FUZZY IDEALS OF IMPLICATION GROUPOIDS

Ravi Kumar Bandaru

Department of Engineering Mathematics
GITAM University
Hyderabad Campus, Hyderabad, 502329
India
e-mail: ravimaths83@gmail.com

K.P. Shum

Institute of Mathematics
Yunnan University
Kunming-650091
China
e-mail: kpshum@ynu.edu.cn

N. Rafi

Department of Mathematics
Bapatla Engg. College
Bapatla, Andhra Pradesh, 522 101
India
e-mail: rafimaths@gmail.com

Abstract. In this paper, we introduce the concept of fuzzy ideals in implication groupoids and investigate its properties.

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1. Introduction

In 50-ties, L. Henkin and T. Skolem introduced the notion of Hilbert algebra as an algebraic counterpart of intuitionistic logic. The structure of Hilbert algebras has been later studied by by D. Busneag [2] and Y.B. Jun [13]. It is well known that the filters of a Hilbert algebra forms a deductive system. Since there exist various modifications of the Hilbert algebra, we now cite the one given in [2]. Recall that a Hilbert algebra is an algebra $\mathcal{H} = (H, *, 1)$ of type $(2, 0)$ satisfying the following axioms.

$(H1)$ $x * (y * x) = 1$.
$(H2)$ $(x * (y * z)) * ((x * y) * (x * z)) = 1$.
$(H3)$ $x * y = 1$ and $y * x = 1$ imply $x = y$.

In [6], I. Chajda and R. Halas further studied the properties of ideals and congruences of Hilbert algebras. Later, I. Chajda and R. Halas [7] introduced the concept of implication groupoid as a generalization of the implication reduct of intuitionistic logic, i.e. a Hilbert algebra and studied some connections among ideals, deductive systems and congruence kernels whenever the implication groupoid is distributive. In [10], [11], [13], [12], W.A. Dudek, Y.B. Jun et al studied the concept of fuzzy ideal, fuzzy deductive systems in Hilbert algebras and discuss the relation between the fuzzy ideals and fuzzy deductive systems.

In this paper, we give a characterization theorem of fuzzy ideals of a distributive implication groupoid. We also consider to characterize the fuzzy ideals of a distributive implication groupoid in terms of their level ideals. Our results strengthen and enrich many known results in the literature concerning fuzzy ideals and fuzzy filters of implicative semigroups, for example, see[14], [9], [16], [15]. It is noted that some results given in this paper are extended results of implicative fuzzy ideals of a distributive implication groupoid recently given by Bandaru and Shum in [3].

2. Preliminaries

We first recall some definitions and basic results which were discussed in [9], [7], [4] for the development of the paper.

Definition 2.1. An algebra $(A, *, 1)$ of type (2,0) is called an Implication groupoid if it satisfies the following identities:
(1) $x * x = 1$
(2) $1 * x = x$ for all $x, y \in A$.

Example 2.2. Let $A = \{1, a, b\}$ in which $*$ is defined by

*	1	a	b
1	1	a	b
a	a	1	b
b	a	b	1

Then $(A, *, 1)$ is an implication groupoid.

Example 2.3. Let $A = \{1, a, b, c\}$ in which $*$ is defined by

*	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	a	b	1

Then $(A, *, 1)$ is an implication groupoid.

Definition 2.4. An Implication groupoid $(A, *, 1)$ of type (2,0) is called a distributive implication groupoid if it satisfies the following identity:

(LD) $$x * (y * z) = (x * y) * (x * z) \quad \text{(left distributivity)}$$

for all $x, y, z \in A$.

Example 2.5. Let $A = \{1, a, b, c, d\}$ in which $*$ is defined by

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	b	1
b	1	a	1	1	d
c	1	a	1	1	d
d	1	1	c	c	1

Then $(A, *, 1)$ is a distributive implication groupoid.

In every implication groupoid, one can introduce the so called induced relation $\leq$ by the setting

$$x \leq y \text{ if and only if } x * y = 1.$$

Lemma 2.6. *Let $(A, *, 1)$ be a distributive implication groupoid. Then A satisfies the identities*

$$x * 1 = 1 \textit{ and } x * (y * x) = 1$$

Moreover, the induced relation $\leq$ is a quasiorder on A and the following relationships are satisfied:
(i) $x \leq 1$
(ii) $x \leq y * x$
(iii) $x * ((x * y) * y) = 1$
(iv) $1 \leq x$ *implies* $x = 1$
(v) $y * z \leq (x * y) * (x * z)$
(vi) $x \leq y$ *implies* $y * z \leq x * z$
(vii) $x * (y * z) \leq y * (x * z)$
(viii) $x * y \leq (y * z) * (x * z)$

Definition 2.7. Let $\mathcal{A} = (A, *, 1)$ be an implication groupoid. A subset $I \subseteq A$ is called an ideal of $\mathcal{A}$ if
(I1) $1 \in I$
(I2) $x \in A, y \in I$ imply $x * y \in I$.
(I3) $x \in A, y_1, y_2 \in I$ imply $(y_2 * (y_1 * x)) * x \in I$

Remark 2.8. If I is an ideal of an implication groupoid $\mathcal{A} = (A, *, 1)$ and $a \in I$, $x \in A$, then $(a * x) * x \in I$.

Definition 2.9. Let $\mathcal{A} = (A, *, 1)$ be an implication groupoid. A subset $D \subseteq A$ is called a deductive system of $\mathcal{A}$ if
(D1) $1 \in D$
(D2) $x \in D$ and $x * y \in D$ imply $y \in D$.

Lemma 2.10. *Let $\mathcal{A}$ be an implication groupoid. Then every ideal of $\mathcal{A}$ is a deductive system of $\mathcal{A}$.*

It is noted that the converse of the above lemma does not hold in general.

Example 2.11. From Example 2.2, we can easily see that $\{1, a\}$ is its deductive system which is not an ideal since $b * a = b \notin \{1, a\}$.

Theorem 2.12. *A nonempty subset I of a distributive implication groupoid $\mathcal{A}$ is an ideal if and only if it is a deductive system of $\mathcal{A}$.*

Definition 2.13. Let X be a set. A fuzzy set in X is a function $\mu : X \longrightarrow [0, 1]$.

Definition 2.14. Let μ be a fuzzy set in a set X. For $\alpha \in [0, 1]$, the set $\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$ is called a level subset of μ.

Definition 2.15. If μ is a fuzzy relation on a set X and ν is a fuzzy set in X, then μ is called a fuzzy relation on ν if

$$\mu(x, y) \leq \min\{\nu(x), \nu(y)\} \text{ for all } x, y \in X.$$

Definition 2.16. The Cartesian product of two fuzzy sets μ and ν in X is defined by

$$(\mu \times \nu)(x, y) = \min\{\nu(x), \nu(y)\} \text{ for all } x, y \in X.$$

Lemma 2.17. *Let μ and ν be fuzzy sets in a set X. Then*
(i) *$\mu \times \nu$ is a fuzzy relation on X.*
(ii) *$(\mu \times \nu)_\alpha = \mu_\alpha \times \nu_\alpha$ for all $\alpha \in [0, 1]$.*

Definition 2.18. Let ν be a fuzzy set in a set X. The strongest fuzzy relation on X is a fuzzy relation μ_ν defined by μ_ν defined by

$$\mu_\nu(x, y) = \min\{\nu(x), \nu(y)\} \text{ for all } x, y \in X.$$

Lemma 2.19. *For a given fuzzy set ν in a set X, let μ_ν be the strongest fuzzy relation on X. Then for $\alpha \in [0, 1]$, we have $(\mu_\nu)_\alpha = \nu_\alpha \times \nu_\alpha$.*

3. Fuzzy ideals

In this section we introduce the concept of fuzzy ideal in a distributive implication groupoid and study their properties.

In what follows, X is a distributive implication groupoid unless otherwise specified.

Definition 3.1. A fuzzy set μ in X is called a fuzzy ideal of X if it satisfies the following conditions:
(i) $\mu(1) \geq \mu(x)$
(ii) $\mu(y) \geq \min\{\mu(x), \mu(x * y)\}$, for all $x, y \in X$.

Example 3.2. Let $A = \{1, a, b, c, d\}$ in which $*$ is defined by

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	1	1	1	d
c	1	1	1	1	d
d	1	a	b	c	1

Then $(A, *, 1)$ is a distributive implication groupoid. Let $t_1, t_2 \in [0, 1]$ be such that $t_1 > t_2$. Define a mapping $\mu : X \longrightarrow [0, 1]$ by $\mu(1) = \mu(d) = t_1$ and $\mu(a) = \mu(b) = \mu(c) = t_2$. Then μ is a fuzzy ideal of X.

We now give a characterization theorem of fuzzy ideals of a distributive implication groupoid.

Theorem 3.3. *Let μ be a fuzzy set in a distributive implication groupoid X. Then μ is a fuzzy ideal of X if and only if for every $\alpha \in [0, 1]$, the level subset μ_α is an ideal of X, when $\mu_\alpha \neq \emptyset$.*

Proof. Let μ be a fuzzy ideal of X. Then $\mu(1) \geq \mu(x)$ for all $x \in X$.

In particular, $\mu(1) \geq \mu(x) \geq \alpha$ for every $x \in \mu_\alpha$. Hence $1 \in \mu_\alpha$.

Let $x, x * y \in \mu_\alpha$. Then $\mu(x) \geq \alpha$ and $\mu(x * y) \geq \alpha$ and hence $\mu(y) \geq \min\{\mu(x), \mu(x * y)\} \geq \alpha$. Therefore $y \in \mu_\alpha$. Hence μ_α is an ideal of X.

Conversely, assume that μ_α is an ideal of X for every $\alpha \in [0, 1]$ with $\mu_\alpha \neq \emptyset$. Let $x, y \in X$ and $\mu(x * y) = \alpha_1$ and $\mu(x) = \alpha_2$. Then $x * y \in \mu_{\alpha_1}$ and $x \in \mu_{\alpha_2}$. Without loss of generality, we may assume that $\alpha_1 \leq \alpha_2$. Then $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$ and so $x \in \mu_{\alpha_1}$. Since μ_{α_1} is a ideal of X, we have $y \in \mu_{\alpha_1}$. Hence, $\mu(y) \geq \alpha_1 = \min\{\mu(x * y), \mu(x)\}$.

Suppose $\mu(1) < \mu(x_0)$ for some $x_0 \in X$. Let $\alpha_0 = \frac{1}{2}(\mu(1) + \mu(x_0))$. Then $\mu(1) < \alpha_0$ and $0 \leq \alpha_0 < \mu(x_0) \leq 1$. Hence $x_0 \in \mu_{\alpha_0}$ and $\mu_{\alpha_0} \neq \emptyset$. Since μ_{α_0} is a ideal of X, we have $1 \in \mu_{\alpha_0}$ and so $\mu(1) \geq \alpha_0$. This is a contradiction and hence $\mu(1) \geq \mu(x)$ for all $x \in X$. Therefore, μ is a fuzzy ideal of X. ■

Definition 3.4. Let μ be a fuzzy ideal of X. Then for each $\alpha \in [0, 1]$,the ideal μ_α of X, $\alpha \in [0, 1]$, is called a level ideals of μ, when $\mu_\alpha \neq \emptyset$.

Now, we give a crucial lemma concerning the level ideals of a distributive implication groupoid.

Lemma 3.5. *Any ideal of a distributive implication groupoid X can be realized as a level ideal of some fuzzy ideal of X.*

Proof. Let A be an ideal of X and $\mu : X \longrightarrow [0, 1]$ be a fuzzy set defined by

$$\mu(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

where α is a fixed number in $(0, 1)$. Note that $1 \in A$, so that $\mu(1) = \alpha \geq \mu(x)$ for all $x \in A$. Let $x, y \in X$. Now, we verify condition (ii) of Definition 3.1.

If $x \in A$ and $x * y \in A$ then $y \in A$ and whence $\mu(y) = \mu(x) = \mu(x * y) = \alpha$. Hence, we have

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\}.$$

If $x \notin A$ and $x * y \notin A$ then $\mu(x) = \mu(x * y) = 0$. This shows that

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\}.$$

If exactly one of x and $x * y \in A$ then exactly one of $\mu(x)$ and $\mu(x * y)$ is equal to 0. Hence,

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\}$$

Therefore,

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\} \text{ for all } x, y \in X.$$

This proves that μ is fuzzy ideal of X and $\mu_\alpha = A$. ∎

In the following theorems, we consider the level ideals of a distributive implication groupoid X.

Theorem 3.6. *Let μ be a fuzzy ideal of a distributive implication groupoid X. Then two level ideals $\mu_{\alpha_1}, \mu_{\alpha_2}$ (with $\alpha_1 < \alpha_2$) of μ are equal if and only if there is no $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$.*

Proof. Assume that $\mu_{\alpha_1} = \mu_{\alpha_2}$ for $\alpha_1 < \alpha_2$. If there exists $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$ then μ_{α_2} is a proper subset of μ_{α_1}. This is impossible.

Conversely, suppose that there is no $x \in X$ such that $\alpha_1 \leq \mu(x) < \alpha_2$. Note that $\alpha_1 < \alpha_2$ implies $\mu_{\alpha_2} \subseteq \mu_{\alpha_1}$. If $x \in \mu_{\alpha_1}$, then $\mu(x) \geq \alpha_1$ and so $\mu(x) \geq \alpha_2$ because $\mu(x) \not< \alpha_2$. Hence $x \in \mu_{\alpha_2}$ which says that $\mu_{\alpha_1} \subseteq \mu_{\alpha_2}$. Thus $\mu_{\alpha_1} = \mu_{\alpha_2}$. This completes the proof. ∎

Let μ be a fuzzy set in X and denote the image of μ by $Im(\mu)$.

Theorem 3.7. *Let μ be a fuzzy ideal of a distributive implication groupoid X. If $Im(\mu) = \{\alpha_1, \alpha_2, ..., \alpha_n\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_n$, then the family of ideals $\mu_{\alpha_i}(i = 1, 2, ..., n)$ constitutes all the level ideals of μ.*

Proof. Let $\alpha \in [0, 1]$ and $\alpha \notin Im(\mu)$. If $\alpha < \alpha_1$, then $\mu_{\alpha_1} \subseteq \mu_\alpha$. Since $\mu_{\alpha_1} = X$, we have $\mu_\alpha = X$ and $\mu_\alpha = \mu_{\alpha_1}$. If $\alpha_i < \alpha < \alpha_{i+1} (1 \leq i \leq n-1)$, then there is no $x \in X$ such that $\alpha \leq \mu(x) < \alpha_{i+1}$. Using Theorem 3.7, we obtain $\mu_\alpha = \mu_{\alpha_i+1}$. This shows that for any $\alpha \in [0, 1]$ with $\alpha \leq \mu(1)$, the level ideals μ_α is in $\{\mu_{\alpha_i} \mid 1 \leq i \leq n\}$. ∎

The following lemma is obvious and we omit the proof.

Lemma 3.8. *Let X be a distributive implication groupoid and μ a fuzzy ideal of X. If α and β belong to $Im(\mu)$ such that $\mu_\alpha = \mu_\beta$ then $\alpha = \beta$.*

Theorem 3.9. *Let μ and ν be two fuzzy ideals of a distributive implication groupoid X such that μ and ν have the finite images and have the identical family of level ideals. If $Im(\mu) = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ and $Im(\nu) = \{\beta_1, \beta_2, ..., \beta_n\}$ where $\alpha_1 > \alpha_2 > \cdots > \alpha_m$ and $\beta_1 > \beta_2 > \cdots > \beta_n$ then*

(i) $m = n$,

(ii) $\mu_{\alpha_i} = \nu_{\beta_i}$ *for* $i = 1, 2, ..., m$,

(iii) *if* $x \in X$ *such that* $\mu(x) = \alpha_i$ *then* $\nu(x) = \beta_i$ *for* $i = 1, 2, ..., m$.

Proof. (i) By Theorem 3.7, we can say that the only level ideals of μ and ν are μ_{α_i} and ν_{β_i} respectively. Since μ and ν have the identical family of level ideals, it follows that $m = n$ and so (i) holds.

(ii) Again, by Theorem 3.7, we get that

$$\{\mu_{\alpha_1}, \mu_{\alpha_2}, ..., \mu_{\alpha_m}\} = \{\nu_{\beta_1}, \nu_{\beta_2}, ..., \nu_{beta_m}\},$$

and, by Theorem 3.6, we have

$$\mu_{\alpha_1} \subset \mu_{\alpha_2} \subset \cdots \subset \mu_{\alpha_m} = A \text{ and } \nu_{\beta_1} \subset \nu_{\beta_2} \subset \cdots \subset \nu_{\beta_m} = A.$$

Hence $\mu_{\alpha_i} = \beta_i$ for $i = 1, 2, ..., m$ and (ii) holds.

(iii) Let $x \in A$ be such that $\mu(x) = \alpha_i$ and let $\nu(x) = \beta_j$. Then $x \in \mu_{\alpha_i} = \nu_{\beta_i}$ and so $\nu(x) \geq \beta_i$. Hence $\beta_j \geq \beta_i$ which implies $\nu_{\beta_j} \subseteq \nu_{\beta_i}$. Since $x \in \nu_{\beta_j} = \mu_{\alpha_j}$, therefore $\alpha_i = \mu(x) \geq \alpha_j$. It follows that $\mu_{\alpha_i} \subseteq \mu_{\alpha_j}$. By (ii), $\nu_{\beta_i} = \mu_{\alpha_i}, \mu_{\alpha_j} = \nu_{\beta_j}$. Consequently $\nu_{\beta_i} = \nu_{\beta_j}$ and by Lemma 3.8 we have $\beta_i = \beta_j$. Thus $\nu(x) = \beta_i$. ∎

The following theorem can be proved easily.

Theorem 3.10. *Let μ and ν be as in Theorem* 3.9. *Then $\mu = \nu$ if and only if $Im(\mu) = Im(\nu)$.*

Theorem 3.11. *Let X be a distributive implication groupoid and let μ be a fuzzy set in X with $Im(\mu) = \{\alpha_0, \alpha_1, ..., \alpha_k\}$ where $\alpha_0 > \alpha_1 > \cdots > \alpha_k$. Suppose that there exists a chain of ideals of $X : A_0 \subset A_1 \subset \cdots \subset A_k = A$ such that $\mu(\overline{A_n}) = \alpha_n$ where $\overline{A_n} = A_n - A_{n-1}, A_{-1} = \emptyset$ for $n = 0, 1, ..., k$. Then μ is a fuzzy ideal of X.*

Proof. Since $1 \in A_0$, we have $\mu(1) = \alpha_0 \geq \mu(x)$ for all $x \in A$. In order to prove that μ satisfies the condition (ii) of Definition 3.1, we divide into the following cases:

If x and y belong to the same $\overline{A_n}$, then $\mu(x) = \mu(y) = \alpha_n$ and so

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\}$$

Assume that $x \in \overline{A_i}$ and $y \in \overline{A_j}$ for every $i \neq j$. Without loss of generality, we may assume that $i < j$. Then $\mu(x) = \alpha_i > \alpha_j = \mu(y)$ and so

$$\min\{\mu(y), \mu(y * x)\} \leq \mu(y) < \mu(x).$$

Since $x \in \overline{A_i}$, we have $x \in A_i$. It follows that $x \in A_{j-1}$ as $i \leq j-1$. Now, we assert that $x * y \notin D_{j-1}$. In fact, if not, then $x * y \in A_{j-1}$ and $x \in A_{j-1}$ imply $y \in A_{j-1}$, which contradicts to $y \in \overline{A_j} = A_j - A_{j-1}$. Hence $\mu(x * y) \leq \alpha_j$ and so

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\}.$$

Summarizing the above results, we obtain that $\mu(y) \geq \min\{\mu(x), \mu(x * y)\}$ for all $x, y \in X$. Therefore, μ is a fuzzy ideal of X. ∎

Theorem 3.12. *Let μ be a fuzzy ideal of a distributive implication groupoid X. If $Im(\mu) = \{\alpha_0, \alpha_1, ..., \alpha_k\}$ with $\alpha_0 > \alpha_1 > \cdots > \alpha_k$, then $A_n = \mu_{\alpha_n}, n = 0, 1, ..., k$ are ideals of X and $\mu(\overline{A_n}) = \alpha_n, n = 0, 1, 2, ..., k$ where $\overline{A_n} = A_n - A_{n-1}$ and $A_{-1} = \emptyset$.*

Proof. By Theorem 3.7, $A_n = \mu_{\alpha_n}(n = 0, 1, ..., k)$ is an ideal of X. Clearly, $\mu(A_0) = \alpha_0$. Since $\mu(A_1) = \{\alpha_0, \alpha_1\}$ for $x \in \overline{A_1}$ we have $\mu(x) = \alpha_1$, namely $\mu(\overline{A_1}) = \alpha_1$. Repeating the above argument, we have $\mu(\overline{A_n}) = \alpha_n (0 \leq n \leq k)$. ∎

Theorem 3.13. *If μ is a fuzzy ideal of a distributive implication groupoid X, then the set $X_\mu = \{x \in X \mid \mu(x) = \mu(1)\}$ is an ideal of X.*

Proof. Clearly, $1 \in X_\mu$. Assume that $x \in X_\mu$ and $x * y \in X_\mu$. Then

$$\mu(x) = \mu(1) = \mu(x * y).$$

Since μ is a fuzzy ideal of X, we have

$$\mu(y) \geq \min\{\mu(x), \mu(x * y)\} = \mu(1).$$

Therefore, $\mu(y) = \mu(1)$. Hence $y \in X_\mu$. ∎

Using a given fuzzy ideal, we construct a new fuzzy ideal.

Let $\alpha \geq 0$ be a real number. If $m \in [0, 1], m^\alpha$ shall mean the positive root in case $\alpha < 1$. We define $\mu^\alpha : X \to [0, 1]$ by $\mu^\alpha(x) = (\mu(x))^\alpha$.

Finally, we conclude this section with the following theorem.

Theorem 3.14. *If μ is a fuzzy ideal of a distributive implication groupoid X, then μ^α is also a fuzzy ideal of X and $X_{\mu^\alpha} = X_\mu$.*

Proof. We have that $\mu^\alpha(1) = (\mu(1))^\alpha \geq (\mu(x))^\alpha = \mu^\alpha(x)$ for all $x \in X$. Let $x, y \in X$. We assert that $\mu^\alpha(y) \geq \min\{\mu^\alpha(x), \mu^\alpha(x * y)\}$. In fact, suppose that $\mu(x) \leq \mu(x * y)$. It follows from Definition 3.1(ii) that

$$\mu(y) \geq \mu(x).$$

Hence $\mu^\alpha(x) \leq \mu^\alpha(x * y)$ and $\mu^\alpha \leq \mu^\alpha(y)$ which imply that

$$\mu^\alpha(y) \geq \min\{\mu^\alpha(x), \mu^\alpha(x * y)\}.$$

The argument is similar if $\mu(x) \geq \mu(x * y)$. Finally,

$$\begin{aligned} X_{\mu^\alpha} &= \{x \in X \mid \mu^\alpha(x) = \mu^\alpha(1)\} \\ &= \{x \in X \mid (\mu(x))^\alpha = (\mu(1))^\alpha\} \\ &= \{x \in X \mid \mu(x) = \mu(1)\} \\ &= X_\mu \end{aligned}$$

■

4. Cartesian product of fuzzy ideals

Let $(X, *, 1)$ and $(Y, *, 1)$ be distributive implication groupoids. Define an operation $\rightarrow$ on $X \times Y$ by

$$(x, y) \rightarrow (s, t) = (x * s, y * t) \text{ for all } (x, y), (s, t) \in X \times Y.$$

Then we can easily verify that $(X \times Y, \rightarrow, (1, 1))$ is a distributive implication groupoid.

The following proposition can be proved easily.

Proposition 4.1. *Let A_1 and A_2 be ideals of distributive implication groupoids X and Y respectively. Then $A_1 \times A_2$ is a ideal of $X \times Y$.*

Proposition 4.2. *For a given fuzzy set ν in a distributive implication groupoid X, let μ_ν be the strongest fuzzy relation on X. If μ_ν is a fuzzy ideal of $X \times X$ then $\nu(x) \leq \nu(1)$ for all $x \in X$.*

Proof. Since μ_ν is a fuzzy ideal of $X \times X$, we have

$$\mu_\nu(x, y) \leq \mu_\nu(1, 1) \text{ for all } (x, y) \in X \times X.$$

Hence $\min\{\nu(x), \nu(y)\} \leq \min\{\nu(1), \nu(1)\}$ which implies that $\nu(x) \leq \nu(1)$ for all $x \in X$. ■

The following proposition follows from Lemma 2.19 and we omit the proof.

Proposition 4.3. *If ν is a fuzzy ideal of a distributive implication groupoid X then the level ideals of μ_ν are given by $(\mu_\nu)_\alpha = \nu_\alpha \times \nu_\alpha$ for all $\alpha \in [0, 1]$.*

Theorem 4.4. *Let μ and ν be fuzzy ideals of a distributive implication groupoid X. Then $\mu \times \nu$ is a fuzzy ideal of $X \times X$.*

Proof. For any $(x, y) \in X \times X$, we have

$$(\mu \times \nu)(1, 1) = \min\{\mu(1), \nu(1)\} \geq min\{\mu(x), \nu(y)\} = (\mu \times \nu)(x, y).$$

Now, let $(x, y), (r, s) \in X \times X$. Then

$$\begin{aligned}
&\min\{(\mu\times\nu)(x,y),(\mu,\nu)((x,y)\to(r,s))\}\\
&\qquad=\min\{(\mu\times\nu)(x,y),(\mu,\nu)((x*r,y*s))\}\\
&\qquad=\min\{\min\{\mu(x),\nu(y)\},min\{\mu(x*r),\nu(y*s)\}\}\\
&\qquad=\min\{\min\{\mu(x),\mu(x*r)\},min\{\nu(y),\nu(y*s)\}\}\\
&\qquad\le\min\{\mu(r),\nu(s)\}\\
&\qquad=(\mu\times\nu)(r,s).
\end{aligned}$$

■

Theorem 4.5. *Let μ and ν be fuzzy sets in a distributive implication groupoid X such that $\mu\times\nu$ is a fuzzy ideal of $X\times X$. Then*
(i) *either $\mu(1)\ge\mu(x)$ or $\nu(1)\ge\nu(x)$ for all $x\in X$.*
(ii) *if $\mu(1)\ge\mu(x)$ for all $x\in X$ then $\nu(1)\ge\mu(x)$ or $\nu(1)\ge\nu(x)$ for all $x\in X$.*
(iii) *if $\nu(1)\ge\nu(x)$ for all $x\in X$ then $\mu(1)\ge\nu(x)$ or $\mu(1)\ge\mu(x)$ for all $x\in X$.*
(iv) *either μ or ν is a fuzzy ideal of X.*

Proof. (i) If both μ and ν do not satisfy $\mu(1)\ge\mu(x)$ and $\nu(1)\ge\nu(x)$ for all $x\in X$ then there exist $x,y\in X$ such that $\mu(x)>\mu(1)$ and $\nu(y)>\nu(1)$. Then

$$(\mu\times\nu)(x,y)=\min\{\mu(x),\nu(y)\}>\min\{\mu(1),\nu(1)\}=(\mu\times\nu)(1,1)$$

which is contradiction. Hence (i) proved.

(ii) Again, we use reduction to absurdity. Let $x,y\in X$ be such that $\mu(x)>\nu(1)$ and $\nu(y)>\nu(1)$. Then

$$(\mu\times\nu)(1,1)=\min\{\mu(1),\nu(1)\}=\nu(1)$$

and

$$(\mu\times\nu)(x,y)=\min\{\mu(x),\nu(y)\}>\nu(1)=(\mu\times\nu)(1,1)$$

which is a contradiction. Hence (ii) is proved.

(iii) The proof is similar to (ii).

(iv) Since, by (i), either μ or ν satisfies Definition 3.1(i), without loss of generality we may assume that μ satisfies Definition 3.1(i). Using (ii), we have that either $\mu(x)\le\nu(1)$ or $\nu(x)\le\nu(1)$ for all $x\in X$.

If $\mu(x)\le\nu(1)$ for all $x\in X$ then

$$(\mu\times\nu)(x,1)=\min\{\mu(x),\nu(1)\}=\mu(x)\text{ for all }x\in X.$$

Let $(x,y),(r,s)\in X\times X$. Since $\mu\times\nu$ is a fuzzy ideal of $X\times X$ by Definition 3.1(ii) we have

$$\begin{aligned}(\mu\times\nu)(r,s)&\ge\min\{(\mu\times\nu)(x,y),(\mu\times\nu)((x,y)\to(r,s))\}\\&=\min\{(\mu\times\nu)(x,y),(\mu\times\nu)(x*r,y*s)\}.\end{aligned}\tag{I}$$

If we take $y = s = 1$, then

$$\begin{aligned}\mu(r) &= (\mu \times \nu)(r, 1)\\ &\geq \min\{(\mu \times \nu)(x, 1), (\mu \times \nu)(x * r, 1 * 1)\}\\ &= \min\{(\mu \times \nu)(x, 1), (\mu \times \nu)(x * r, 1)\}\\ &= \min\{\min\{\mu(x), \nu(1)\}, \min\{\mu(x * r), \nu(1)\}\}\\ &= \min\{\mu(x), \mu(x * r)\}\end{aligned}$$

showing that μ satisfies Definition 3.1(ii). Hence μ is a fuzzy ideal of X.

Now, we consider the case $\nu(x) \leq \nu(1)$ for all $x \in X$. Suppose that $\mu(y) > \nu(1)$ for some $y \in X$. Then $\mu(1) \geq \mu(y) > \nu(1)$. Since $\nu(x) \leq \nu(1)$ for all $x \in X$, it follows that $\mu(1) > \nu(x)$ for all $x \in X$. Hence $(\mu \times \nu)(1, x) = \min\{\mu(1), \nu(x)\} = \nu(x)$ for all $x \in X$.

Taking $x = r = 1$ in (I), then

$$\begin{aligned}\nu(s) &= (\mu \times \nu)(1, s)\\ &\geq min\{(\mu \times \nu)(1, y), (\mu \times \nu)(1 * 1, y * s)\}\\ &= \min\{(\mu \times \nu)(1, y), (\mu \times \nu)(1, y * s)\}\\ &= \min\{\min\{\mu(1), \nu(y)\}, \min\{\mu(1), \nu(y * s)\}\}\\ &= \min\{\nu(y), \nu(y * s)\},\end{aligned}$$

which proves that ν satisfies Definition 3.1(ii). Hence ν is a fuzzy ideal of X. ■

Now, we give an example to show that if $\mu \times \nu$ is a fuzzy ideal of $X \times X$ then μ and ν both need not be fuzzy ideals of X.

Example 4.6. Let X be a distributive implication groupoid with $|A| \geq 2$ and let $\alpha, \beta \in [0, 1]$ be such that $0 \leq \alpha \leq \beta < 1$. Define the fuzzy sets μ and $\nu : X \to [0, 1]$ by $\mu(x) = \alpha$ and

$$\nu(x) = \begin{cases} \beta, & \text{if } x = 1;\\ 1, & \text{if } x \neq 1.\end{cases}$$

for all $x \in X$, respectively. Then $(\mu \times \nu)(x, y) = min\{\mu(x), \nu(y)\} = \alpha$ for all $(x, y) \in X \times X$ that is $\mu \times \nu : X \times X \to [0, 1]$ is a constant function. Hence $\mu \times \nu$ is a fuzzy ideal of $X \times X$. Now μ is a fuzzy ideal of X but ν is not a fuzzy ideal of X because ν does not satisfy Definition 3.1(i).

In the following theorem, we characterize the fuzzy ideal of a distributive implication groupoid X.

Theorem 4.7. *Let ν be a fuzzy set in a distributive implication groupoid X and let μ_ν be the strongest fuzzy relation on X. Then ν is a fuzzy ideal of X if and only if μ_ν is a fuzzy ideal of $X \times X$.*

Proof. Assume that ν is a fuzzy ideal of X. We note from Definition 3.1(i) that for all $(x,y) \in X \times X$,

$$\mu_\nu(x,y) = \min\{\nu(x), \nu(y)\} \leq \min\{\nu(1), \nu(1)\} = \mu_\nu(1,1)$$

showing that μ_ν satisfies Definition 3.1(i). Let $(x,y),(r,s) \in X \times X$. Then

$$\begin{aligned}
&\min\{\mu_\nu(x,y), \mu_\nu((x,y) \to (r,s))\} \\
&\qquad = \min\{\mu_\nu(x,y), \mu_\nu((x*r, y*s))\} \\
&\qquad = \min\{\min\{\nu(x), \nu(y)\}, \min\{\nu(x*r), \nu(y*s)\}\} \\
&\qquad = \min\{\min\{\nu(x), \nu(x*r)\}, \min\{\nu(y), \nu(y*s)\}\} \\
&\qquad \leq \min\{\nu(r), \nu(s)\} \\
&\qquad = \mu_\nu(r,s)
\end{aligned}$$

This proves that μ_ν satisfies Definition 3.1(ii). Hence μ_ν is a fuzzy ideal of $X \times X$.

Conversely, suppose that μ_ν is a fuzzy ideal of $X \times X$. Then

$$min\{\nu(x), \nu(y)\} = \mu_\nu(x,y) \leq \mu_\nu(1,1) = min\{\nu(1), \nu(1)\} = \nu(1),$$

for all $x, y \in X$. It follows that $\nu(x) \leq \nu(1)$ for all $x \in X$.

For any $(x,y),(r,s) \in X \times X$, we have

$$\begin{aligned}
\min\{\nu(r), \nu(s)\} &= \mu_\nu(r,s) \\
&\geq \min\{\mu_\nu(x,y), \mu_\nu((x,y) \to (r,s))\} \\
&= \min\{\mu_\nu(x,y), \mu_\nu((x*r, y*s))\} \\
&= \min\{\min\{\nu(x), \nu(y)\}, \min\{\nu(x*r), \nu(y*s)\}\} \\
&= \min\{\min\{\nu(x), \nu(x*r)\}, \min\{\nu(y), \nu(y*s)\}\}.
\end{aligned}$$

In particular, if we take $y = s = 1$ (resp. $x = r = 1$), then

$$\nu(r) \geq \min\{\nu(x), \nu(x*r)\}(\text{ resp. } \nu(s) \geq \min\{\nu(y), \nu(y*s)\}).$$ ■

Definition 4.8. Let $(X, *, 1)$ and $(Y, \Delta, 1')$ be two distributive implication groupoids. Then a mapping $f : X \to Y$ is called a homomorphism if $f(x*y) = f(x)\Delta f(y)$ for all $x, y \in X$.

Note that if $f : X \to Y$ is homomorphism of distributive implication groupoids, then $f(1) = 1'$.

Definition 4.9. Let $f : X \to Y$ be a mapping of distributive implication groupoids and μ be a fuzzy set of Y. The map μ^f is the pre-image of μ under f, if $\mu^f(x) = \mu(f(x))$ for all $x \in X$.

Theorem 4.10. *Let $f : X \to Y$ be a homomorphism of distributive implication groupoids. If μ is a fuzzy ideal of Y then μ^f is a fuzzy ideal of X.*

Proof. For any $x \in X$, we have

$$\mu^f(x) = \mu(f(x)) \leq \mu(1') = \mu(f(1)) = \mu^f(1).$$

Let $x, y \in X$. Then

$$\begin{aligned}\min\{\mu^f(x * y), \mu^f(x)\} &= \min\{\mu(f(x * y)), \mu(f(x))\} \\ &= \min\{\mu(f(x) * f(y)), \mu(f(x))\} \\ &\leq \mu(f(y)) = \mu^f(y).\end{aligned}$$

Hence μ^f is a fuzzy ideal of X. ∎

We conclude this paper with the following theorem.

Theorem 4.11. *Let $f : X \to Y$ be an onto homomorphism of distributive implication groupoids. If μ^f is fuzzy ideal of X, then μ is a fuzzy ideal of Y.*

Proof. Let $y \in Y$. Then there exists $x \in X$ such that $f(x) = y$. Then

$$\mu(y) = \mu(f(x)) = \mu^f(x) \leq \mu^f(1) = \mu(f(1)) = \mu(1').$$

Let $x, y \in Y$. Then there exist $a, b \in X$ such that $f(a) = x$ and $f(b) = y$. It follows that

$$\begin{aligned}\mu(y) &= \mu(f(b)) \\ &= \mu^f(b) \\ &\geq \min\{\mu^f(a * b), \mu^f(a)\} \\ &= \min\{\mu(f(a * b)), \mu(f(a))\} \\ &= \min\{\mu(f(a) * f(b)), \mu(f(a))\} \\ &= \min\{\mu(x * y), \mu(x)\}.\end{aligned}$$

Hence μ is a fuzzy ideal of Y. ∎

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