap} L_i) &\subseteq T\Gamma T\Gamma L_i \subseteq L_i, \text{ for all } i \in I. \\ T\Gamma T\Gamma(\underset{i\in I}{\cap} L_i) &\subseteq L_i, \text{ for all } i \in I. \text{ Implies that} \\ T\Gamma T\Gamma(\underset{i\in I}{\cap} L_i) &\subseteq \underset{i\in I}{\cap} L_i. \end{aligned}$$

Hence $\underset{i\in I}{\cap} L_i$ is a biΓ-left ideal of T. Similarly, we can prove for biΓ-right and biΓ-lateral ideal of T. ∎

Definition 3.2.5 A nonempty subset Q of a biΓ-ternary semigroup T is called a biΓ-quasi-ideal of T if

$$Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q$$

and

$$Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q.$$

Definition 3.2.6 A biΓ-bi-ideal B of a biΓ-ternary semigroup T is a biΓ-ternary subsemigroup of T satisfying,

$$B\Gamma T\Gamma B\Gamma T\Gamma B \subseteq B.$$

We will write $B\Gamma QI(s)$ and $B\Gamma BI(s)$ for biΓ-quasi-ideal(s) and biΓ-bi-ideal(s), respectively.

Proposition 3.2.7 *Let T be a $B\Gamma TS$. Then every $B\Gamma QI$ of T is a $B\Gamma TSS$ of T.*

Proof. We suppose that Q is a biΓ-quasi-ideal of T. Since

$$\begin{aligned} Q\Gamma Q\Gamma Q &\subseteq Q\Gamma T\Gamma T, \ Q\Gamma Q\Gamma Q \subseteq T\Gamma Q\Gamma T \text{ and } Q\Gamma Q\Gamma Q \subseteq T\Gamma T\Gamma Q. \\ &\Rightarrow Q\Gamma Q\Gamma Q \subseteq Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q. \\ &\Rightarrow Q\Gamma Q\Gamma Q \subseteq Q, \text{ since } Q \text{ is bi}\Gamma\text{-quasi-ideal.} \end{aligned}$$

Implies that Q is biΓ-ternary subsemigroup of T. ∎

Proposition 3.2.8 *The arbitrary intersection of $B\Gamma QI(s)$ of T is a $B\Gamma QI$ of T.*

Proof. Straightforward. ∎

Remark 3.2.9 Note that a $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$)of T is also $B\Gamma QI$ of T but any $B\Gamma QI$ of T may not be a $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$)of T, so we have following lemma.

Lemma 3.2.10 *Let T be a $B\Gamma TS$. Then every $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$) of T is a $B\Gamma QI$ of T.*

Proof. Let L be a biΓ-left ideal of T, then

$$T\Gamma T\Gamma L \subseteq L,$$

which implies that

$$L\Gamma T\Gamma T \cap T\Gamma L\Gamma T \cap T\Gamma T\Gamma L \subseteq L,$$

also

$$L\Gamma T\Gamma T \cap T\Gamma T\Gamma L\Gamma T\Gamma T \cap T\Gamma T\Gamma L \subseteq L.$$

Hence L is biΓ-quasi-ideal of T. Other cases are similar. ∎

Lemma 3.2.11 *A nonempty subset Q of T is a $B\Gamma QI$ of T if and only if it is an intersection of a $B\Gamma LI$, a $B\Gamma MI$ and a $B\Gamma RI$ of T.*

Proof. Let L, M and R be the biΓ-left, biΓ-lateral and biΓ-right ideals of T. Let $Q = R \cap M \cap L$, then

$$\begin{aligned}
&Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q \\
&= (R \cap M \cap L)\Gamma T\Gamma T \cap T\Gamma(R \cap M \cap L)\Gamma T \cap T\Gamma T\Gamma(R \cap M \cap L) \\
&\subseteq R\Gamma T\Gamma T \cap T\Gamma M\Gamma T \cap T\Gamma T\Gamma L \\
&\subseteq R \cap M \cap L, \text{ since, } L, M \text{ and } R, \text{ are bi}\Gamma\text{-left, bi}\Gamma\text{-lateral, bi}\Gamma - right \text{ ideals.} \\
&= Q.
\end{aligned}$$

Similarly, $Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q \subseteq Q$. Hence Q is a biΓ-quasi-ideal of T.

Conversely, let Q be a biΓ-quasi-ideal of T. For any $q \in Q$, $(q)_l, (q)_m, (q)_r$, be the biΓ-left, biΓ-lateral and biΓ-right ideals of T generated by q, then

$$\begin{aligned}
q &\in (q)_r \cap (q)_m \cap (q)_l \\
\bigcup_{q\in Q}\{q\} &\subseteq \bigcup_{a\in Q}(q)_r \cap \bigcup_{q\in Q}(q)_m \cap \bigcup_{q\in Q}(q)_l \\
Q &\subseteq (Q)_r \cap (Q)_m \cap (Q)_l.
\end{aligned}$$

Since, $(Q)_l = Q \cup Q\Gamma T\Gamma T, (Q)_m = Q \cup T\Gamma Q\Gamma T \cup T\Gamma T\Gamma Q\Gamma T\Gamma T$ and $(Q)_r = Q \cup T\Gamma T\ \Gamma Q$, then

$$\begin{aligned}
&(Q)_r \cap (Q)_m \cap (Q)_l \\
&= (Q \cup Q\Gamma T\Gamma T) \cap (Q \cup T\Gamma Q\Gamma T \cup T\Gamma T\Gamma Q\Gamma T\Gamma T) \cap Q \cup T\Gamma T\Gamma Q \\
&= Q \cup (Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q) \cup (Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q) \\
&\subseteq Q, \text{ Since } Q \text{ is bi}\Gamma\text{-quasi-ideal of } T,
\end{aligned}$$

which implies that $Q = (Q)_r \cap (Q)_m \cap (Q)_l$. Where, $(Q)_r$, $(Q)_m$, and $(Q)_l$ are biΓ-left ideal, biΓ-lateral ideal and a biΓ-right ideal of T. Hence the proof. ∎

Lemma 3.2.12 *Let T be a $B\Gamma TS$ and L_s, M_s, R_s be the smallest $B\Gamma LI$, $B\Gamma MI$, $B\Gamma RI$ of T. The $R_s \cap M_s \cap L_s$ is the smallest $B\Gamma QI$ of T.*

Proof. Straightforward. ∎

Lemma 3.2.13 *Let T be a $B\Gamma TS$. If Q be a $B\Gamma QI$ of T and S be a $B\Gamma TSS$ of T, then $Q \cap S$ is a $B\Gamma QI$ of S.*

Proof. Let Q be the biΓ-quasi-ideal and S be a biΓ-ternary subsemigroup of T. If $Q \cap S \neq \phi$, then as

$$\begin{aligned} & (Q\cap S)\Gamma S\Gamma S \cap S\Gamma(Q\cap S)\Gamma S \cap S\Gamma S\Gamma(Q\cap S) \\ \subseteq\ & S\Gamma S\Gamma S \cap S\Gamma S\Gamma S \cap S\Gamma S\Gamma S, \text{ since } Q\cap S \subseteq S. \\ =\ & S\Gamma S\Gamma S \subseteq S. \end{aligned}$$

Also,

$$\begin{aligned} & (Q\cap S)\Gamma S\Gamma S \cap S\Gamma(Q\cap S)\Gamma S \cap S\Gamma S\Gamma(Q\cap S) \\ \subseteq\ & Q\Gamma S\Gamma S \cap S\Gamma Q\Gamma S \cap S\Gamma S\Gamma Q, \text{ since } Q\cap S \subseteq Q. \\ \subseteq\ & Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q, \text{ since } S \subseteq T. \\ \subseteq\ & Q, \text{ since } Q \text{ is bi}\Gamma\text{-quasi-ideal of } T, \end{aligned}$$

which implies that

$$(Q\cap S)\Gamma S\Gamma S \cap S\Gamma(Q\cap S)\Gamma S \cap S\Gamma S\Gamma(Q\cap S) \subseteq Q\cap S.$$

Similarly,

$$(Q\cap S)\Gamma S\Gamma S \cap S\Gamma S\Gamma(Q\cap S)\Gamma S\Gamma S \cap S\Gamma S\Gamma(Q\cap S) \subseteq Q\cap S.$$

Hence, $Q \cap S$ is a biΓ-quasi-ideal of S. ∎

Proposition 3.2.14 *Let T be a $B\Gamma TS$ and $X, Y(\neq \phi) \subseteq T$, then $X\Gamma T\Gamma Y$ is a $B\Gamma BI$ of T.*

Proof. Let $B = X\Gamma T\Gamma Y$, as

$$\begin{aligned} B\Gamma B\Gamma B &= (X\Gamma T\Gamma Y)\Gamma(X\Gamma T\Gamma Y)\Gamma(X\Gamma T\Gamma Y) \\ &= X\Gamma T\Gamma Y\Gamma X\Gamma T\Gamma Y\Gamma X\Gamma T\Gamma Y \\ &\subseteq X\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma Y \\ &\subseteq X\Gamma T\Gamma Y = B \end{aligned}$$

which implies that $B = X\Gamma T\Gamma Y$ is biΓ-ternary subsemigroup of T. Also

$$\begin{aligned} B\Gamma T\Gamma B\Gamma T\Gamma B &= (X\Gamma T\Gamma Y)\Gamma T\Gamma(X\Gamma T\Gamma Y)\Gamma T\Gamma(X\Gamma T\Gamma Y) \\ &= X\Gamma T\Gamma Y\Gamma T\Gamma X\Gamma T\Gamma Y\Gamma T\Gamma X\Gamma T\Gamma Y \\ &\subseteq X\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma Y \\ &\subseteq X\Gamma T\Gamma Y = B \end{aligned}$$

Hence B is a biΓ-bi-ideal of T. ∎

Theorem 3.2.15 *Let $X, Y, Z(\neq \phi) \subseteq T$, then $X\Gamma Y\Gamma Z$ is a biΓ-bi-ideal of T if any one of X, Y or Z is either a biΓ-left ideal or a biΓ-right ideal or a biΓ-lateral ideal of T.*

Proof. We suppose that Z is biΓ-left ideal of T then $T\Gamma T\Gamma Z \subseteq Z$. Let $B = X\Gamma Y\Gamma Z$ then as,

$$\begin{aligned} B\Gamma B\Gamma B &= (X\Gamma Y\Gamma Z)\Gamma(X\Gamma Y\Gamma Z)\Gamma(X\Gamma Y\Gamma Z) \\ &\subseteq X\Gamma Y\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma Z \\ &\subseteq X\Gamma Y\Gamma T\Gamma T\Gamma Z, \\ &\subseteq X\Gamma Y\Gamma Z = B, \text{ since } T\Gamma T\Gamma Z \subseteq Z. \end{aligned}$$

Implies that, $B = X\Gamma Y\Gamma Z$ is a biΓ-ternary subsemigroup of T.

Also

$$\begin{aligned} B\Gamma T\Gamma B\Gamma T\Gamma B &= (X\Gamma Y\Gamma Z)\Gamma T\Gamma(X\Gamma Y\Gamma Z)\Gamma T\Gamma(X\Gamma Y\Gamma Z) \\ &\subseteq X\Gamma Y\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma Z \\ &\subseteq X\Gamma Y\Gamma T\Gamma T\Gamma Z \\ &\subseteq X\Gamma Y\Gamma Z = B, \text{ since } T\Gamma T\Gamma Z \subseteq Z. \end{aligned}$$

Hence $B = X\Gamma Y\Gamma Z$ is a biΓ-bi-ideal of T. ∎

Lemma 3.2.16 *Let T be a $B\Gamma TS$ then every $B\Gamma QI$ of T is a $B\Gamma BI$ of T.*

Proof. Let Q be a biΓ-quasi-ideal of a biΓ-ternary semigroup T then

$$\begin{aligned} Q\Gamma T\Gamma T \cap T\Gamma Q\Gamma T \cap T\Gamma T\Gamma Q &\subseteq Q \text{ and} \\ Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q &\subseteq Q. \end{aligned}$$

Now, as

$$\begin{aligned} Q\Gamma T\Gamma Q\Gamma T\Gamma Q &\subseteq Q\Gamma T\Gamma T\Gamma T\Gamma T \subseteq Q\Gamma T\Gamma T, \text{ and} \\ Q\Gamma T\Gamma Q\Gamma T\Gamma Q &\subseteq T\Gamma T\Gamma T\Gamma T\Gamma Q \subseteq T\Gamma T\Gamma Q, \text{ also} \\ Q\Gamma T\Gamma Q\Gamma T\Gamma Q &\subseteq T\Gamma T\Gamma Q\Gamma T\Gamma T, \end{aligned}$$

which implies that

$$Q\Gamma T\Gamma Q\Gamma T\Gamma Q \subseteq Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q\Gamma T\Gamma T \cap T\Gamma T\Gamma Q,$$

$$\Rightarrow Q\Gamma T\Gamma Q\Gamma T\Gamma Q \subseteq Q.$$

Hence Q is a biΓ-bi-ideal of T. ∎

Note that the converse of above lemma is not true (see Example 3.3.10).

Corollary 3.2.17 *Let T be a $B\Gamma TS$ then every $B\Gamma LI$ ($B\Gamma RI$, $B\Gamma MI$)of a T is a $B\Gamma BI$ of T.*

Proof. Follows from *Lemma* 3.2.10 and *Lemma* 3.2.16. ∎

Theorem 3.2.18 *Let T be a $B\Gamma TS$ and A be a $B\Gamma I$ and Q be a $B\Gamma QI$ of T then $A \cap Q$ is a $B\Gamma BI$ of T.*

Proof. Since $A \cap Q \subseteq A$ and $A \cap Q \subseteq Q$, where A is a biΓ-ternary subsemigroup of T and Q is a biΓ-quasi-ideal of T then as,

$$\begin{aligned}(A \cap Q)\Gamma(A \cap Q)\Gamma(A \cap Q) &\subseteq A\Gamma A\Gamma A \subseteq A, \text{ and}\\ (A \cap Q)\Gamma(A \cap Q)\Gamma(A \cap Q) &\subseteq Q\Gamma Q\Gamma Q \subseteq Q,\end{aligned}$$

$$\Rightarrow (A \cap Q)\Gamma(A \cap Q)\Gamma(A \cap Q) \subseteq A \cap Q,$$

implies that $A \cap Q$ is a biΓ-ternary subsemigroup of T.

Since Q is biΓ-quasi-ideal and hence a biΓ-bi-ideal then,

$$(A \cap Q)\Gamma T\Gamma(A \cap Q)\Gamma T\Gamma(A \cap Q) \subseteq Q\Gamma T\Gamma Q\Gamma T\Gamma Q \subseteq Q.$$

Also, since A is a biΓ-ideal and hence a biΓ-lateral ideal of T then

$$(A \cap Q)\Gamma T\Gamma(A \cap Q)\Gamma T\Gamma(A \cap Q) \subseteq A\Gamma(T\Gamma A\Gamma T)\Gamma A \subseteq A\Gamma A\Gamma A \subseteq A.$$

This implies that $(A \cap Q)\Gamma T\Gamma(A \cap Q)\Gamma T\Gamma(A \cap Q) \subseteq A \cap Q$. Hence $A \cap Q$ is biΓ-bi-ideal of T. ∎

Lemma 3.2.19 *Let T be a $B\Gamma TS$, then the arbitrary intersection of $B\Gamma BI(s)$ of T is a $B\Gamma BI$ of T.*

Proof. Straightforward. ∎

3.3. Regular biΓ-ternary semigroup

Definition 3.3.1 Let T be a $B\Gamma TS$. An element $a \in T$ is called a biΓ-regular element of T if $a \in a\Gamma T\Gamma a\Gamma T\Gamma a$, i.e. there exists $x, y \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = a\alpha x\beta a\gamma y\delta a$. A $B\Gamma TS$, T is called a regular biΓ-ternary semigroup if its every element is a biΓ-regular element.

Lemma 3.3.2 *Every $B\Gamma MI$ ideal of a regular $B\Gamma TS$ is a regular $B\Gamma TS$.*

Proof. Let T be a regular $B\Gamma TS$ and M be a $B\Gamma MI$ of T then $T\Gamma M\Gamma T \subseteq M$. Let $a \in M$ then $a \in T$ and T is regular, so there exist $x, y \in T$, $\alpha, \beta, \gamma, \delta \in T$, such that

$$\begin{aligned}a &= a\alpha x\beta a\gamma y\delta a\\ &= a\alpha x\beta a\gamma y\delta a\alpha x\beta a\gamma y\delta a = a\alpha(x\beta a\gamma y)\delta a\alpha(x\beta a\gamma y)\delta a\\ &= a\alpha m\delta a\alpha m\delta a, \text{ where } m = x\beta a\gamma y \in T\Gamma M\Gamma T \subseteq M.\\ &\in a\Gamma M\Gamma a\Gamma M\Gamma a,\end{aligned}$$

which implies that a is regular in M. Hence M is regular biΓ-ternary semigroup. ∎

Note that a $B\Gamma LI$ and a $B\Gamma RI$ of a regular $B\Gamma TS$ may not be a regular $B\Gamma TS$.

Corollary 3.3.3 *Every $B\Gamma I$ of a regular $B\Gamma TS$ is a regular $B\Gamma TS$.*

Proof. Straightforward. ∎

Definition 3.3.4 Let T be a $B\Gamma TS$ and I be a $B\Gamma I$ of T. Then I is called an idempotent $B\Gamma I$ of T if $I\Gamma I\Gamma I = I$.

Lemma 3.3.5 *Let T be a regular $B\Gamma TS$. Then every $B\Gamma MI$ of T is an idempotent $B\Gamma I$ of T.*

Proof. let M be a biΓ-lateral ideal of a regular biΓ-ternary semigroup T then $M\Gamma M\Gamma M \subseteq T\Gamma M\Gamma T \subseteq M$. For any $m \in M$, $m \in T$, (Since $M \subseteq T$) and T is regular, $m \in m\Gamma T\Gamma m\Gamma T\Gamma m$ implies that

$$\begin{aligned} m &= m\alpha x\beta m\gamma y\delta m, \text{ for, } x, y \in T \text{ and } \alpha, \beta, \gamma, \delta \in \Gamma. \\ &= m\alpha(x\beta m\gamma y)\delta m \\ &\in M\Gamma M\Gamma M, \text{ implies that} \\ M &\subseteq M\Gamma M\Gamma M. \end{aligned}$$

Hence $M\Gamma M\Gamma M = M$, implies that M is idempotent. ∎

Theorem 3.3.6 *Let T be a $B\Gamma TS$, then the following statements are equivalent,*

(i) *T is regular.*

(ii) *$R\Gamma M\Gamma L = R \cap M \cap L$, where, L, R and M are $B\Gamma LI$, $B\Gamma RI$ and $B\Gamma MI$ of T.*

(iii) *$(a)_r\Gamma(b)_m\Gamma(c)_l = (a)_r \cap (b)_m \cap (c)_l$, for every $a, b, c \in T$.*

(iv) *$(t)_r\Gamma(t)_m\Gamma(t)_l = (t)_r \cap (t)_m \cap (t)_l$, for each $t \in T$.*

Proof. (i) ⇒ (ii) Let T be a regular $B\Gamma TS$ and R, M, L be the biΓ-right, biΓ-lateral and biΓ-left ideals of T then as

$$\begin{aligned} R\Gamma M\Gamma L &\subseteq R\Gamma T\Gamma T \subseteq R \\ R\Gamma M\Gamma L &\subseteq T\Gamma M\Gamma T \subseteq M \text{ and} \\ R\Gamma M\Gamma L &\subseteq T\Gamma T\Gamma L \subseteq L, \text{ implies that} \\ R\Gamma M\Gamma L &\subseteq R \cap M \cap L. \end{aligned}$$

Now let $a \in R\cap M\cap L \subseteq T$ and T is regular then there exist $x, y \in T, \alpha, \beta, \gamma, \delta \in \Gamma$ such that $a = a\alpha x\beta a\gamma y\delta a$. Also

$$\begin{aligned} a &= a\alpha x\beta a\gamma y\delta a = a\alpha(x\beta a\gamma y)\delta a \in R\Gamma M\Gamma L \\ &\Rightarrow R \cap M \cap L \subseteq R\Gamma M\Gamma L. \end{aligned}$$

Hence $R \cap M \cap L = R\Gamma M\Gamma L$.

(ii) $\Rightarrow$ (iii) Let $R \cap M \cap L = R\Gamma M\Gamma L$, for every bi$\Gamma$-right R, biΓ-lateral M and biΓ-left ideal L of T. For $a, b, c \in T$, taking $R = (a)_r$, $M = (b)_m$ and $L = (c)_l$, by (ii), we have $(a)_r\Gamma(b)_m\Gamma(c)_l = R\Gamma M\Gamma L = R \cap M \cap L = (a)_r \cap (b)_m \cap (c)_l$.

(iii) $\Rightarrow$ (iv) Taking $a = b = c = t$, then (iii) becomes $(t)_r\Gamma(t)_m\Gamma(t)_l = (t)_r \cap (t)_m \cap (t)_l$.

(iv) $\Rightarrow$ (i) For any $t \in T$, the biΓ-right ideal, biΓ-lateral ideal and biΓ-left ideal of T generated by t are given as,

$$\begin{aligned}(t)_r &= t \cup t\Gamma T\Gamma T,\\ (t)_m &= t \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T\\ (t)_l &= t \cup T\Gamma T\Gamma t.\end{aligned}$$

By given condition

$$\begin{aligned}(t)_r \cap (t)_m \cap (t)_l &= (t)_r\Gamma(t)_m\Gamma(t)_l\\ &= (t \cup t\Gamma T\Gamma T)\Gamma(t \cup T\Gamma t\Gamma T \cup T\Gamma T\Gamma t\Gamma T\Gamma T)\Gamma t \cup T\Gamma T\Gamma t\\ &= (t\Gamma t\Gamma t) \cup (t\Gamma T\Gamma t\Gamma T\Gamma t) \cup (t\Gamma T\Gamma T\Gamma t\Gamma t) \cup (t\Gamma t\Gamma T\Gamma T\Gamma t)\\ &\quad \cup(t\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T\Gamma t).\end{aligned}$$

Since, $t \in (t)_r \cap (t)_m \cap (t)_l$.

If $t \in t\Gamma t\Gamma t$, then

$$\begin{aligned}t &= t\alpha t\beta t, \text{ for } \alpha, \beta \in \Gamma.\\ &= t\alpha t\beta t\alpha t\beta t \in t\Gamma T\Gamma t\Gamma T\Gamma t, \ t \text{ is regular.}\end{aligned}$$

If $t \in t\Gamma T\Gamma t\Gamma T\Gamma t$, then t is regular.

If $t \in t\Gamma T\Gamma T\Gamma t\Gamma t$, then

$$\begin{aligned}t &= t\alpha x\beta y\gamma t\delta t, \text{ for } x, y \in T, \ \alpha, \beta, \gamma, \delta \in \Gamma.\\ &= t\alpha(x\beta y\gamma t)\delta t\alpha(x\beta y\gamma t)\delta t\\ &\in t\Gamma T\Gamma t\Gamma T\Gamma t, \text{ since, } x\beta y\gamma t \in T, \ \Rightarrow t \text{ is regular.}\end{aligned}$$

If $t \in t\Gamma t\Gamma T\Gamma T\Gamma t$, then

$$\begin{aligned}t &= t\alpha t\beta x\gamma y\delta t, \text{ for } x, y \in T, \ \alpha, \beta, \gamma, \delta \in \Gamma.\\ &= t\alpha(t\beta x\gamma y)\delta t\alpha(t\beta x\gamma y)\delta t\\ &\in t\Gamma T\Gamma t\Gamma T\Gamma t, \text{ since, } t\beta x\gamma y \in T, \ \Rightarrow t \text{ is regular.}\end{aligned}$$

If $t \in t\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T\Gamma t$, then as

$$\begin{aligned}t\Gamma T\Gamma T\Gamma t\Gamma T\Gamma T\Gamma t &\subseteq t\Gamma T\Gamma T\Gamma T\Gamma T\Gamma T\Gamma t, \text{ since, } t \in T.\\ &\subseteq t\Gamma T\Gamma t, \text{ since } T\Gamma T\Gamma T \subseteq T.\\ t &\in t\Gamma T\Gamma t, \text{then}\\ t &= t\alpha x\beta t, \ x \in T, \ \alpha, \beta \in \Gamma.\\ &= t\alpha x\beta t\alpha x\beta t \in t\Gamma T\Gamma t\Gamma T\Gamma t, \ \Rightarrow t \text{ is regular.}\end{aligned}$$

Since $t \in T$ is arbitrary. Hence T is regular biΓ-ternary semigroup. ∎

Theorem 3.3.7 *Let T be a $B\Gamma TS$, then the following statements are equivalent,*

(i) *T is regular*

(ii) *$R\Gamma T\Gamma L = R \cap L$, for every $B\Gamma RI$, R and $B\Gamma LI$, L of T.*

(iii) *$(s)_r\Gamma T\Gamma(t)_l = (s)_r \cap (t)_l$, for every $s, t \in T$.*

(iv) *$(t)_r\Gamma T\Gamma(t)_l = (t)_r \cap (t)_l$, for each $t \in T$.*

Proof. Straightforward. ∎

Theorem 3.3.8 *Let T be a $B\Gamma TS$ then the following statements are equivalent,*

(i) *T is regular.*

(ii) *$B\Gamma T\Gamma B\Gamma T\Gamma B = B$, for every $B\Gamma BI$, B of T.*

(iii) *$Q\Gamma T\Gamma Q\Gamma T\Gamma Q = Q$, for every $B\Gamma QI$, Q of T.*

Proof. (i) ⇒ (ii) Let T be a $B\Gamma TS$ and B be a $B\Gamma BI$ of T then $B\Gamma T\Gamma B\Gamma T\Gamma B \subseteq B$. Now, for $b \in B \subseteq T$, where T is regular, $b \in b\Gamma T\Gamma b\Gamma T\Gamma b \subseteq B\Gamma T\Gamma B\Gamma T\Gamma B$, implies that, $B \subseteq B\Gamma T\Gamma B\Gamma T\Gamma B$. Hence $B\Gamma T\Gamma B\Gamma T\Gamma B = B$.

(ii) ⇒ (iii) We suppose that (ii) holds and Q be a *bi*Γ-*quasi-ideal* of T then by *Lemma* 3.2.16, Q is a *bi*Γ-*bi-ideal* of T and by (ii), $Q\Gamma T\Gamma Q\Gamma T\Gamma Q = Q$, holds.

(iii) ⇒ (i) We suppose that for any *bi*Γ-*quasi-ideal* Q of T, $Q\Gamma T\Gamma Q\Gamma T\Gamma Q = Q$ holds. Let R, M and L be the the biΓ-right, biΓ-lateral and biΓ-left ideals of T respectively. Then $R \cap M \cap L = Q_1$ be a *bi*Γ-*quasi-ideal* of T and by the supposition

$$\begin{aligned} Q_1\Gamma T\Gamma Q_1\Gamma T\Gamma Q_1 &= Q_1 = R \cap M \cap L, \text{ and} \\ Q_1\Gamma T\Gamma Q_1\Gamma T\Gamma Q_1 &\subseteq R\Gamma T\Gamma M\Gamma T\Gamma L \subseteq R\Gamma M\Gamma L, \text{ since } M \text{ is lateral ideal.} \\ \text{This implies that, } R \cap M \cap L &\subseteq R\Gamma M\Gamma L. \end{aligned}$$

Also,

$$\begin{aligned} R\Gamma M\Gamma L &\subseteq R\Gamma T\Gamma T \subseteq R,\ R\Gamma M\Gamma L \subseteq M \text{ and } R\Gamma M\Gamma L \subseteq L, \\ \text{implies that, } R\Gamma M\Gamma L &\subseteq R \cap M \cap L. \end{aligned}$$

Hence $R\Gamma M\Gamma L = R \cap M \cap L$, which implies that by *Theorem* 3.3.6, T is regular. ∎

Lemma 3.3.9 *Let T be a $B\Gamma TS$. Then T is regular if and only if every $B\Gamma I$ of T is an idempotent $B\Gamma I$.*

Proof. Let T be a regular $B\Gamma TS$ and A be a biΓ-ideal of T. Then $A\Gamma A\Gamma A \subseteq A$. Now, let $a \in A \subseteq T$ and T is regular then there exist $x, y \in T$ and $\alpha, \beta, \gamma, \delta \in \Gamma$ such that

$$\begin{aligned} a &= a\alpha(x\beta a\gamma y)\delta a, \\ &\in A\Gamma A\Gamma A, \text{ since, } x\beta a\gamma y \in T\Gamma A\Gamma T \subseteq A. \\ A &\subseteq A\Gamma A\Gamma A. \end{aligned}$$

Hence $A\Gamma A\Gamma A = A$ i.e. A is idempotent.

Conversely, we suppose that every biΓ-ideal of T is idempotent. Let A, B, C be three biΓ-ideals of T then $A \cap B \cap C$ is also a biΓ-ideal of T and hence by supposition

$$(A \cap B \cap C)\Gamma(A \cap B \cap C)\Gamma(A \cap B \cap C) = (A \cap B \cap C).$$

Since, A, B, C are biΓ-ideals of T then $A\Gamma B\Gamma C \subseteq A\Gamma T\Gamma T \subseteq A$, $A\Gamma B\Gamma C \subseteq T\Gamma B\Gamma T \subseteq B$ and $A\Gamma B\Gamma C \subseteq T\Gamma T\Gamma C \subseteq C$, implies that $A\Gamma B\Gamma C \subseteq A \cap B \cap C$.

Also, $A \cap B \cap C \subseteq A, A \cap B \cap C \subseteq B$ and $A \cap B \cap C \subseteq C$, implies that

$$\begin{aligned} (A \cap B \cap C)\Gamma(A \cap B \cap C)\Gamma(A \cap B \cap C) &\subseteq A\Gamma B\Gamma C, \\ \text{implies that, } A \cap B \cap C &\subseteq A\Gamma B\Gamma C. \end{aligned}$$

Hence $A\Gamma B\Gamma C = A \cap B \cap C$ and by *Theorem* 3.3.6, T is regular. ∎

Example 3.3.10 Let T be a $B\Gamma TS$. Then a $B\Gamma BI$ of T may not be a $B\Gamma QI$ of T.

Proof. Let T be a $B\Gamma TS$, which is not regular. Let L_s, M_s and R_s be the smallest *bi*Γ*-left ideal, bi*Γ*-lateral ideal and bi*Γ*-right ideal of T* then by Lemma 3.2.12 and 3.2.16, $R_s\Gamma M_s\Gamma L_s$ is a *bi*Γ*-bi-ideal of T*. We claim that $R_s\Gamma M_s\Gamma L_s$ is not a *bi*Γ*-quasi ideal of T*, otherwise, consider as

$$\begin{aligned} R_s\Gamma M_s\Gamma L_s &\subseteq R_s\Gamma T\Gamma T \subseteq R_s, \text{ since } R_s \text{ is bi}\Gamma\text{-right ideal.} \\ R_s\Gamma M_s\Gamma L_s &\subseteq T\Gamma M_s\Gamma T \subseteq M_s, \text{ since } M_s \text{ is bi}\Gamma\text{-lateral ideal.} \\ R_s\Gamma M_s\Gamma L_s &\subseteq T\Gamma T\Gamma L_s \subseteq L_s, \text{ since } L_s \text{ is bi}\Gamma\text{-left ideal,} \end{aligned}$$

implies that, $R_s\Gamma M_s\Gamma L_s \subseteq R_s \cap M_s \cap L_s$.

Now, if $R_s\Gamma M_s\Gamma L_s$ is a *bi*Γ*-quasi ideal of T* then by Lemma 3.2.12, $R_s \cap M_s \cap L_s$ is the smallest *bi*Γ*-quasi ideal of T*. Which implies that $R_s \cap M_s \cap L_s \subseteq R_s\Gamma M_s\Gamma L_s$. Hence $R_s \cap M_s \cap L_s = R_s\Gamma M_s\Gamma L_s$, where L_s, M_s and R_s be the biΓ-left ideal , biΓ-lateral ideal and biΓ-right ideal of T. But this hold only if T is a regular biΓ-ternary semigroup, which is a contradiction. Hence $R_s\Gamma M_s\Gamma L_s$ is a *bi*Γ*-bi-ideal of T* but not a *bi*Γ*-quasi ideal of T*. ∎

From the above example, we can write the following lemma.

Lemma 3.3.11 *Let T be a regular $B\Gamma TS$. Then every $B\Gamma BI$ of T is a $B\Gamma QI$ of T.*

Proof. Straightforward. ∎

By combining Lemmas 3.2.16 and 3.3.11, we can write the following theorem.

Theorem 3.3.12 *Let T be a regular $B\Gamma TS$. Then a nonempty subset A of T is a $B\Gamma BI$ of T if and only if it is a $B\Gamma QI$ of T.*

Also, in view of Lemmas 3.2.11 and 3.3.11, we can write the following theorem.

Theorem 3.3.13 *Let T be a regular $B\Gamma TS$. Then a $B\Gamma TSS$ of T is a $B\Gamma BI$ of T if and only if it is an intersection of a $B\Gamma LI$, a $B\Gamma MI$ and a $B\Gamma RI$ of T.*

References

[1] Clifford, A.H., Preston, G.B., *The Algebraic Theory of Semigroups*, vol. I. *Mathematical Surveys*, No. 7, American Mathematical Society, Providence, R.I., 1961.

[2] Clifford, A.H., Preston, G.B., *The Algebraic Theory of Semigroups*, vol. II, *Mathematical Surveys*, No. 7, American Mathematical Society, Providence, R.I., 1967.

[3] Chinram, R., Jirojkul, C., *On bi-*Γ*-ideals in* Γ*-semigroups*, Songklanakarin J. Sci. Technol., 29 (2007), 231-234.

[4] Dutta, T.K., Kar, S., Maity, B.K., *On ideals in regular ternary semigroups*, Math. Gen. Algebra Appl., 28 (2008), 147-159.

[5] Dixit, V.N., Dewan, S., *A note on quasi and bi-ideals in ternary semigroups*, Int. J. Math. Math. Sci., 18 (1995), 501-508.

[6] Good, R.A., Hughes, D.R., *Associated groups for a semigroup*, Bull. Amer. Math. Soc., 58 (1952), 624-625.

[7] Iampan, A., *2009. Note on bi-ideals in* Γ*-semigroups*, Int. J. Algebra Comput., 3 (4) (2009), 181-188.

[8] Islam, B., *Relation Q and Bi-Ideals in* Γ*-Semigroups*, International Mathematical Forum, 4 (1) (2009), 33-38.

[9] Jirojkul, C., Sripakorn, R., Chinram, R., *Minimal quasi-ideals in* Γ*-semigroups*, Int. Math. Forum., 4 (1) (2009), 7-11.

[10] Kasner, E. , *An extension of the group concept*, Bull. Amer. Math. Soc., 10 (1904), 290-291.

[11] Lajos, S., *On the Bi-ideals in Semigroups*, Proc. Japan Acad. 45 (1969), 710-712.

[12] Lajos, S., $(m;k;n)$*-ideals in semigroups. Notes on Semigroups* II, Karl Marx Univ. Econ.,Dept. Math. Budapest, No. 1 (1976), 12-19.

[13] Lehmer, D.H., *A ternary analogue of abelian groups*, Amer. J. Math., (1932), 329-338.

[14] Ljapin, E.S., *Semigroups*, Translations of Math. Monographs, vol. 3, Amer. Math. Soc., 1974.

[15] Petrich, M., *Introduction to semigroups*, Merrill, Columbus, Ohio, 1973.

[16] Petrich, M., *Lectures in semigroups*, Wiley, 1977.

[17] Petrich, M., *Inverse semigroups*, Wiley, 1984.

[18] Prufer, H., *Theorie der AbelschenGruppen*, Mathematische Zeitschrift, 20 (1) (1924), 166-187.

[19] Saha, N.K., *On* Γ*-semigroup* II. Bull. Cal. Math. Soc., 79 (1987), 331-335.

[20] Saha, N.K., *On* Γ*-semigroup*. III, Bull. Cal. Math. Soc., 80 (1988), 1-12.

[21] Santiago, M.L., *Some contribution to the study of ternary semigroups and semi-heaps*, Ph.D. Thesis, University of Madras, India, 1983.

[22] Sen. M.K., *On* Γ*-semigroups*, Proc. Int. Conf. Algebra Appl. Decker Publication. New York 301, 1982.

[23] Sen, M.K., Saha, N.K., *On* Γ*-semigroup*. I. Bull. Cal. Math. Soc., 78 (1986), 180-186.

[24] Sioson, F.M., *Ideal theory in ternary semigroups*, Math. Japonica., 10 (1965), 63-84.

[25] Steinfeld, O., *Uber die quasi-ideals von Halbgruppen*, Publ. Math. (Debrecen), 4 (1956), 262-275.

[26] Szasz, G., *Interior ideals in semigroups, Notes on semigroups* IV, Karl Marx Univ. Econ., Dept. Math. Budapest, no. 5 (1977), 1-7.

[27] Szasz, G., *Remark on interior ideals of semigroups*, Studia Scient. Math. Hung., 16 (1981), 61-63.

Accepted: 01.01.2015