COMPUTATION OF TOPOLOGICAL INDICES OF NON-COMMUTING GRAPHS

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Abstract. Let G be a non-abelian finite group. The non-commuting graph Γ_G of G is defined as a graph with vertex set $G - Z(G)$ in which two distinct vertices x and y are joined if and only if $xy \neq yx$. Various topological indices have been defined for simple and connected graphs. Since non-commuting graph is a simple and connected graph, topological indices could be defined for it. The main object of this article is to calculate various indices like Wiener index, Hyper-Wiener index, Schultz index and Gutman index for the non-commuting graph of the group G.

Keywords: non-commuting graph, Wiener index, hyper-Wiener index, Schultz index, Gutman index.

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1. Introduction

Let G be a non-abelian finite group. Various graphs could be attributed to G, one of which is the non-commuting graph denoted by Γ_G. The set of vertices and edges of Γ_G are $V(\Gamma_G)$ and $E(\Gamma_G)$ respectively so that $V(\Gamma_G) = G - Z(G)$ in which $Z(G)$ is the center of G and for every $x, y \in V(\Gamma_G)$ we have:

$$\{x, y\} \in E(\Gamma_G) \Longleftrightarrow xy \neq yx$$

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It is apparent that if G is an abelian group, Γ_G would turn to a null graph. For this, G is assumed to be a non-abelian group. The centralizer of x within G which is denoted by $C_G(x)$ is a subset of G which is defined as $\{g \in G|\ gx = xg\}$.

According to [8] the non-commuting graph of a finite group G was first introduced by Paul Erdos in connection with the following problem: let G be a group whose non-commuting graph Γ_G has no infinite complete subgraphs. Is it true that there is a finite bound on the cardinalities of complete subgraphs of Γ_G? By [8], the answer to this question is positive and this was the origin of many similar questions and research.

Assume that $\Gamma = (V, E)$ is a graph in which V is the set of vertices and E is the set of edges. This graph is assumed to be a finite graph whenever $|V|, |E|$ are finite. We assume this graph is connected, the distance between two x and y the vertex is shown as $d(x, y)$. It is the shortest path between the two vertices x and y. The degree of the vertex x which is shown by $\rho(x)$ equals to the number of edges through x. The largest distance between all pairs of the vertices of G is called the diameter of G.

The Wiener index of the graph G which is shown as $W(G)$ is defined as follows:

$$W(G) = \frac{1}{2}\sum_{x,y\in V} d(x, y) = \frac{1}{2}\sum_{x\in V} d(x).$$

Where $d(x) = \sum_{v\in V} d(x, v)$.

The Wiener index is one of the oldest descriptors concerned with the molecular graph. This index first was proposed by H. Wiener [9] and it is concerned with the determination of the boiling points of Paraffins. In mathematical research, the Wiener index has been first studied in [4]. It is an invariant of the graph, it is invariant under the automorphism group of the graph.

The Hyper-Wiener index of the graph G which is shown as $WW(G)$ is defined as follows:

$$WW(G) = \frac{1}{2}\sum_{\{x,y\}\subset V} (d(x, y) + d^2(x, y)).$$

The Hyper-Wiener index is one of the recently introduced distance-based molecular structure-descriptors. It was put forward in 1993 and since then it has attracted much attention of theoretical chemists. In parallel with the symbol W for the Wiener index the hyper-Wiener index is traditionally denoted by WW [3], [6].

Schultz in 1989 introduced a graph-theoretical descriptor for characterizing alkanes by an integer, namely the Schultz index, defined as

$$S(G) = \sum_{\{x,y\}\subset V} (\rho(x) + \rho(y))d(x, y).$$

The Gutman index (also known as Schultz index of the second kind [2], [3], [5], [7]) of a graph G is defined as

$$Gut(G) = \sum_{\{x,y\}\subset V} \rho(x)\rho(y)d(x, y).$$

Our main goal is to calculate the above mentioned indices for the non-commuting graph of G in terms of the order of $G, Z(G)$ and the number of conjugacy classes of G. The following lemmas will be used repeatedly in calculating process:

Lemma 1.1. *Let G be a non-abelian finite group, then* $\operatorname{diam}(\Gamma_G) = 2$.

Proof. This is Proposition 2.1 in [1]. ∎

Lemma 1.2. *Let G be a non-abelian finite group and k be the number of conjugacy classes of G, then*

$$|E(\Gamma_G)| = \frac{1}{2}|G|(|G| - k(G)).$$

Proof. See [1] Lemma 3.27. ∎

Lemma 1.3. *Let G be a non-abelian finite group. If x is one of the vertices of Γ_G, then*

$$\rho(x) = |G| - |C_G(x)|.$$

Proof. See Lemma 3.1 in [1]. ∎

2. The Wiener index of the non-commuting graph of a group

Before we calculate the Wiener index, we prove the following lemma.

Lemma 2.1. *Let G be a finite group and k be the number of conjugacy classes of G, then*

$$\sum_{x \notin Z(G)} |C_G(x)| = |G|(k - |Z(G)|).$$

Proof. We know that G is the union of its conjugacy classes and assume that $\{x_i\}_{i=1}^{k}$ is the set of the representatives of the conjugacy classes of G. Then we have:

$$G = \bigcup_{i=1}^{k} \text{class}(x_i)$$

Now, let $\{x_i\}_{i=1}^{t}$ be the set of non-central of G class representatives and then we have $k = t + |Z(G)|$. Every x which is not placed within would be placed within one of $Z(G)$ in which $\text{class}(x_i)s$ in which $1 \leq i \leq t$. Therefore we have:

$$\sum_{x \notin Z(G)} |C_G(x)| = \sum_{i=1}^{t} |\text{class}(x_i)|\ |C_G(x_i)| = |G|t = |G|(k - |Z(G)|). \qquad ∎$$

Now, we calculate the Wiener index of the non-commuting graph of a group G. Assume that $x, y \in G - Z(G)$ are two arbitrary distinct vertices of the graph Γ_G. According to the Lemma 1.1 we have $d(x, y) = 1$ or 2.

If $d(x, y) = 1$, then $xy \neq yx$ and $y \in G - C_G(x)$.

If $d(x, y) = 2$, then $xy = yx$ and $x \neq y \in C_G(x) - Z(G)$.

So, we have

$$W(\Gamma_G) = \frac{1}{2} \sum_{x \in G - Z(G)} d(x)$$

for all $x \in G - Z(G)$. Therefore:

$$d(x) = 2 \text{ (The number of vertices whose distance from } x \text{ is 2)}$$
$$+ 1\text{(The number of vertices whose distance from } x \text{ is 1).}$$

Then

$$d(x) = 2(|C_G(x)| - |Z(G)| - 1) + (|G| - |C_G(x)|) = |G| + |C_G(x)| - 2|Z(G)| - 2$$

Now, we can calculate the Wiener index:

$$\begin{aligned} W(\Gamma_G) &= \frac{1}{2} \sum_{x \in G - Z(G)} d(x) \\ &= \frac{1}{2} \left(\sum_{x \in G - Z(G)} (|G| - 2|Z(G)| - 2) + |C_G(x)| \right) \\ &= \frac{1}{2} \left[(|G| - 2|Z(G)| - 2)(|G| - |Z(G)|) + \sum_{x \in G - Z(G)} |C_G(x)| \right] \end{aligned}$$

By Lemma 2.1, we have

$$W(\Gamma_G) = \frac{1}{2}[(|G| - 2|Z(G)| - 2)(|G| - |Z(G)|) + |G|(k - |Z(G)|)]$$

So

$$W(\Gamma_G) = \frac{1}{2}[(|G| - 2|Z(G)|)^2 - |G|(|G| - k) + |G|(|G| - 2) - 2|Z(G)|(|Z(G)| - 1)]$$

or

$$W(\Gamma_G) = \frac{1}{2}[(|G| - 2|Z(G)|)^2 - 2|E(\Gamma_G)| + |G|(|G| - 2) - 2|Z(G)|(|Z(G)| - 1)].$$

Therefore, we have proved the following:

Theorem 2.2. *Let G be a non-abelian finite group and Γ_G be its non-commuting graph. Then*

$$W(\Gamma_G) = \frac{1}{2}[(|G| - 2|Z(G)|)^2 - 2|E(\Gamma_G)| + |G|(|G| - 2) - 2|Z(G)|(|Z(G)| - 1)].$$

3. The Hyper-Wiener index of non-commuting graph of a group

Assume that x, y are two arbitrary vertices of the non-commuting graph of the group G. The Hyper-Wiener index of this graph is defined as follows:

$$WW(G) = \frac{1}{4} \sum_{x,y \in G - Z(G)} (d(x,y) + d^2(x,y)).$$

In order to calculate the Hyper-Wiener index; first we calculate $d^2(x,y)$ for every $x, y \in G - Z(G)$.

Let us set $G - Z(G) = \{x_i\}_{i=1}^{m}$ where $m = |G| - |Z(G)|$. So we have:

$$\sum_{x,y \in G-Z(G)} d^2(x,y) = \sum_{x_i \in G-Z(G)} d^2(x_1, x_i) + \ldots + \sum_{x_i \in G-z(G)} d^2(x_m, x_i)$$

Without loss of generality, we calculate $\sum_{x_i \in G-Z(G)} d^2(x_j, x_i)$ for a constant x_j.

1) If $d^2(x_j, x_i) = 1$, then $x_j x_i \neq x_i x_j$ and $x_i \in G - C_G(x_j)$.

2) If $d^2(x_j, x_i) = 4$, then $x_j x_i = x_i x_j$ and $x_j \neq x_i \in C_G(x_j) - Z(G)$.

Therefore:

$$\begin{aligned}\sum_{x_i \in G-Z(G)} d^2(x_j, x_i) &= 4 \text{ (The number of vertices whose distances from } x \text{ is 2)} \\ &+ \text{(The number of vertices whose distances from } x \text{ is 1)} \\ &= 4(|C_G(x_1)| - |Z(G)| - 1) + (|G| - |C_G(x_j)|) \\ &= |G| + 3|C_G(x_j)| - 4|Z(G)| - 4\end{aligned}$$

But, x_j is an arbitrary vertex. So we can write this formula for all x_i in which $1 \leq i \leq m$. Now we calculate $\sum_{x,y \in G-Z(G)} d^2(x,y)$.

$$\begin{aligned}\sum_{x,y \in G-Z(G)} d^2(x,y) &= \sum_{i=1}^{m} (|G| + 3|C_G(x_i)| - 4|Z(G)| - 4) \\ &= (|G| - 4|Z(G)| - 4)(|G| - |Z(G)|) + 3\sum_{i=1}^{m} |C_G(x_i)| \\ &= (|G| - 4|Z(G)| - 4)(|G| - |Z(G)|) + 3|G|(k - |Z(G)|) \\ &= (|G| - 4|Z(G)|)^2 + |G|(3|G| - 4) \\ &- 6|E(\Gamma_G)| - 4|Z(G)|(3|Z(G)| - 1).\end{aligned}$$

So, the Hyper-Wiener index is as follows:

$$\begin{aligned}WW(\Gamma_G) = \frac{1}{2} W(\Gamma_G) + \frac{1}{4}[&(|G| - 4|Z(G)|)^2 + |G|(3|G| - 4) \\ &-6|E(\Gamma_G)| - 4|Z(G)|(3|Z(G)| - 1)].\end{aligned}$$

By Lemma 2.1, we have:

$$WW(\Gamma_G) = \frac{1}{4}[(|G| - 4|Z(G)|)^2 + (|G| - 2|Z(G)|)^2 - 8|E(\Gamma_G)| + 2|G|(2|G| - 3) - 2|Z(G)|(7|Z(G)| - 3)].$$

Therefore, we have proved the following:

Theorem 3.1. *Let G be a non-abelian finite group and Γ_G be its non-commuting graph. Then*

$$WW(\Gamma_G) = \frac{1}{4}[(|G| - 4|Z(G)|)^2 + (|G| - 2|Z(G)|)^2 - 8|E(\Gamma_G)| + 2|G|(2|G| - 3) - 2|Z(G)|(7|Z(G)| - 3)].$$

4. The Schultz index of the non-commuting graph of a group

The Schultz index of a general graph $\Gamma = (V, E)$ is as follows:

$$S(\Gamma_G) = \sum_{\{x,y\}\subset V} (\rho(x) + \rho(y))d(x, y).$$

Without loss of generality we calculate $\sum_{x_i \in G-Z(G)} (\rho(x_j) + \rho(x_i))d(x_j, x_i)$ for a fixed x_j.

$$\sum_{x_i\in G-Z(G)} (\rho(x_j) + \rho(x_i))d(x_j, x_i) = \sum_{x_i\in G-C_G(x_j)} \rho(x_j) + \rho(x_i) + 2 = \sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} \rho(x_j) + \rho(x_i)$$

Therefore, we have:

$$\sum_{x_i\in G-Z(G)} (\rho(x_j) + \rho(x_i))d(x_j, x_i) = \sum_{x_i\in G-C_G(x_j)} (2|G| - |C_G(x_j)| - |C_G(x_i)|)$$

$$+ 2 \sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} (2|G| - |C_G(x_j)| - |C_G(x_i)|)$$

$$= (2|G| - |C_G(x_j)|)(|G| + |C_G(x_j)| - 2|Z(G)| - 2)$$

$$- \left(\sum_{x_i\in G-C_G(x_j)} |C_G(x_j)| + \sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} |C_G(x_i)| \right) - \sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} |C_G(x_i)|$$

Such that $G - Z(G) = \{x_i\}_{i=1}^m$ by Lemma 2.1, we have:

$$\sum_{x_i \in G-Z(G)} (\rho(x_j)+\rho(x_i))d(x_j,x_i) = (|G|+\rho(x_j))(2|G|-2|Z(G)|-2) + |G|(|Z(G)|+1-|G|) - \rho(x_j)(|G|+1+\rho(x_j)) + 2|E(\Gamma_G)| - \sum_{x_j \neq x_i \in C_G(x_j)-Z(G)} |C_G(x_i)|$$

x_j is arbitrary. So we can write this formula for all x_i in which $1 \leq i \leq m$.

Now, we continue to calculate the Schultz index of the non-commuting graph:

$$\begin{aligned} S(\Gamma_G) &= \frac{1}{2}\left[\sum_{x_i \in G-Z(G)} (\rho(x_1)+\rho(x_i))d(x_1,x_i) + \dots \right. \\ &\left. + \sum_{x_i \in G-Z(G)} (\rho(x_m)+\rho(x_i))d(x_m,x_i)\right] \\ &= \frac{1}{2}[(2|G|-2|Z(G)|-2)(|G|(|G|-|Z(G)|)+2|E(\Gamma_G)|) \\ &+ |G|(|G|-|Z(G)|)(|Z(G)|+1-|G|) + 2|E(\Gamma_G)|(|G|-|Z(G)|) \\ &- (|G|+1)(2|E(\Gamma_G)|) - \sum_{i=1}^{m}(\rho(x_i))^2 \\ &- \left(\sum_{x_1 \neq x_i \in C_G(x_1)-Z(G)} |C_G(x_i)| + \dots + \sum_{x_m \neq x_i \in C_G(x_m)-Z(G)} |C_G(x_i)|\right)] \end{aligned}$$

We can calculate $\sum_{i=1}^{m}(\rho(x_i))^2$ as follows:

$$\begin{aligned} \sum_{i=1}^{m}(\rho(x_i))^2 &= \sum_{i=1}^{m}(|G|-|C_G(x_i)|)^2 \\ &= 4|G|\ |E(\Gamma_G)| - |G|^2(|G|-|Z(G)|) + \sum_{i=1}^{m}|C_G(x_i)|^2 \end{aligned}$$

So, we have:

$$\begin{aligned} S(\Gamma_G) &= \frac{1}{2}[(2|G|-2|Z(G)|-2)(|G|(|G|-|Z(G)|)+2|E(\Gamma_G)|) \\ &+ |G|(|G|-|Z(G)|)(|Z(G)|+1) - 2|E(\Gamma_G)|(2|G| \\ &+ |Z(G)|+1) - \sum_{i=1}^{m}|C_G(x_i)|^2 \\ &- \left(\sum_{x_1 \neq x_i \in C_G(x_1)-Z(G)} |C_G(x_i)| + \dots + \sum_{x_m \neq x_i \in C_G(x_m)-Z(G)} |C_G(x_i)|\right)]. \end{aligned}$$

Therefore, we have the following theorem:

Theorem 4.1. *Let G be a non-abelian finite group and Γ_G be its non-commuting graph. Then*

$$\begin{aligned} S(\Gamma_G) = \frac{1}{2}[&(2|G| - 2|Z(G)| - 2)(|G|(|G| - |Z(G)|) + 2|E(\Gamma_G)|) \\ &+ |G|(|G| - |Z(G)|)(|Z(G)| + 1) - 2|E(\Gamma_G)|(2|G| \\ &+ |Z(G)| + 1) - \sum_{i=1}^{m} |C_G(x_i)|^2 \\ &- \left(\sum_{i=1}^{m} \sum_{x_i \neq x_j \in C_G(x_i)-Z(G)} |C_G(x_j)|\right)]. \end{aligned}$$

Definition 4.2. Let G be a non-abelian group. G is called an AC-group if $C_G(x)$ is abelian for all $x \in G - Z(G)$.

The following characterization of AC-group may be useful in some points.

Theorem 4.3. *Let G be an AC-group, then*

$$\begin{aligned} S(\Gamma_G) = \frac{1}{2}\Bigg[&(2|G| - 2|Z(G)| - 2)|G|(|G| - |Z(G)|) \\ &+ 2|G|(|G| - |Z(G)|)(|Z(G)| + 1) \\ &- 4|E(\Gamma_G)|(|G| + |Z(G)| + 1) - 2\sum_{x \in G-Z(G)} (|G| - \rho(x))^2\Bigg]. \end{aligned}$$

Proof. We have:

$$\begin{aligned} S(\Gamma_G) = \frac{1}{2}\Bigg[&(2|G| - 2|Z(G)| - 2)(|G|(|G| - |Z(G)|) + 2|E(\Gamma_G)|) \\ &+ |G|(|G| - |Z(G)|)(|Z(G)| + 1) - 2|E(\Gamma_G)|(2|G| \\ &+ |Z(G)| + 1) - \sum_{i=1}^{m} |C_G(x_i)|^2 \\ &- \Big(\sum_{i=1}^{m} \sum_{x_i \neq x_j \in C_G(x_i)-Z(G)} |C_G(x_j)|\Big)\Bigg]. \end{aligned}$$

G is an AC-group. So we can calculate $\sum_{x_j \neq x_i \in C_G(x_j)-Z(G)} |C_G(x_i)|$ for all x_i. It is easy to prove that for all $x_j \neq x_i \in C_G(x_j) - Z(G)$ that $|C_G(x_i)| = |C_G(x_j)|$.

So, we have:

$$\sum_{x_1 \neq x_i \in C_G(x_1)-Z(G)} |C_G(x_i)| + \ldots + \sum_{x_m \neq x_i \in C_G(x_m)-Z(G)} |C_G(x_i)|$$
$$= |C_G(x_1)|(|C_G(x_1)| - |Z(G)| - 1) + \ldots$$
$$+ |C_G(x_1)|(|C_G(x_1)| - |Z(G)| - 1)$$
$$= \sum_{i=1}^{m} |C_G(x_i)|^2 - (|Z(G)| + 1) \sum_{i=1}^{m} |C_G(x_i)|$$
$$= \sum_{i=1}^{m} |C_G(x_i)|^2 - (|Z(G)| + 1)|G|(k - |Z(G)|)$$
$$= \sum_{i=1}^{m} |C_G(x_i)|^2 - (|Z(G)| + 1) - 2|E(\Gamma_G)| + |G|(|G| - |Z(G)|).$$

By using this formulation, we calculate the Schultz index of the non-commuting graph of G.

$$\begin{aligned} S(\Gamma_G) = \frac{1}{2}\Big[&(2|G| - 2|Z(G)| - 2)(|G|(|G| - |Z(G)|) + 2|E(\Gamma_G)|) \\ &+ |G|(|G| - |Z(G)|)(|Z(G)| + 1) - 2|E(\Gamma_G)|(2|G| \\ &+ |Z(G)| + 1) - 2\sum_{i=1}^{m} |C_G(x_i)|^2 \\ &+ (|Z(G)| + 1) - 2|E(\Gamma_G)| + |G|(|G| - |Z(G)|)\Big]. \end{aligned}$$

By easy calculation, we have:

$$\begin{aligned} S(\Gamma_G) &= \frac{1}{2}\Big[(2|G| - 2|Z(G)| - 2)|G|(|G| - |Z(G)|) \\ &+2|G|(|G| - |Z(G)|)(|Z(G)| + 1) \\ &-4|E(\Gamma_G)|(|G| + |Z(G)| + 1) - 2\sum_{x \in G-Z(G)} (|G| - \rho(x))^2\Big]. \end{aligned}$$

∎

5. The Gutman index of non-commuting graph of a group

The Gutman index of non-commuting graph is as follows:

$$\mathrm{Gut}(\Gamma_G) = \sum_{\{x,y\} \subset V(\Gamma_G)} \rho(x)\rho(y)d(x, y).$$

We have

$$G - Z(G) = \{x_i\}_{i=1}^{m}.$$

We can write:

$$\mathrm{Gut}(\Gamma_G) = \frac{1}{2}\left[\sum_{i=1}^{m} \rho(x_1)\rho(x_i)d(x_1,x_i) + \ldots + \sum_{i=1}^{m} \rho(x_m)\rho(x_i)d(x_m,x_i)\right]$$

Without loss of generality, we calculate $\sum_{i=1}^{m} \rho(x_j)\rho(x_i)d(x_j,x_i)$ for a fixed x_j.

$$\begin{aligned}\sum_{i=1}^{m} \rho(x_j)\rho(x_i)d(x_j,x_i) &= \sum_{x_i\in G-C_G(x_j)} \rho(x_j)\rho(x_i) + 2\sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} \rho(x_j)\rho(x_i) \\ &= \rho(x_j)\left[\sum_{x_i\in G-C_G(x_j)} \rho(x_i) + 2\sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} \rho(x_i)\right] \\ &= \rho(x_j)\left[\sum_{x_i\in G-C_G(x_j)} (|G|-|C_G(x_i)|)\right. \\ &\left. +2\sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} (|G|-|C_G(x_i)|)\right] \\ &= \rho(x_j)[|G|(|G|-|Z(G)|-1) - \rho(x_j)(|G|+1) + 2|E(\Gamma_G)| \\ &- \sum_{x_j\neq x_i\in C_G(x_j)-Z(G)} |C_G(x_i)|]\end{aligned}$$

We can write this formula for all x_i in which $1 \leq i \leq m$. Now, we calculate the Gutman index:

$$\begin{aligned}&\mathrm{Gut}(\Gamma_G) = \frac{1}{2}\left[\sum_{i=1}^{m} \rho(x_1)\rho(x_i)d(x_1,x_i) + \ldots + \sum_{i=1}^{m} \rho(x_m)\rho(x_i)d(x_m,x_i)\right] \\ &= \frac{1}{2}\left[2|E(\Gamma_G)|\ |G|(|G|-|Z(G)|-1) + (2|E(\Gamma_G)|)^2 - (|G|+1)\sum_{i=1}^{m}(\rho(x_i))^2\right. \\ &\left. - \left(\rho(x_1)\sum_{x_1\neq x_i\in C_G(x_1)-Z(G)} |C_G(x_i)| + \ldots + \rho(x_m)\sum_{x_m\neq x_i\in C_G(x_m)-Z(G)} |C_G(x_i)|\right)\right]\end{aligned}$$

By using the quality of $\sum_{i=1}^{m}(\rho(x_i))^2$ and $\rho(x_j)$, we have:

$$\begin{aligned}\mathrm{Gut}(\Gamma_G) = \frac{1}{2}[&(2|E(\Gamma_G)|)^2 + |G|^2(|G|+1)(|G|-|Z(G)|)\\
&- 2|E(\Gamma_G)|\ |G|(|G|+|Z(G)|+3) - (|G|+1)\sum_{i=1}^{m}|C_G(x_i)|^2\\
&- |G|\left(\sum_{x_1\neq x_i\in C_G(x_1)-Z(G)}|C_G(x_i)| + \ldots + \sum_{x_m\neq x_i\in C_G(x_m)-Z(G)}|C_G(x_i)|\right)\\
&+ |C_G(x_1)|\sum_{x_1\neq x_i\in C_G(x_1)-Z(G)}|C_G(x_i)| + \ldots\\
&+ |C_G(x_m)|\sum_{x_m\neq x_i\in C_G(x_m)-Z(G)}|C_G(x_i)|].\end{aligned}$$

Theorem 5.1. *Let G ba a non-abelian finite group and Γ_G be its non-commuting graph. Then*

$$\begin{aligned}\mathrm{Gut}(\Gamma_G) = \frac{1}{2}\Bigg[&(2|E(\Gamma_G)|)^2 + |G|^2(|G|+1)(|G|-|Z(G)|)\\
&- 2|E(\Gamma_G)|\ |G|(|G|+|Z(G)|+3) - (|G|+1)\sum_{i=1}^{m}|C_G(x_i)|^2\\
&- |G|\Big(\sum_{x_1\neq x_i\in C_G(x_1)-Z(G)}|C_G(x_i)| + \ldots + \sum_{x_m\neq x_i\in C_G(x_m)-Z(G)}|C_G(x_i)|\Big)\\
&+ \sum_{i=1}^{m}|C_G(x_i)|\sum_{x_i\neq x_j\in C_G(x_i)-Z(G)}|C_G(x_j)|\Bigg].\end{aligned}$$

Theorem 5.2. *Let G be an AC-group, then*

$$\begin{aligned}\mathrm{Gut}(\Gamma_G) = \frac{1}{2}\Big[&(2|E(\Gamma_G)|)^2 + |G|^2(|G|+1)(|G|-|Z(G)|)\\
&- 2|E(\Gamma_G)|\ |G|(|G|+|Z(G)|+3)\\
&+ |G|(|Z(G)|+1)(|G|(|G|-|Z(G)|) - 2|E(\Gamma_G)|) + \sum_{i=1}^{m}|C_G(x_i)|^3\\
&- (2|G|+|Z(G)|+2)\sum_{i=1}^{m}|C_G(x_i)|^2\Big].\end{aligned}$$

Proof. Using Theorem 5.1 and Definition 4.2 the result follows easily. ∎

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