ON 2-ABSORBING PRIMARY AND WEAKLY 2-ABSORBING ELEMENTS IN MULTIPLICATIVE LATTICES

Fethi Çallialp

Beykent University
Faculty of Science and Art
Ayazağa-Maslak, Istanbul
Turkey
e-mail: fethicallialp@beykent.edu.tr

Ece Yetkin
Unsal Tekir

Marmara University
Department of Mathematics
Ziverbey, Goztepe, 34722, Istanbul
Turkey
e-mails: yetkinece@gmail.com and
utekir@marmara.edu.tr

Abstract. In this paper, we introduce the concept of 2-absorbing primary and weakly 2-absorbing primary elements which are generalizations of primary and weakly primary elements in multiplicative lattices. Let L be a multiplicative lattice. A proper element q of L is said to be a (weakly) 2-*absorbing primary element* of L if whenever $a, b, c \in L$ with $(0 \neq abc \leq q)$ $abc \leq q$ implies either $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$. Some properties of 2-absorbing primary and weakly 2-absorbing primary elements are presented and relations among prime, primary, 2-absorbing, weakly 2-absorbing, 2-absorbing primary and weakly 2-absorbing primary elements are investigated. Furthermore, we determine 2-absorbing primary elements in some special lattices and give a new characterization for principal element domains in terms of 2-absorbing primary elements.

Keywords: prime element, primary element, 2-absorbing element, 2-absorbing primary element, multiplicative lattice.

1991 Mathematics Subject Classification: Primary 16F10; Secondary 16F05, 13A15.

1. Introduction

The concept of 2-absorbing ideal in a commutative ring with identity, which is a generalization of prime ideal, was introduced by Badawi in [7] and studied in [8], [12], and [1]. Various generalizations of prime ideals are also studied in

[5], [11], [14] and [6]. As a generalization of primary ideals the concept of 2-absorbing primary ideals and weakly 2-absorbing primary ideals are introduced in [9] and [10]. Our aim is to extend the concept of 2-absorbing primary ideals of commutative rings to 2-absorbing primary elements of non modular multiplicative lattices and give a characterization for principal element domains in terms of 2-absorbing primary elements.

A *multiplicative lattice* is a complete lattice L with the least element 0 and compact greatest element 1, on which there is defined a commutative, associative, completely join distributive product for which 1 is a multiplicative identity. An element a of L is said to be compact if whenever $a \leq \bigvee_{\alpha \in I} a_\alpha$ implies $a \leq \bigvee_{\alpha \in I_0} a_\alpha$ for some finite subset I_0 of I. By a *C-lattice* we mean a (not necessarily modular) multiplicative lattice which is generated under joins by a multiplicatively closed subset C of compact elements. C-lattices can be localized. For any prime element p of L, L_p denotes the localization at $F = \{x \in C \mid x \nleq p\}$. For details on C-lattices and their localization theory, the reader is referred to [15] and [19]. We note that in a C-lattice, a finite product of compact elements is again compact. Throughout this paper, L denotes a C-lattice and the set of all compact elements of L is shown by L_*. An element $e \in L$ is said to be *principal* [13], if it satisfies the meet principal property (i) $a \wedge be = ((a : e) \wedge b)e$ and join principal property (ii) $(ae \vee b) : e = (b : e) \vee a$. A finite product of meet (join) principal elements of L is again meet (join) principal from [13, Lemma 3.3 and Lemma 3.4].

If every element of L is principal, then L is called a *principal element lattice*. For more information about principal element lattices, the reader is referred to [3], [16] and [17]. L is called a *totally ordered lattice*, if any two elements of L are comparable. L is said to be a Prüfer lattice if every compact element is principle.

An element $a \in L$ is said to be *proper* if $a < 1$. A proper element p of L (*weakly, [4]) prime* if $(0 \neq ab \leq p)$ $ab \leq p$ implies either $a \leq p$ or $b \leq p$. If 0 is prime, then L is said to be a *domain*. An element $m < 1$ in L is said to be *maximal* if $m < x \leq 1$ implies $x = 1$. It can be easily shown that maximal elements are prime. A maximal element m of L is said to be *simple*, if there is no element $a \in L$ such that $m^2 < a < m$. L is said to be *quasi-local* if it contains a unique maximal element. If $L = \{0, 1\}$, then L is called a *field*. An element $a \in L$ is said to be a *strong compact* element if both a and $a^\omega = \bigwedge_{n=1}^{\infty} a^n$ are compact elements of L. Strong compact elements have been studied in [16]. For $a \in L$, we define radical of a as $\sqrt{a} = \wedge\{p \in L : p \text{ is prime and } a \leq p\}$. Note that in a C-lattice L, $\sqrt{a} = \wedge\{p \in L : p \text{ is prime and } a \leq p\} = \vee\{x \in L_* \mid x^n \leq a \text{ for some } n \in \mathbb{Z}^+\}$. (See also Theorem 3.6 of [21]). A proper element q is said to be (weakly) *primary* if for every $a, b \in L$, $(0 \neq ab \leq q)$ $ab \leq q$ implies either $a \leq q$ or $b^n \leq q$ for some $n \in \mathbb{Z}^+$, [6]. If q is primary and if $\sqrt{q} = p$ is a prime element, then q is called a p-primary element. A principally generated C-lattice domain L is said to be a *Dedekind domain*, if every element of L is a finite product of prime elements of L.

Recall from [18] that a proper element q of L is called a (weakly) 2-*absorbing* element of L if whenever $a, b, c \in L$ with $(0 \neq abc \leq q)$ $abc \leq q$, then $ab \leq q$ or

$ac \leq q$ or $bc \leq q$. In this paper, we introduce the concepts of 2-absorbing primary and weakly 2-absorbing primary element which are generalizations of primary and weakly primary elements. A proper element q of L is said to be a (weakly) 2-*absorbing primary* element of L if whenever $a, b, c \in L$ with $(0 \neq abc \leq q)$ $abc \leq q$, then $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$.

Among many results in this paper, it is shown (Theorem 2.4) that the radical of a 2-absorbing primary element of L is a 2-absorbing element of L. It is shown (Theorem 2.6) that if q_1 is a p_1-primary element of L for some prime element p_1 of L and q_2 is a p_2-primary element of L for some prime element p_2 of L, then q_1q_2 and $q_1 \wedge q_2$ are 2-absorbing primary elements of L. It is shown (Theorem 2.7) that if radical of q is primary, then q is a 2-absorbing primary element. 2-absorbing primary and weakly 2-absorbing primary elements of cartesian product of multiplicative lattices are presented (Theorem 2.20-2.24). A new characterization for principal element domains in terms of 2-absorbing primary elements is established (Theorem 3.30).

2. 2-absorbing primary and Weakly 2-absorbing primary elements

Definition 2.1

(1) A proper element q of L is called a 2-absorbing primary element of L if whenever $a, b, c \in L$ and $abc \leq q$, then $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$.

(2) A proper element q of L is called a *weakly 2-absorbing primary* element of L if whenever $a, b, c \in L$ and $0 \neq abc \leq q$, then $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$.

The following theorem is obvious from the definitions, so the proof is omitted.

Theorem 2.2 *Let q be a proper element of L. Then*

(1) *If q is a (weakly) prime element, then q is a (weakly) 2-absorbing primary element.*

(2) *If q is a (weakly) primary element, then q is a (weakly) 2-absorbing primary element.*

(3) *If q is a (weakly) 2-absorbing element, then q is a (weakly) 2-absorbing primary element.*

(4) *If q is a 2-absorbing primary element, then q is a weakly 2-absorbing primary element.*

It is known from [Theorem 1, [15]] that if L is a Prüfer lattice and p is a prime element of L, then p^n is p-primary element. Thus p^n is a 2-absorbing primary element of L for all $n > 0$.

Theorem 2.3

(1) *An element $q \in L$ is a 2-absorbing primary element if and only if for any $a, b, c \in L_*$, $abc \leq q$ implies either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$.*

(2) *An element $q \in L$ is a weakly 2-absorbing primary element if and only if for any $a, b, c \in L_*$, $0 \neq abc \leq q$ implies either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$.*

Proof. (1) Assume that for any $a, b, c \in L_*$, $abc \leq q$ implies either $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$. Let $a, b, c \in L$, $abc \leq q$, $bc \not\leq \sqrt{q}$ and $ac \not\leq \sqrt{q}$. Then there exist compact elements $a' \leq a$, $b' \leq b$ and $c' \leq c$ such that $a'b'c' \leq q$. Since $ac \not\leq \sqrt{q}$ and $bc \not\leq \sqrt{q}$, there exist compact elements $a_1 \leq a$, $c_1 \leq c$, $c_2 \leq c$ and $b_1 \leq b$ such that $a_1c_1 \not\leq \sqrt{q}$ and $b_1c_2 \not\leq \sqrt{q}$. Put $c_3 = c_1 \vee c_2 \vee c'$, $a_2 = a_1 \vee a'$, $b_2 = b_1 \vee b'$. We show that $ab \leq q$. Choose compact elements $a_\alpha \leq a$ and $b_\alpha \leq b$. Then $(a_2 \vee a_\alpha)c_3(b_2 \vee b_\alpha) \leq q$, $(a_2 \vee a_\alpha)c_3 \not\leq \sqrt{q}$, $c_3(b_2 \vee b_\alpha) \not\leq \sqrt{q}$ and hence by the hypothesis, $(a_2 \vee a_\alpha)(b_2 \vee b_\alpha) \leq q$. So $a_\alpha b_\alpha \leq q$. Consequently, $ab \leq q$. Therefore q is a 2-absorbing element of L. The converse part is obvious.

(2) It can be easily shown similar to (1). ∎

Theorem 2.4 *If q is a 2-absorbing primary element of L, then $\sqrt{q}$ is a 2-absorbing element of L.*

Proof. Let $a, b, c \in L$ such that $abc \leq \sqrt{q}$, $ac \not\leq \sqrt{q}$ and $bc \not\leq \sqrt{q}$. Since $abc \leq \sqrt{q}$, there exists a positive integer n such that $(abc)^n = a^nb^nc^n \leq q$. We obtain $a^nc^n \not\leq \sqrt{q}$ and $b^nc^n \not\leq \sqrt{q}$. Since q is 2-absorbing primary, we conclude that $a^nb^n = (ab)^n \leq q$, and hence $ab \leq \sqrt{q}$. Thus $\sqrt{q}$ is a 2-absorbing element of L. ∎

Theorem 2.5 *Let q be a proper element of L. Then $\sqrt{q}$ is a (weakly) 2-absorbing element of L if and only if $\sqrt{q}$ is a (weakly) 2-absorbing primary element of L.*

Proof. Since $\sqrt{\sqrt{q}} = \sqrt{q}$, the proof is clear. ∎

Theorem 2.6 *If q is a 2-absorbing primary element of L, then one of the following statements must hold.*

(1) *$\sqrt{q} = p$ is a prime element,*

(2) *$\sqrt{q} = p_1 \wedge p_2$, where p_1 and p_2 are the only distinct prime elements of L that are minimal over q.*

Proof. Suppose that q is a 2-absorbing primary element of L. Then $\sqrt{q}$ is a 2-absorbing element by Theorem 2.4. Since $\sqrt{\sqrt{q}} = \sqrt{q}$, the claim follows from Theorem 3 in [18]. ∎

Let q be a proper element of L. It is known that if $\sqrt{q}$ is a maximal element of L, then q is a primary element of L. The following theorem states that it is sufficient that if $\sqrt{q}$ is a primary element of L, then q is a 2-absorbing primary element of L. Note that $\sqrt{q}$ is a (weakly) prime element of L if and only if $\sqrt{q}$ is a (weakly) primary element of L as $\sqrt{q} = \sqrt{\sqrt{q}}$.

Theorem 2.7 *Let q be a proper element of L.*

(1) *If $\sqrt{q}$ is a primary element of L, then q is a 2-absorbing primary element of L.*

(2) *If $\sqrt{q}$ is a weakly primary element of L, then q is a weakly 2-absorbing primary element of L.*

Proof. (1) Suppose that $abc \leq q$ for some $a, b, c \in L$ and $ab \not\leq q$. Since $(ac)(bc) = abc^2 \leq q \leq \sqrt{q}$ and $\sqrt{q}$ is a primary element of L, we have $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$. Hence q is a 2-absorbing primary element of L.

(2) Suppose that $0 \neq abc \leq q$ for some $a, b, c \in L$ and $ab \not\leq q$. Suppose that $ab \not\leq \sqrt{q}$. Since $\sqrt{q}$ is a weakly primary element of L, we have $c \leq \sqrt{q}$, and thus $ac \leq \sqrt{q}$. Suppose that $ab \leq \sqrt{q}$. Since $0 \neq abc \leq q$ and $ab \leq \sqrt{q}$, we have $0 \neq ab \in \sqrt{q}$. Since $\sqrt{q}$ is a weakly primary element of L and $0 \neq ab \leq \sqrt{q}$, we have $a \leq \sqrt{q}$ or $b \leq \sqrt{q}$. Thus $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$. Thus q is a weakly 2-absorbing primary element of L. ∎

Definition 2.8 Let q be a 2-absorbing primary element of L. Then $p = \sqrt{q}$ is a 2-absorbing element by Theorem 2.2. We say that q is a p-2-absorbing primary element of L.

Theorem 2.9 *Let q_1 is a p_1-primary element of L and q_2 is a p_2-primary element of L for some prime elements p_1 and p_2 of L. Then the following statements hold.*

(1) *$q_1 q_2$ is a 2-absorbing primary element of L.*

(2) *$q_1 \wedge q_2$ is a 2-absorbing primary element of L.*

Proof. (1). Suppose that $abc \leq q_1 q_2$ for some $a, b, c \in L$, $ac \not\leq \sqrt{q_1 q_2}$, and $bc \not\leq \sqrt{q_1 q_2}$. Then $a, b, c \not\leq \sqrt{q_1 q_2}$. As $\sqrt{q_1 q_2} = p_1 \wedge p_2$, $\sqrt{q_1 q_2}$ is a 2-absorbing element of L by [18]. Since $ac, bc \not\leq \sqrt{q_1 q_2}$, we have $ab \leq \sqrt{q_1 q_2}$. We show that $ab \leq q_1 q_2$. Since $ab \leq \sqrt{q_1 q_2} \leq p_1$, we may assume that $a \leq p_1$. Since $a \not\leq \sqrt{q_1 q_2} = p_1 \wedge p_2$ and $ab \leq \sqrt{q_1 q_2} \leq p_2$, we conclude that $a \not\leq p_2$ and $b \leq p_2$. Since $b \leq p_2$ and $b \not\leq \sqrt{q_1 q_2}$, we have $b \not\leq p_1$. If $a \leq q_1$ and $b \leq q_2$, then $ab \leq q_1 q_2$, so we are done. Thus assume that $a \not\leq q_1$. Since q_1 is a p_1-primary element of L and $a \not\leq q_1$, we have $bc \leq p_1$. Since $b \leq p_2$ and $bc \leq p_1$, we have $bc \leq \sqrt{q_1 q_2}$, a contradiction. Thus $a \leq q_1$. Similarly, if $b \not\leq q_2$, we conclude $ac \leq \sqrt{q_1 q_2}$, which is again a contradiction. So $a \leq q_1$ and $b \leq q_2$ and thus $ab \leq q_1 q_2$.

(2). Let $q = q_1 \wedge q_2$. Then $\sqrt{q} = p_1 \wedge p_2$ is a 2-absorbing element of L. Suppose that $abc \leq q$ for some $a, b, c \in L$, $ac \not\leq \sqrt{q}$, and $bc \not\leq \sqrt{q}$. Then $a, b, c \not\leq \sqrt{q} = p_1 \wedge p_2$ and $ab \leq \sqrt{q} \leq p_1$. We show that $ab \leq q$. Since $ab \leq \sqrt{q} \leq p_1$, we may assume that $a \leq p_1$. Since $a \not\leq \sqrt{q}$ and $ab \leq \sqrt{q} \leq p_2$, we conclude that $a \not\leq p_2$ and $b \leq p_2$. Since $b \leq p_2$ and $b \not\leq \sqrt{q}$, we get $b \not\leq p_1$. If $a \leq q_1$ and $b \leq q_2$, then $ab \leq q$ and we are done. So suppose that $a \not\leq q_1$. Since q_1 is a p_1-primary element of L and $a \not\leq q_1$, we have $bc \leq p_1$. Since $b \leq p_2$ and $bc \leq p_1$, we have $bc \leq \sqrt{q}$, a contradiction. Hence we have $a \leq q_1$. By the similar argument, we conclude $a \leq q_1$ and $b \leq q_2$. Thus $ab \leq q$. ■

As a consequence of Theorem 2.9, we have the following corollary.

Corollary 2.10 *Let p_1, p_2 be prime elements of L. If p_1^n is a p_1-primary element of L and p_2^m is a p_2-primary element of L for some positive integers n, m, then $p_1^n p_2^m$ and $p_1^n \wedge p_2^m$ are 2-absorbing primary elements of L.*

Theorem 2.11 *Let $q_1, q_2, ..., q_n$ be p-2-absorbing primary elements of L for some 2-absorbing element p of L. Then $q = \bigwedge_{i=1}^{n} q_i$ is a p-2-absorbing primary element of L.*

Proof. Let $a, b, c \in L$ with $abc \leq q$. Suppose that $ab \nleq q$. Then $ab \nleq q_i$ for some $i \in \{1, 2, ..., n\}$. It implies either $bc \leq \sqrt{q_i} = p$ or $ac \leq \sqrt{q_i} = p$. Since $\sqrt{q} = \bigwedge_{i=1}^{n} \sqrt{q_i} = p$, we are done. ■

Definition 2.12 Let q be a weakly 2-absorbing primary element of L. We say (a, b, c) is a *triple-zero* of q if $abc = 0$, $ab \nleq q$, $bc \nleq \sqrt{q}$, and $ac \nleq \sqrt{q}$.

Note that if q is a weakly 2-absorbing primary element of L that is not 2-absorbing primary element, then there exists a triple-zero (a, b, c) of q for some $a, b, c \in L$.

Theorem 2.13 *Let q be a weakly 2-absorbing primary element of L and suppose that (a, b, c) is a triple-zero of q for some $a, b, c \in L$. Then*

(1) $abq = bcq = acq = 0$,

(2) $aq^2 = bq^2 = cq^2 = 0$.

Proof. (1) Suppose that $abq \neq 0$. Then there exists a compact element $x \leq q$ such that $abx \neq 0$. Hence $0 \neq ab(c \vee x) \leq q$. Since $ab \nleq q$ and q is weakly 2-absorbing primary, we have $a(c \vee x) \leq \sqrt{q}$ or $b(c \vee x) \leq \sqrt{q}$. So $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$, a contradiction. Thus $abx = 0$, and so $abq = 0$. Similarly, it can be easily verified that $bcq = acq = 0$.

(2) Suppose that $aq_1q_2 \neq 0$ for some compact elements $q_1, q_2 \leq q$. Hence from (1) we have $0 \neq a(b \vee q_1)(c \vee q_2) = aq_1q_2 \leq q$. It implies either $a(b \vee q_1) \leq q$ or $a(c \vee q_2) \leq \sqrt{q}$ or $(b \vee q_1)(c \vee q_2) \leq \sqrt{q}$. Thus $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$, a contradiction. Therefore $aq^2 = 0$. Similarly, one can easily show that $bq^2 = cq^2 = 0$. ■

Theorem 2.14 *If q is a weakly 2-absorbing primary element of L that is not 2-absorbing primary, then $q^3 = 0$.*

Proof. Suppose that q is a weakly 2-absorbing primary element that is not a 2-absorbing primary element of L. Then there exists (a, b, c) a triple-zero of q for some $a, b, c \in L$. Assume that $q^3 \neq 0$. Hence $q_1q_2q_3 \neq 0$, for some compact elements $q_1, q_2, q_3 \leq q$. By Theorem 2.13, we obtain $(a \vee q_1)(b \vee q_2)(c \vee q_3) = q_1q_2q_3 \neq 0$. This implies that $(a \vee q_1)(b \vee q_2) \leq q$ or $(a \vee q_1)(c \vee q_3) \leq \sqrt{q}$ or $(b \vee q_2)(c \vee q_3) \leq \sqrt{q}$. Thus we have $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$, a contradiction. Thus $q^3 = 0$. ■

Corollary 2.15 *If q is a weakly 2-absorbing primary element of L that is not 2-absorbing primary, then $\sqrt{q} = \sqrt{0}$.*

Theorem 2.16 *Let $q_1, q_2, ..., q_n$ be weakly 2-absorbing primary elements of L that are not 2-absorbing primary. Then $q = \bigwedge_{i=1}^{n} q_i$ is a weakly 2-absorbing primary element of L.*

Proof. Since q_i's are weakly 2-absorbing primary that are not 2-absorbing primary, we get $\sqrt{q_i} = \sqrt{0}$ for each $1 \leq i \leq n$ by Corollary 2.15. So the result is obtained easily similar to the argument in the proof of Theorem 2.11. ■

Theorem 2.17 *Suppose that 0 has a triple-zero (a, b, c) for some $a, b, c \in L$ such that $ab \nleq \sqrt{0}$. Let q be a weakly 2-absorbing primary element of L. Then q is not a 2-absorbing primary element of L if and only if $q \leq \sqrt{0}$.*

Proof. Suppose that q is not a 2-absorbing primary element of L. Then $q \leq \sqrt{0}$ by Corollary 2.15. Conversely, suppose that $q \leq \sqrt{0}$. By hypothesis, we conclude that $ab \nleq q$, $ac \nleq \sqrt{0}$, and $bc \nleq \sqrt{0}$. Thus (a, b, c) is a triple-zero of q. Hence q is not a 2-absorbing primary element of L. ■

Recall that L is said to be *reduced* if $\sqrt{0} = 0$.

Corollary 2.18 *Let L be a reduced lattice and $q \neq 0$ be a proper element of L. Then q is a weakly 2-absorbing primary element if and only if q is a 2-absorbing primary element of L.*

Theorem 2.19 *Let m be a maximal element of L and q be a proper element of L. If q is a 2-absorbing primary element of L, then q_m is a 2-absorbing primary element of L_m.*

Proof. Let $a, b, c \in L_*$ such that $a_mb_mc_m \leq q_m$. Then $abc \leq q_m$, so $uabc \leq q$ for some $u \nleq m$. Hence we get either $uab \leq q$ or $bc \leq \sqrt{q}$ or $uac \leq \sqrt{q}$. Since $(\sqrt{q})_m = \sqrt{q_m}$ by [15], and $u_m = 1_m$, it follows either $a_mb_m \leq q_m$ or $b_mc_m \leq \sqrt{q_m}$ or $a_mc_m \leq \sqrt{q_m}$.It completes the proof. ■

Recall that for any $a \in L$, $L/a = \{b \in L : a \leq b\}$ is a multiplicative lattice with multiplication $c \circ d = cd \vee a$. For more details, the reader is referred to [2].

Lemma 1 *Let a and q be proper elements of L with $a \leq q$. If q is a 2-absorbing primary element of L, then $\overline{q}$ is a weakly 2-absorbing primary element of L/a.*

Proof. The proof is clear. ∎

Theorem 2.20 *Let $L = L_1 \times L_2$, where L_1 and L_2 are C- lattices. Then a proper element q is a 2-absorbing primary element of L if and only if it has one of the following three forms.*

(1) *$q = (q_1, 1_{L_2})$ for some 2-absorbing primary element q_1 of L_1,*

(2) *$q = (1_{L_1}, q_2)$ for some 2-absorbing primary element q_2 of L_2,*

(3) *$q = (q_1, q_2)$ for some primary element q_1 of L_1 and some primary element q_2 of L_2.*

Proof. If $q = (q_1, 1_{L_2})$ for some 2-absorbing primary element q_1 of L_1 or $q = (1_{L_1}, q_2)$ for some 2-absorbing primary element q_2 of L_2, then it is clear that q is a 2-absorbing primary element of L. Hence assume that $q = (q_1, q_2)$ for some primary element q_1 of L_1 and some primary element q_2 of L_2. Then $q_1' = (q_1, 1_{L_2})$ and $q_2' = (1_{L_1}, q_2)$ are primary elements of L. Hence $q_1' \wedge q_2' = (q_1, q_2) = q$ is a 2-absorbing primary element of L by Theorem 2.9.

Conversely, suppose that q is a 2-absorbing primary element of L. Then $q = (q_1, q_2)$ for some element q_1 of L_1 and some element q_2 of L_2. Suppose that $q_2 = 1_{L_2}$. Since q is a proper element of L, $q_1 \neq 1_{L_1}$. Let $L' = L/\{0\} \times L_2$. Then $\overline{q} = (q_1, 1_{L_2})$ is a 2-absorbing primary element of L' by Lemma 1. Now, we show that q_1 is a 2-absorbing primary element of L_1. Let $abc \leq q_1$ for some $a, b, c \in L_1$. Hence $(a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) = (abc, 1_{L_2}) \leq \overline{q}$, which implies that $(a, 1_{L_2})(b, 1_{L_2}) \leq \overline{q}$ or $(b, 1_{L_2})(c, 1_{L_2}) \leq \sqrt{\overline{q}}$ or $(a, 1_{L_2})(c, 1_{L_2}) \leq \sqrt{\overline{q}}$. It means that either $ab \leq q_1$ or $bc \leq \sqrt{q_1}$ or $ac \leq \sqrt{q_1}$. Thus q_1 is a 2-absorbing primary element of L_1.

If $q_1 = 1_{L_1}$, then q_2 can be obtained as a 2-absorbing primary element of L_2 by the similar way. Hence assume that $q_1 \neq 1_{L_1}$ and $q_2 \neq 1_{L_2}$. Then $\sqrt{q} = (\sqrt{q_1}, \sqrt{q_2})$. On the contrary, suppose that q_1 is not a primary element of L_1. Then there are a, $b \in L_1$ such that $ab \leq q_1$ but neither $a \leq q_1$ nor $b \leq \sqrt{q_1}$. Let $x = (a, 1)$, $y = (1, 0)$, and $z = (b, 1)$. Then $xyz = (ab, 0) \leq q$ implies that either $xy = (a, 0) \leq q$ and $xz = (ab, 1) \leq \sqrt{q}$ and $yz = (b, 0) \leq \sqrt{q}$, a contradiction. Therefore q_1 is a primary element of L_1. Similarly it can be easily seen that q_2 is a primary element of L_2, as needed. ∎

Theorem 2.21 *Let L_1 and L_2 be C-lattices, q be a proper element of L_1, and $L = L_1 \times L_2$. Then the following statements are equivalent.*

(1) *$(q, 1_{L_2})$ is a weakly 2-absorbing primary element of L.*

(2) *$(q, 1_{L_2})$ is a 2-absorbing primary element of L.*

(3) *q is a 2-absorbing primary element of L_1.*

Proof. (1)⇒(2) Since $(q, 1_{L_2}) \not\leq \sqrt{0}$, we conclude that $(q, 1_{L_2})$ is a 2-absorbing primary element of L by Corollary 2.15.

(2)⇒(3) Suppose that q is not a 2-absorbing primary element of L_1. Then there exist $a, b, c \in L_1$ such that $abc \leq q$, but $ab \not\leq q$, $bc \not\leq \sqrt{q}$, and $ac \not\leq \sqrt{q}$. Since $(a, 1_{L_2})(b, 1_{L_2})(c, 1_{L_2}) \leq (q, 1_{L_2})$, we have $(a, 1_{L_2})(b, 1_{L_2}) = (ab, 1_{L_2}) \leq (q, 1_{L_2})$ or $(a, 1_{L_2})(c, 1_{L_2}) = (ac, 1_{L_2}) \leq \sqrt{(q, 1_{L_2})} = (\sqrt{q}, 1_{L_2})$ or $(b, 1_{L_2})(c, 1_{L_2}) = (bc, 1_{L_2}) \leq \sqrt{(q, 1_{L_2})} = (\sqrt{q}, 1_{L_2})$. It follows that $ab \leq q$ or $bc \leq \sqrt{q}$ or $ac \leq \sqrt{q}$, a contradiction. Thus q is a 2-absorbing primary element of L_1.

(3)⇒(1) Let q be a 2-absorbing primary element of L_1. Then it can be easily shown that $(q, 1_{L_2})$ is a 2-absorbing primary element of L, therefore (1) holds. ■

Theorem 2.22 *Let L_1 and L_2 be C-lattices, q_1, q_2 be nonzero elements of L_1 and L_2, respectively, and let $L = L_1 \times L_2$. If (q_1, q_2) is a proper element of L, then the following statements are equivalent.*

(1) *(q_1, q_2) is a weakly 2-absorbing primary element of L.*

(2) *$q_1 = 1_{L_1}$ and q_2 is a 2-absorbing primary element of L_1 or $q_2 = 1_{L_2}$ and q_1 is a 2-absorbing primary element of L_1 or q_1, q_2 are primary elements of L_1 and L_2, respectively.*

(3) *(q_1, q_2) is a 2-absorbing primary element of L.*

Proof. (1)⇒(2) Assume that (q_1, q_2) is a weakly 2-absorbing primary element of L. If $q_1 = 1_{L_1}$ ($q_2 = 1_{L_2}$), then q_2 is a 2-absorbing primary element of L_2 (q_1 is a 2-absorbing primary element of L_1) by Theorem 2.21. So we may assume that $q_1 \neq 1_{L_1}$ and $q_2 \neq 1_{L_2}$. Let $a, b \in L_2$ such that $ab \leq q_2$ and let $x \in L_*$ with $0 \neq x \leq q_1$. Then $0 \neq (x, 1)(1, a)(1, b) = (x, ab) \leq (q_1, q_2)$. Since q_1 is proper, $(1, a)(1, b) = (1, ab) \not\leq \sqrt{(q_1, q_2)}$. Hence we have $(x, 1)(1, a) = (x, a) \leq (q_1, q_2)$ or $(x, 1)(1, b) = (x, b) \leq \sqrt{(q_1, q_2)}$, and so $a \leq q_2$ or $b \leq \sqrt{q_2}$. Thus q_2 is a primary element of L_2. Similarly, it can be easily shown that q_1 is a primary element of L_1.

(2)⇒(3) The proof is clear by Theorem 2.20.

(3)⇒(1) It is clear. ■

Theorem 2.23 *Let L_1 and L_2 be C-lattices and $L = L_1 \times L_2$. Then a nonzero proper element q of L is a weakly 2-absorbing primary element of L that is not 2-absorbing primary if and only if one of the following conditions holds.*

(1) *$q = (q_1, q_2)$, where q_1 is a nonzero weakly primary element of L_1 that is not primary and $q_2 = 0$ is a primary element of L_2.*

(2) *$q = (q_1, q_2)$, where q_2 is a nonzero weakly primary element of L_2 that is not primary and $q_1 = 0$ is a primary element of L_1.*

Proof. Suppose that q is a nonzero weakly 2-absorbing primary element of L that is not 2-absorbing primary element. Then $q = (q_1, q_2)$ for some elements q_1, q_2 of L_1 and L_2 respectively. Assume that $q_1 \neq 0$ and $q_2 \neq 0$. Then q is a 2-absorbing primary element of L by Theorem 2.22, a contradiction. Therefore $q_1 = 0$ or $q_2 = 0$. Without loss of generality we may assume that $q_2 = 0$. We show that $q_2 = 0$ is a primary element of L_2. Let $a, b \in L_2$ such that $ab \leq q_2$, and let $x \in L_*$ such that $0 \neq x \leq q_1$. Since $0 \neq (x, 1)(1, a)(1, b) = (x, ab) \leq q$ and $(1, a)(1, b) = (1, ab) \nleq \sqrt{q}$, we obtain $(x, a) = (x, 1)(1, a) \leq q$ or $(x, b) = (x, 1)(1, b) \leq \sqrt{q}$, and so $a \leq q_2$ or $b \leq \sqrt{q_2}$. Thus $q_2 = 0$ is a primary element of L_2. Next, we show that q_1 is a weakly primary element of L_1. Let $0 \neq ab \leq q_1$, for some $a, b \in L_1$. Since $0 \neq (a, 1)(b, 1)(1, 0) \leq (q_1, 0)$ and $(ab, 1) \nleq (q_1, 0)$, we conclude $(a, 0) = (a, 1)(1, 0) \leq \sqrt{(q_1, 0)} = \sqrt{q}$ or $(b, 0) = (b, 1)(1, 0) \leq \sqrt{(q_1, 0)} = \sqrt{q}$. Thus $a \leq q_1$ or $b \leq \sqrt{q_1}$, and therefore q_1 is a weakly primary element of L_1. Now, we show that q_1 is not primary. Suppose that q_1 is a primary element of L_1. Since $q_2 = 0$ is a primary element of L_2, we conclude that $q = (q_1, q_2)$ is a 2-absorbing primary element of L by Theorem 2.20, a contradiction. Thus q_1 is a weakly primary element of L_1 that is not primary.

Conversely, suppose that (1) holds. Assume that $(0, 0) \neq (a, a')(b, b')(c, c') \leq q = (q_1, 0)$. Since $a'b'c' = 0$ and $(0, 0) \neq (a, a')(b, b')(c, c') \leq (q_1, 0)$, we conclude that $abc \neq 0$. Assume $(a, a')(b, b') \nleq q$. We consider three cases.

Case one: Suppose that $ab \nleq q_1$, but $a'b' = 0$. Since q_1 is a weakly primary element of L_1, we have $c \leq \sqrt{q_1}$. Since $q_2 = 0$ is a primary element of L_2, we have $a' = 0$ or $b' \leq \sqrt{q_2}$. Thus $(a, a')(c, c') \leq \sqrt{q}$ or $(b, b')(c, c') \leq \sqrt{q}$.

Case two: Suppose that $ab \nleq q_1$ and $a'b' \neq 0$. Then $(c, c') \leq (\sqrt{q_1}, \sqrt{0}) = \sqrt{q}$. Thus $(a, a')(c, c') \leq \sqrt{q}$ or $(b, b')(c, c') \leq \sqrt{q}$.

Case three: Suppose that $ab \leq q_1$, but $a'b' \neq 0$. Since $0 \neq ab \leq q_1$ and q_1 is a weakly primary element of L_1, we have $a \leq q_1$ or $b \leq \sqrt{q_1}$. Since $a'b' \neq 0$ and $q_2 = 0$ is a primary element of L_2, we have $c' \leq \sqrt{q_2}$. Thus $(a, a')(c, c') \leq \sqrt{q}$ or $(b, b')(c, c') \leq \sqrt{q}$. Hence q is a weakly 2-absorbing primary element of L. Since q_1 is not a primary element of L_1, q is not a 2-absorbing primary element of L by Theorem 2.22. ■

Theorem 2.24 *Let $L = L_1 \times L_2 \times ... \times L_n$, where $2 < n < \infty$, and $L_1, L_2, ..., L_n$ are C-lattices and let q be a nonzero proper element of L. Then the following statements are equivalent.*

(1) *q is a weakly 2-absorbing primary element of L.*

(2) *q is a 2-absorbing primary element of L.*

(3) *Either $q = (q_t)_{t=1}^n$ such that for some $k \in \{1, 2, ..., n\}$, q_k is a 2-absorbing primary element of L_k, and $q_t = 1_{L_t}$ for every $t \in \{1, 2, ..., n\} \setminus \{k\}$ or $q = (q_t)_{t=1}^n$ such that for some $k, m \in \{1, 2, ..., n\}$, q_k is a primary element of L_k, q_m is a primary element of L_m, and $q_t = 1_{L_t}$ for every $t \in \{1, 2, ..., n\} \setminus \{k, m\}$.*

Proof. (1)$\Leftrightarrow$ (2) Since q is a proper element of L, we have $q = (q_1, \cdots, q_n)$, where every q_i's are element of L_i, and $q_j \neq 1_{L_j}$ for some $j \in \{1, ..., n\}$. Suppose that $q = (q_1, q_2, ..., q_n) \neq 0$ is a weakly 2-absorbing primary element of L. Then there is a compact element $0 \neq (a_1, a_2, ..., a_n) \leq q$. Hence $0 \neq (a_1, a_2, ..., a_n) = (a_1, 1, 1, ..., 1)(1, a_2, 1, ..., 1)...(1, 1, ..., a_n) \leq q$ implies there is a $j \in \{1, ..., n\}$ such that $b_j = 1_{L_j}$ and $(b_1, ..., b_n) \leq \sqrt{q} = (\sqrt{q_1}, ..., \sqrt{q_n})$, where $b_1, ..., b_n \in \{a_1, ..., a_n\}$. Hence $\sqrt{q_j} = 1_{L_j}$, and so $q_j = 1_{L_j}$. Thus $\sqrt{q} \neq \sqrt{0}$, and hence by Corollary 2.15, q is a 2-absorbing primary element. The converse is obvious.

(2)$\Leftrightarrow$ (3) We use induction on n. If $n = 2$, then we are done by Theorem 2.22. Hence let $3 \leq n < \infty$ and assume that the result is satisfied when $S = L_1 \times \cdots \times L_{n-1}$. Thus $L = S \times L_n$. Theorem 2.22 implies that q is a 2-absorbing primary element of L if and only if either $q = (s, 1_{L_n})$ for some 2-absorbing primary element s of S or $q = (1_s, t)$ for some 2-absorbing primary element t of L_n or $q = (s, t)$ for some primary element s of S and some primary element t of L_n. Since a proper element s of S is a primary element of S if and only if $s = (q_k)_{k=1}^{n-1}$ such that for some $k \in \{1, 2, ..., n-1\}$, we conclude that q_k is a primary element of L_k, and $q_t = 1_{L_t}$ for every $t \in \{1, 2, ..., n-1\} \setminus \{k\}$. So this completes the proof of the theorem. ∎

3. 2-absorbing primary elements in some special lattices

Theorem 3.25 *Suppose that $\sqrt{0}$ is a prime (primary) element of L. Let q be a proper element of L. Then q is a weakly 2-absorbing primary element of L if and only if q is a 2-absorbing primary element of L.*

Proof. Suppose that q is a weakly 2-absorbing primary element of L. Assume that $abc \leq q$ for some $a, b, c \in L$. If $0 \neq abc \leq q$, then $ab \leq q$ or $ac \leq \sqrt{q}$ or $bc \leq \sqrt{q}$. Hence assume that $abc = 0$ and $ab \nleq q$. Since $abc = 0 \leq \sqrt{0}$ and $\sqrt{0}$ is a prime element of L, we conclude that $a \leq \sqrt{0}$ or $b \leq \sqrt{0}$ or $c \leq \sqrt{0}$. Since $\sqrt{0} \leq \sqrt{q}$, we conclude that $ac \leq \sqrt{0} \leq \sqrt{q}$ or $bc \in \sqrt{0} \leq \sqrt{q}$. Thus q is a 2-absorbing primary element of L. The converse is clear. ∎

Recall that L is called *quasilocal* if it has exactly one maximal element.

Theorem 3.26 *Let L be a quasilocal lattice with maximal element $\sqrt{0}$. The following statements hold.*

1. *Every element of L is a weakly 2-absorbing primary element of L.*
2. *A proper element q of L is a weakly 2-absorbing primary element if and only if q is a 2-absorbing primary element.*

Proof. It is obvious by Theorem 3.25. ∎

Theorem 3.27 *Let L_1, L_2 and L_3 be C-lattices and let $L = L_1 \times L_2 \times L_3$. Then every proper element of L is a weakly 2-absorbing primary element of L if and only if L_1, L_2 and L_3 are fields.*

Proof. Suppose that every proper element of L is a weakly 2-absorbing primary element of L. Without loss of generality, we may assume that L_1 is not a field. Then there exists a nonzero proper element q of L_1. Thus $a = (q, 0, 0)$ is a weakly 2-absorbing primary element of L, which contradicts with Theorem 2.24.

Conversely, suppose that L_1, L_2, L_3 are fields. Then every nonzero proper element of L is a 2-absorbing element by Theorem 2.24. Since 0 is always weakly 2-absorbing primary, the proof is completed. ∎

Theorem 3.28 *Suppose that every proper element of L is a weakly 2-absorbing primary element. Then L has at most three incomparable prime elements.*

Proof. Assume that there are p_1, p_2, p_3 and p_4 incomparable prime elements of L. Let $q = p_1 \wedge p_2 \wedge p_3$. Hence $\sqrt{q} = \sqrt{p_1} \wedge \sqrt{p_2} \wedge \sqrt{p_3}$. Thus $\sqrt{q}$ is not a 2-absorbing element of L by Theorem 2.6. So q is not a 2-absorbing primary element of L by Theorem 2.2. Hence $q^3 = 0$ by Theorem 2.14. Thus $q^3 = p_1^3 p_2^3 p_3^3 = 0 < p_4$ implies that $p_1 < p_4$ or $p_2 < p_4$ or $p_3 < p_4$, a contradiction. Thus L has at most three incomparable prime elements. ∎

In view of Theorem 3.28, we have the following result.

Corollary 3.29 *Suppose that every proper element of L is a weakly 2-absorbing primary element. Then L has at most three maximal elements.*

Theorem 3.30 *Let L is a principally generated domain that is not a field. Then the following statements are equivalent.*

(1) *L is a principal element domain.*

(2) *Every maximal element is strong compact and a nonzero proper element q of L is a 2-absorbing primary element of L if and only if either $q = m^n$ for some maximal element m of L and some positive integer n or $q = m_1^n m_2^k$ for some maximal elements m_1, m_2 of L and some positive integers n, k.*

(3) *Every maximal element is strong compact and a nonzero proper element q of L is a 2-absorbing primary element of L if and only if either $q = p^n$ for some prime element p of L and some positive integer n or $q = p_1^n p_2^k$ for some prime elements p_1, p_2 of L and some positive integers n, k.*

Proof. (1) $\Rightarrow$ (2). Let L be a principal element domain. Then every maximal element is strong compact by [16, Theorem 2]. Suppose q is a nonzero 2-absorbing primary element of L that is not maximal. Then $q = m_1^{n_1} m_2^{n_2} . \cdots . m_k^{n_k}$ for some distinct maximal elements $m_1, ..., m_k$ of L and some integers $n_1, ..., n_k \geq 1$. Since every nonzero prime element of L is maximal and $\sqrt{q}$ is either a maximal element of L or $q_1 \wedge q_2$ for some maximal elements q_1, q_2 of L by Theorem 2.6, we conclude that either $q = m^n$ for some maximal element m of L and some $n \geq 1$ or $q = m_1^n m_2^k$ for some maximal elements m_1, m_2 of L and some $n, m \geq 1$. Conversely, suppose that $q = m^n$ for some maximal element m of L and some positive integer $n \geq 1$

or $q = m_1^n m_2^k$ for some maximal elements m_1, m_2 of L and some integers $n, k \geq 1$. Then q is a 2-absorbing primary element of L by Theorem 2.9 and Corollary 2.10.

(2)⇒(3) It is clear.

(3)⇒(1) Suppose that m is a maximal element of L and $q \in L$ with $m^2 \leq q \leq m$. Then q is an m-primary element. Hence q is a 2-absorbing primary element. From the hypothesis (3), either $q = m$ or $q = m^2$, so there is no element $a \in L$ such that $m^2 < a < m$ which shows that m is simple. Therefore, by [16, Theorem 2], L is a principal element domain. ■

Suppose that L is principally generated. Then L is a Dedekind domain if and only if L is a principal element lattice by Theorem 2.7 in [3]. So we have the following result as a consequence of Theorem 3.30.

Corollary 3.31 *Let L be a principally generated domain. If L is a Dedekind domain, then $1_L \neq q \in L$ is 2-absorbing primary if and only if $q = p^n$ for some prime element p of L, a positive integer n or $q = p_1^n p_2^m$ for some prime elements p_1, p_2 of L, some positive integers n, m.*

Acknowledgements. This work is supported by the Scientific Research Project Program of Marmara (BAPKO).

References

[1] Anderson, D.D., Badawi, A., *On n-absorbing ideals of commutative rings*, Comm. Algebra., 39 (2011), 1646-1672.

[2] Anderson, D.D., *Abstract commutative ideal theory without chain condition*, Algebra Universalis, 6 (1976), 131-145.

[3] Anderson, D.D., Jayaram, C., *Principal element lattices*, Czechoslovak Mathematical Journal, 46 (1996), 99-109.

[4] Anderson, D.D., Smith, E., *Weakly prime ideals*, Houston Journal of Mathematics, 29 (2003), 831-840.

[5] Anderson, D.D., Bataineh, N., *Generalizations of prime ideals*, Comm. Algebra, 36 (2008), 686-696.

[6] Atani, S.E., Farzalipour, F., *On weakly primary ideals*, Georgian Mathematical Journal, 12 (2005), 423-429.

[7] Badawi, A., *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc., 75 (2007), 417-429.

[8] Badawi, A., Yousefian Darani, A., *On weakly 2-absorbing ideals of commutative rings*, Houston J. Math., 39 (2013), 441–452.

[9] Badawi, A., Tekir, Ü., Yetkin, E., *On 2-absorbing primary ideals in commutative rings*, Bull. Korean Math. Soc., 51 (4) (2014), 1163–1173.

[10] Badawi, A., Tekir, Ü., Yetkin, E., *On weakly 2-absorbing primary ideals in commutative rings*, Journal of the Korean Mathematical Society (in press).

[11] Çallialp, F., Jayaram, C., Tekir, U., *Weakly prime elements in multiplicative lattices*, Communications in Algebra, 40 (2012), 2825-2840.

[12] Darani, A.Y., Puczylowski, E.R., *On 2-absorbing commutative semigroups and their applications to rings*, Semigroup Forum, 86 (2013), 83-91.

[13] Dilworth, R.P., *Abstract commutative ideal theory*, Pacific Journal of Mathematics, 12 (1962), 481-498.

[14] Ebrahimpour, M., Nekooei, R., *On generalizations of prime ideals*, Comm. Algebra, 40 (2012), 1268-279.

[15] Jayaram, C., Johnson, E.W., *s-prime elements in multiplicative lattices*, Periodica Mathematica Hungarica, 31 (1995), 201-208.

[16] Jayaram, C., Johnson, E.W., *Strong compact elements in multiplicative lattices*, Czechoslovak Mathematical Journal, 47 (122) (1997), 105-112.

[17] Jayaram, C., *Primary elements in Prüfer lattices*, Czechoslovak Mathematical Journal, 52 (127) (2002), 585-593.

[18] Jayaram, C., Tekir, U., Yetkin, E., *2-absorbing and weakly 2-absorbing elements in multiplicative lattices*, Communications in Algebra, 42 (2014), 1-16.

[19] Johnson, J.A., Sherette, G.R., *Structural properties of a new class of CM-lattices*, Canadian Journal of Mathematics, 38 (1986), 552-562.

[20] Payrovi, S., Babaei, S., *On the 2-absorbing ideals*, Int. Math. Forum, 7 (2012), 265-271.

[21] Thakare, N.K., Manjarekar, C.S., Maeda, S., *Abstract spectral theory* II, Minimal characters and minimal spectrums of multiplicative lattices, Acta. Sci. Math. (Szeged), 52 (1988), 53-67.

Accepted: 06.12.2014