SOME NEW OPERATIONS ON INTERVAL-VALUED INTUITIONISTIC FUZZY SOFT SETS

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Abstract. Interval-valued intuitionistic fuzzy soft set theory is an intuitionistic fuzzy extension of the interval-valued fuzzy soft set theory or an interval-valued fuzzy extension of the intuitionistic fuzzy soft set theory. In this paper, we further consider interval-valued intuitionistic fuzzy soft sets. Some new operations on interval-valued intuitionistic fuzzy soft sets, i.e., "·", "+" and Cartesian product, are defined, and some related properties are investigated.

Keywords: Soft sets; interval-valued intuitionistic fuzzy soft sets.

1. Introduction

Probability theory, fuzzy sets [35], interval mathematics [11], and other mathematical tools are often useful approaches to dealing with uncertainties [24]. However, all of these theories have their own difficulties, and one of the major reasons is the inadequacy of their parametrization [23]. Soft set theory is a new mathematical tool for modeling uncertainties, which is free from the difficulties existing in those theories. At present, soft set theory has proven useful in many fields, such as prediction [30], [31], rules ming [12], decision making [7], [16], [36], mobile cloud computing [25], data analysis [8], [37]. Recently, many researches focused on theoretical aspect of soft sets. As a continuation of Molodtsov's pioneer work [23], Maji et al. [20] gave a detailed theoretical study on soft sets. Furthermore, Ali et al. [5] proposed some new operations on soft sets, such as restricted intersection, restricted union and restricted difference. Çağman and Enginoğlu [6] defined soft matrices, which are representatives of soft sets. In [10], Gong et al. presented the

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bijective soft sets, which are special soft sets. As an extended concept of bijective soft sets, the exclusive disjunctive soft sets [29] were introduced. Furthermore, Jiang et al. [14] analyzed the existing problems of soft set theory, and presented an extended soft set theory by using the concepts of description logics to act as the parameters of soft sets. Also, the algebraic structures and hyperalgebraic structures of soft sets have been studied increasingly. Aktaş and Çağman [2] introduced soft groups and considered the relationship between fuzzy groups and soft groups. Feng et al. [9] presented soft semirings, Acar et al. [1] defined soft rings, and Jun et al. [15] discussed the soft ordered semigroups. Yamak et al. [33] considered soft hypergroupoids, and we studied soft polygroups [26] and soft hypermodules [28].

Due to the fuzzy characters of parameters, the situation may be more complex in the practical applications of soft sets [34]. By combining fuzzy sets with soft sets, Maji et al. [18] defined fuzzy soft sets. Majumdar and Samanta [22] further generalised the concept of fuzzy soft sets, and introduced generalised fuzzy soft sets, in which a degree is attached with the parametrization of fuzzy sets while defining a fuzzy soft set. Yang et al. [32] presented interval-valued fuzzy soft sets, which is based on a combination of interval-valued fuzzy sets and soft sets. Maji et al. [17], [19], [21] defined intuitionistic fuzzy soft sets by combining intuitionistic fuzzy sets with soft sets. Moreover, Jiang et al. [13] introduced the notion of interval-valued intuitionistic fuzzy soft sets, which is a combination of interval-valued intuitionistic fuzzy sets and soft sets. Also, interval-valued intuitionistic fuzzy soft sets can be considered as intuitionistic fuzzy extension of interval-valued fuzzy soft sets or interval-valued fuzzy extension of intuitionistic fuzzy soft sets. Furthermore, We [27] discussed the necessity and possibility operations on interval-valued intuitionistic fuzzy soft sets, and Zhang et al. [36] developed an adjustable approach to decision making problems based on interval-valued intuitionistic fuzzy soft sets. In this paper, we further consider interval-valued intuitionistic fuzzy soft sets. Some new operations on interval-valued intuitionistic fuzzy soft sets are defined, and some related properties are investigated.

2. Preliminaries

We review some notions about interval-valued intuitionistic fuzzy sets and interval-valued intuitionistic fuzzy soft sets. Let U be an initial universe set, E be the universe set of parameters with respect to U and $A \subseteq E$. $\mathcal{IVIF}(U)$ denotes the set of all interval-valued intuitionistic fuzzy sets of U.

Let $D[0,1]$ denote the set of all closed subintervals of $[0,1]$. An interval-valued fuzzy set [11] A on a universe X is defined by $A = \{\langle x, \mu_A(x)\rangle | x \in X\}$, where $\mu_A : X \to D[0,1]$. For every $x \in X$, $\mu_A(x) = [\underline{\mu}_A(x), \overline{\mu}_A(x)]$ is called the degree of membership of an element x to A. $\underline{\mu}_A(x)$ and $\overline{\mu}_A(x)$ are referred to as the lower and upper degrees of membership of x to A, where $0 \leq \underline{\mu}_A(x) \leq \overline{\mu}_A(x) \leq 1$.

Definition 2.1. ([3]) An interval-valued intuitionistic fuzzy set A over a universe

X is an object having the form $A = \{\langle x, \mu_A(x), \gamma_A(x)\rangle | x \in X\}$, where $\mu_A : X \to D[0,1]$ and $\gamma_A : X \to D[0,1]$ satisfying the condition $\overline{\mu}_A(x) + \overline{\gamma}_A(x) \leq 1$ for all $x \in X$. The intervals $\mu_A(x)$ and $\gamma_A(x)$ denote the degree of membership and the degree of nonmembership of an element x to A, respectively.

Definition 2.2. ([4]) Let A and B be two interval-valued intuitionistic fuzzy sets of a universe X, then

(1) $A \subseteq B$ iff $\underline{\mu}_A(x) \leq \underline{\mu}_B(x)$, $\overline{\mu}_A(x) \leq \overline{\mu}_B(x)$, $\underline{\gamma}_A(x) \geq \underline{\gamma}_B(x)$ and $\overline{\gamma}_A(x) \geq \overline{\gamma}_B(x)$, for all $x \in X$;

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$;

(3) $A^C = \{\langle x, \gamma_A(x), \mu_A(x)\rangle | x \in X\}$;

(4) $A \cup B = \{\langle x, [\sup(\underline{\mu}_A(x), \underline{\mu}_B(x)), \sup(\overline{\mu}_A(x), \overline{\mu}_B(x))], [\inf(\underline{\gamma}_A(x), \underline{\gamma}_B(x)), \inf(\overline{\gamma}_A(x), \overline{\gamma}_B(x))]\rangle | x \in X\}$;

(5) $A \cap B = \{\langle x, [\inf(\underline{\mu}_A(x), \underline{\mu}_B(x)), \inf(\overline{\mu}_A(x), \overline{\mu}_B(x))], [\sup(\underline{\gamma}_A(x), \underline{\gamma}_B(x)), \sup(\overline{\gamma}_A(x), \overline{\gamma}_B(x))]\rangle | x \in X\}$;

(6) $A + B = \{\langle x, [\underline{\mu}_A(x) + \underline{\mu}_B(x) - \underline{\mu}_A(x) \cdot \underline{\mu}_B(x), \overline{\mu}_A(x) + \overline{\mu}_B(x) - \overline{\mu}_A(x) \cdot \overline{\mu}_B(x)], [\underline{\gamma}_A(x) \cdot \underline{\gamma}_B(x), \overline{\gamma}_A(x) \cdot \overline{\gamma}_B(x)]\rangle | x \in X\}$;

(7) $A \cdot B = \{\langle x, [\underline{\mu}_A(x) \cdot \underline{\mu}_B(x), \overline{\mu}_A(x) \cdot \overline{\mu}_B(x)], [\underline{\gamma}_A(x) + \underline{\gamma}_B(x) - \underline{\gamma}_A(x) \cdot \underline{\gamma}_B(x), \overline{\gamma}_A(x) + \overline{\gamma}_B(x) - \overline{\gamma}_A(x) \cdot \overline{\gamma}_B(x)]\rangle | x \in X\}$;

(8) $\Box A = \{\langle x, \mu_A(x), [\underline{\gamma}_A(x), 1 - \overline{\mu}_A(x)]\rangle | x \in X\}$;

(9) $\Diamond A = \{\langle x, [\underline{\mu}_A(x), 1 - \overline{\gamma}_A(x)], \gamma_A(x)\rangle | x \in X\}$.

If A and B are two interval-valued intuitionistic fuzzy sets over universes X_1 and X_2, respectively, then $A \times B = \{\langle\langle x, y\rangle, [\underline{\mu}_A(x) \cdot \underline{\mu}_B(y), \overline{\mu}_A(x) \cdot \overline{\mu}_B(y)], [\underline{\gamma}_A(x) \cdot \underline{\gamma}_B(y), \overline{\gamma}_A(x) \cdot \overline{\gamma}_B(y)]\rangle | x \in X_1, y \in X_2\}$.

Definition 2.3. ([13]) A pair $\langle F, A\rangle$ is called an interval-valued intuitionistic fuzzy soft set over U, where F is a mapping given by $F : A \to \mathcal{IVIF}(U)$.

An interval-valued intuitionistic fuzzy soft set is a parameterized family of interval-valued intuitionistic fuzzy subsets of U. For any parameter $\varepsilon \in A$, $F(\varepsilon)$ is referred as the interval intuitionistic fuzzy value set of parameter ε. It is actually an interval-valued intuitionistic fuzzy set of U, written as

$$F(\varepsilon) = \{\langle x, \mu_{F(\varepsilon)}(x), \gamma_{F(\varepsilon)}(x)\rangle \mid x \in U\},$$

where $\mu_{F(\varepsilon)}(x)$ is the interval-valued fuzzy membership degree that object x holds on parameter ε, and $\gamma_{F(\varepsilon)}(x)$ is the interval-valued fuzzy membership degree that object x does not hold on parameter ε.

Definition 2.4. ([13]) Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. Then $\langle F, A\rangle$ is called an interval-valued intuitionistic fuzzy soft subset of $\langle G, B\rangle$ denoted by $\langle F, A\rangle \Subset \langle G, B\rangle$, if the following conditions are satisfied: (1) $A \subseteq B$; (2) for all $\varepsilon \in A$, $F(\varepsilon)$ is an interval-valued intuitionistic fuzzy subset of $G(\varepsilon)$, that is, for all $x \in U$ and $\varepsilon \in A$, $\underline{\mu}_{F(\varepsilon)}(x) \leq \underline{\mu}_{G(\varepsilon)}(x)$, $\overline{\mu}_{F(\varepsilon)}(x) \leq \overline{\mu}_{G(\varepsilon)}(x)$, $\underline{\gamma}_{F(\varepsilon)}(x) \geq \underline{\gamma}_{G(\varepsilon)}(x)$ and $\overline{\gamma}_{F(\varepsilon)}(x) \geq \overline{\gamma}_{G(\varepsilon)}(x)$.

Two interval-valued intuitionistic fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U are said to be interval-valued intuitionistic fuzzy soft equal, denoted by $\langle F, A\rangle = \langle G, B\rangle$, if $\langle F, A\rangle \Subset \langle G, B\rangle$ and $\langle G, B\rangle \Subset \langle F, A\rangle$.

Let $E = \{e_1, ..., e_n\}$ be a set of parameters. The *not* set of E, denoted by $\rceil E$, is defined by $\rceil E = \{\rceil e_1, ..., \rceil e_n\}$, where $\rceil e_i = \textit{not } e_i$ for all $i \in \{1, ..., n\}$, which holds the opposite meaning of parameter e_i.

Definition 2.5. ([13]) The complement of an interval-valued intuitionistic soft set $\langle F, A\rangle$, denoted by $\langle F, A\rangle^C$, is defined by $\langle F, A\rangle^C = \langle F^C, \rceil A\rangle$, where $F^C : \rceil A \to \mathcal{IVIF}(U)$ is a mapping given by $F^C(\varepsilon) = \{\langle x, \gamma_{F(\rceil\varepsilon)}(x), \mu_{F(\rceil\varepsilon)}(x)\rangle| x \in U\}$ for all $\varepsilon \in \rceil A$.

Definition 2.6. ([13]). The union of two interval-valued intuitionistic fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U is the interval-valued intuitionistic fuzzy soft set $\langle H, C\rangle = \langle F, A\rangle \Cup \langle G, B\rangle$, where $C = A \cup B$, and for all $\varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} \{\langle x, \mu_{F(\varepsilon)}(x), \gamma_{F(\varepsilon)}(x)\rangle | x \in U\} & \text{if } \varepsilon \in A - B, \\ \{\langle x, \mu_{G(\varepsilon)}(x), \gamma_{G(\varepsilon)}(x)\rangle | x \in U\} & \text{if } \varepsilon \in B - A, \\ \{\langle x, [\sup(\underline{\mu}_{F(\varepsilon)}(x), \underline{\mu}_{G(\varepsilon)}(x)), \sup(\overline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{G(\varepsilon)}(x))], & \\ \quad [\inf(\underline{\gamma}_{F(\varepsilon)}(x), \underline{\gamma}_{G(\varepsilon)}(x)), \inf(\overline{\gamma}_{F(\varepsilon)}(x), \overline{\gamma}_{G(\varepsilon)}(x))]\rangle | x \in U\} & \text{if } \varepsilon \in A \cap B. \end{cases}$$

Definition 2.7. ([13]). The intersection of two interval-valued intuitionistic fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U is the interval-valued intuitionistic fuzzy soft set $\langle H, C\rangle = \langle F, A\rangle \Cap \langle G, B\rangle$, where $C = A \cup B$, and for all $\varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} \{\langle x, \mu_{F(\varepsilon)}(x), \gamma_{F(\varepsilon)}(x)\rangle | x \in U\} & \text{if } \varepsilon \in A - B, \\ \{\langle x, \mu_{G(\varepsilon)}(x), \gamma_{G(\varepsilon)}(x)\rangle | x \in U\} & \text{if } \varepsilon \in B - A, \\ \{\langle x, [\inf(\underline{\mu}_{F(\varepsilon)}(x), \underline{\mu}_{G(\varepsilon)}(x)), \inf(\overline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{G(\varepsilon)}(x))], & \\ \quad [\sup(\underline{\gamma}_{F(\varepsilon)}(x), \underline{\gamma}_{G(\varepsilon)}(x)), \sup(\overline{\gamma}_{F(\varepsilon)}(x), \overline{\gamma}_{G(\varepsilon)}(x))]\rangle | x \in U\} & \text{if } \varepsilon \in A \cap B. \end{cases}$$

Definition 2.8. ([13]). Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U, then "$\langle F, A\rangle$ or $\langle G, B\rangle$" is an interval-valued intuitionistic fuzzy soft set $\langle H, A{\times}B\rangle{=}\langle F, A\rangle{\vee}\langle G, B\rangle$, where $H(\alpha, \beta){=}F(\alpha) \cup F(\beta)$ for all $(\alpha, \beta){\in}A{\times}B$, that is $H(\alpha, \beta){=}\{\langle x, [\sup(\underline{\mu}_{F(\alpha)}(x), \underline{\mu}_{G(\beta)}(x)), \sup(\overline{\mu}_{F(\alpha)}(x), \overline{\mu}_{G(\beta)}(x))]$, $[\inf(\underline{\gamma}_{F(\alpha)}(x), \underline{\gamma}_{G(\beta)}(x))$, $\inf(\overline{\gamma}_{F(\alpha)}(x), \overline{\gamma}_{G(\beta)}(x))]\rangle | x \in U\}$ for all $(\alpha, \beta) \in A \times B$.

Definition 2.9. ([13]). Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U, then "$\langle F, A\rangle$ and $\langle G, B\rangle$" is an interval-valued intuitionistic fuzzy soft set $\langle H, A \times B\rangle = \langle F, A\rangle \wedge \langle G, B\rangle$, where $H(\alpha, \beta) = F(\alpha) \cap F(\beta)$ for all $(\alpha, \beta) \in A \times B$, that is, $H(\alpha, \beta) = \{\langle x, [\inf(\underline{\mu}_{F(\alpha)}(x), \underline{\mu}_{G(\beta)}(x)), \inf(\overline{\mu}_{F(\alpha)}(x), \overline{\mu}_{G(\beta)}(x))], [\sup(\underline{\gamma}_{F(\alpha)}(x), \underline{\gamma}_{G(\beta)}(x)), \sup(\overline{\gamma}_{F(\alpha)}(x), \overline{\gamma}_{G(\beta)}(x))]\rangle | x \in U\}$ for all $(\alpha, \beta) \in A \times B$.

Definition 2.10. ([27]) The necessity operation on an interval-valued intuitionistic fuzzy soft set $\langle F, A\rangle$ is denoted by $\Box\langle F, A\rangle$ and is defined as $\Box\langle F, A\rangle = \{\langle x, \mu_{\Box F(\varepsilon)}(x), \gamma_{\Box F(\varepsilon)}(x)\rangle | x \in U \text{ and } \varepsilon \in A\}$, where $\mu_{\Box F(\varepsilon)}(x) = [\underline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x)]$ is the interval-valued fuzzy membership degree that object x holds on parameter ε, and $\gamma_{\Box F(\varepsilon)}(x) = [\underline{\gamma}_{F(\varepsilon)}(x), 1 - \overline{\mu}_{F(\varepsilon)}(x)]$ is the interval-valued fuzzy membership degree that object x does not hold on parameter ε.

Definition 2.11. ([27]) The possibility operation on an interval-valued intuitionistic fuzzy soft set $\langle F, A\rangle$ is denoted by $\Diamond\langle F, A\rangle$ and is defined as $\Diamond\langle F, A\rangle = \{\langle x, \mu_{\Diamond F(\varepsilon)}(x), \gamma_{\Diamond F(\varepsilon)}(x)\rangle | x \in U \text{ and } \varepsilon \in A\}$, where $\gamma_{\Diamond F(\varepsilon)}(x) = [\underline{\gamma}_{F(\varepsilon)}(x), \overline{\gamma}_{F(\varepsilon)}(x)]$ is the interval-valued fuzzy membership degree that object x does not hold on parameter ε, and $\mu_{\Diamond F(\varepsilon)}(x) = [\underline{\mu}_{F(\varepsilon)}(x), 1 - \overline{\gamma}_{F(\varepsilon)}(x)]$ is the interval-valued fuzzy membership degree that object x holds on parameter ε.

3. Some new operations on interval-valued intuitionistic fuzzy soft sets

In this section, we give several new operations on interval-valued intuitionistic fuzzy soft sets, and investigate some related properties.

Definition 3.1. The operation "$\cdot$" of two interval-valued intuitionistic fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U is the intuitionistic fuzzy soft set $\langle H, C\rangle = \langle F, A\rangle \cdot \langle G, B\rangle$, where $C = A \cup B$ and for all $\varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x) \cdot \underline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x) \cdot \overline{\mu}_{F(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{F(\varepsilon)}(x) + \underline{\gamma}_{F(\varepsilon)}(x) - \underline{\gamma}_{F(\varepsilon)}(x) \cdot \underline{\gamma}_{F(\varepsilon)}(x), \\ \quad \overline{\gamma}_{F(\varepsilon)}(x) + \overline{\gamma}_{F(\varepsilon)}(x) - \overline{\gamma}_{F(\varepsilon)}(x) \cdot \overline{\gamma}_{F(\varepsilon)}(x)]\rangle | x \in U\} & \text{if } \varepsilon \in A - B, \\ \{\langle x, [\underline{\mu}_{G(\varepsilon)}(x) \cdot \underline{\mu}_{G(\varepsilon)}(x), \overline{\mu}_{G(\varepsilon)}(x) \cdot \overline{\mu}_{G(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{G(\varepsilon)}(x) + \underline{\gamma}_{G(\varepsilon)}(x) - \underline{\gamma}_{G(\varepsilon)}(x) \cdot \underline{\gamma}_{G(\varepsilon)}(x), \\ \quad \overline{\gamma}_{G(\varepsilon)}(x) + \overline{\gamma}_{G(\varepsilon)}(x) - \overline{\gamma}_{G(\varepsilon)}(x) \cdot \overline{\gamma}_{G(\varepsilon)}(x)]\rangle | x \in U\} & \text{if } \varepsilon \in B - A, \\ \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x) \cdot \underline{\mu}_{G(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x) \cdot \overline{\mu}_{G(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{F(\varepsilon)}(x) + \underline{\gamma}_{G(\varepsilon)}(x) - \underline{\gamma}_{F(\varepsilon)}(x) \cdot \underline{\gamma}_{G(\varepsilon)}(x), \\ \quad \overline{\gamma}_{F(\varepsilon)}(x) + \overline{\gamma}_{G(\varepsilon)}(x) - \overline{\gamma}_{F(\varepsilon)}(x) \cdot \overline{\gamma}_{G(\varepsilon)}(x)]\rangle | x \in U\} & \text{if } \varepsilon \in A \cap B. \end{cases}$$

We can write $\langle F, A\rangle \cdot \langle F, A\rangle = \langle F, A\rangle^2$. For any positive integer n, $\langle F, A\rangle^n = \{\langle x, \mu_{F(\varepsilon)^n}(x), \gamma_{F(\varepsilon)^n}(x)\rangle | x \in U \text{ and } \varepsilon \in A\}$, where $\mu_{F(\varepsilon)^n}(x) = [(\underline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x))^n]$ and $\gamma_{F(\varepsilon)^n}(x) = [1 - (1 - \underline{\gamma}_{F(\varepsilon)}(x))^n, 1 - (1 - \overline{\gamma}_{F(\varepsilon)}(x))^n]$.

Example 3.2. Let $\langle F,A\rangle$ be the interval-valued intuitionistic fuzzy soft set, which describes the "attractiveness of the houses" to Mr. X (say), where U is a set of three houses under the consideration of Mr. X to purchase, denoted by $U=\{h_1,h_2,h_3\}$, $A=\{$convenient traffic, wooden, in good repair$\}$ is a parameter set, and

$$F(\text{convenient traffic})=\{\langle h_1,[0.5,0.7],[0.25,0.3]\rangle,\langle h_2,[0.6,0.75],[0.15,0.25]\rangle,\langle h_3,[0.85,0.9],[0.03,0.1]\rangle\};$$

$$F(\text{wooden})=\{\langle h_1,[0.6,0.75],[0.2,0.25]\rangle,\langle h_2,[0.73,0.82],[0.1,0.15]\rangle,\langle h_3,[0.55,0.65],[0.26,0.35]\rangle\};$$

$$F(\text{in good repair})=\{\langle h_1,[0.76,0.85],[0.08,0.15]\rangle,\langle h_2,[0.55,0.65],[0.2,0.3]\rangle,\langle h_3,[0.7,0.8],[0.15,0.2]\rangle\}.$$

Consider the interval-valued intuitionistic fuzzy soft set $\langle G,B\rangle$ over U, which describes the "attractiveness of the houses" to Mrs. X, where $B=\{$beautiful, wooden, in good repair, in the green surroundings$\}$, and

$$G(\text{beautiful})=\{\langle h_1,[0.8,0.9],[0.05,0.1]\rangle,\langle h_2,[0.65,0.8],[0.1,0.2]\rangle,\langle h_3,[0.7,0.75],[0.2,0.25]\rangle\};$$

$$G(\text{wooden})=\{\langle h_1,[0.72,0.8],[0.15,0.2]\rangle,\langle h_2,[0.6,0.7],[0.2,0.3]\rangle,\langle h_3,[0.65,0.8],[0.15,0.2]\rangle\};$$

$$G(\text{in good repair})=\{\langle h_1,[0.7,0.8],[0.1,0.2]\rangle,\langle h_2,[0.6,0.75],[0.2,0.25]\rangle,\langle h_3,[0.65,0.85],[0.1,0.15]\rangle\};$$

$$G(\text{in the green surroundings})=\{\langle h_1,[0.75,0.85],[0.1,0.15]\rangle,\langle h_2,[0.7,0.8],[0.05,0.2]\rangle,\langle h_3,[0.5,0.6],[0.25,0.35]\rangle\}.$$

According to Definition 3.1, $\langle H,C\rangle=\langle F,A\rangle\cdot\langle G,B\rangle$, where $C=\{$beautiful, convenient traffic, wooden, in good repair$\}$, and

$$H(\text{convenient traffic})=\{\langle h_1,[0.25,0.49],[0.4375,0.51]\rangle,\langle h_2,[0.36,0.5625],[0.2775,0.4375]\rangle,\langle h_3,[0.7225,0.81],[0.0591,0.19]\rangle\};$$

$$H(\text{beautiful})=\{\langle h_1,[0.64,0.81],[0.0975,0.19]\rangle,\langle h_2,[0.4225,0.64],[0.19,0.36]\rangle,\langle h_3,[0.49,0.5625],[0.36,0.4375]\rangle\};$$

$$H(\text{wooden})=\{\langle h_1,[0.432,0.6],[0.32,0.4]\rangle,\langle h_2,[0.438,0.574],[0.28,0.405]\rangle,\langle h_3,[0.3575,0.52],[0.371,0.48]\rangle\};$$

$$H(\text{in good repair})=\{\langle h_1,[0.532,0.68],[0.172,0.32]\rangle,\langle h_2,[0.33,0.4875],[0.36,0.475]\rangle,\langle h_3,[0.455,0.68],[0.235,0.32]\rangle\};$$

$$H(\text{in the green surroundings})=\{\langle h_1,[0.5625,0.7225],[0.19,0.2775]\rangle,\langle h_2,[0.49,0.64],[0.0975,0.36]\rangle,\langle h_3,[0.25,0.36],[0.4375,0.5775]\rangle\}.$$

Theorem 3.3. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer m,n, we have the following properties:*

(1) $\langle F,A\rangle\cdot\langle G,B\rangle=\langle G,B\rangle\cdot\langle F,A\rangle$;

(2) $(\langle F, A\rangle \cdot \langle G, B\rangle)^n = \langle F, A\rangle^n \cdot \langle G, B\rangle^n$;

(3) $\langle F, A\rangle^m \cdot \langle F, A\rangle^n = \langle F, A\rangle^{m+n}$.

Proof. (1) It is straightforward.

(2) By Definition 3.1, we have $(\langle F, A\rangle \cdot \langle G, B\rangle)^n = (H, C)^n$, where $C = A \cup B$, and for all $\varepsilon \in C$,

$$H(\varepsilon)^n = \begin{cases} \{\langle x, [(\underline{\mu}_{F(\varepsilon)}(x) \cdot \underline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x) \cdot \overline{\mu}_{F(\varepsilon)}(x))^n], \\ \quad [1-(1-(\underline{\gamma}_{F(\varepsilon)}(x) + \underline{\gamma}_{F(\varepsilon)}(x) - \underline{\gamma}_{F(\varepsilon)}(x) \cdot \underline{\gamma}_{F(\varepsilon)}(x)))^n, \\ \quad 1-(1-(\overline{\gamma}_{F(\varepsilon)}(x) + \overline{\gamma}_{F(\varepsilon)}(x) - \overline{\gamma}_{F(\varepsilon)}(x) \cdot \overline{\gamma}_{F(\varepsilon)}(x)))^n]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in A - B, \\ \{\langle x, [(\underline{\mu}_{G(\varepsilon)}(x) \cdot \underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{G(\varepsilon)}(x) \cdot \overline{\mu}_{G(\varepsilon)}(x))^n], \\ \quad [1-(1-(\underline{\gamma}_{G(\varepsilon)}(x) + \underline{\gamma}_{G(\varepsilon)}(x) - \underline{\gamma}_{G(\varepsilon)}(x) \cdot \underline{\gamma}_{G(\varepsilon)}(x)))^n, \\ \quad 1-(1-(\overline{\gamma}_{G(\varepsilon)}(x) + \overline{\gamma}_{G(\varepsilon)}(x) - \overline{\gamma}_{G(\varepsilon)}(x) \cdot \overline{\gamma}_{G(\varepsilon)}(x)))^n]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in B - A, \\ \{\langle x, [(\underline{\mu}_{F(\varepsilon)}(x) \cdot \underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x) \cdot \overline{\mu}_{G(\varepsilon)}(x))^n], \\ \quad [1-(1-(\underline{\gamma}_{F(\varepsilon)}(x) + \underline{\gamma}_{G(\varepsilon)}(x) - \underline{\gamma}_{F(\varepsilon)}(x) \cdot \underline{\gamma}_{G(\varepsilon)}(x)))^n, \\ \quad 1-(1-(\overline{\gamma}_{F(\varepsilon)}(x) + \overline{\gamma}_{G(\varepsilon)}(x) - \overline{\gamma}_{F(\varepsilon)}(x) \cdot \overline{\gamma}_{G(\varepsilon)}(x)))^n]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in A \cap B. \end{cases}$$

Since $\langle F, A\rangle^n = \{\langle x, [(\underline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x))^n], [1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n]\rangle | x \in U$ and $\varepsilon \in A\}$ and $\langle G, B\rangle^n = \{\langle x, [(\underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{G(\varepsilon)}(x))^n], [1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n]\rangle | x \in U$ and $\varepsilon \in B\}$, according to Definition 3.1, we can write $\langle F, A\rangle^n \cdot \langle G, B\rangle^n = (O, C)$, where $C = A \cup B$, and for all $\varepsilon \in C$,

$$O(\varepsilon) = \begin{cases} \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x))^n \cdot \underline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x))^n \cdot (\overline{\mu}_{F(\varepsilon)}(x))^n], \\ \quad [1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n + 1 - (1-\underline{\gamma}_{F(\varepsilon)}(x))^n \\ \quad -(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n) \cdot (1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n), 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n + 1 \\ \quad -(1-\overline{\gamma}_{F(\varepsilon)}(x))^n - (1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n) \cdot (1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n)]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in A - B, \\ \{\langle x, [\underline{\mu}_{G(\varepsilon)}(x))^n \cdot \underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{G(\varepsilon)}(x))^n \cdot (\overline{\mu}_{G(\varepsilon)}(x))^n], [1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n + 1 \\ \quad -(1-\underline{\gamma}_{G(\varepsilon)}(x))^n - (1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n) \cdot (1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n), \\ \quad 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n + 1 - (1-\overline{\gamma}_{G(\varepsilon)}(x))^n \\ \quad -(1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n) \cdot (1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n)]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in B - A, \\ \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x))^n \cdot \underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x))^n \cdot (\overline{\mu}_{G(\varepsilon)}(x))^n], [1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n \\ \quad +1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n - (1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n) \cdot (1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n), \\ \quad 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n + 1 - (1-\overline{\gamma}_{G(\varepsilon)}(x))^n \\ \quad -(1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n) \cdot (1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n)]\rangle | x \in U\} \\ \quad \text{if } \varepsilon \in A \cap B. \end{cases}$$

We have that

$$
\begin{aligned}
1-&(1-\underline{\gamma}_{F(\varepsilon)}(x))^n+1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n-(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n)\cdot(1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n)\\
&= (1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n)\cdot(1-(1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n))+1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n\\
&= (1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n)\cdot(1-\underline{\gamma}_{G(\varepsilon)}(x))^n+1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n\\
&= (1-\underline{\gamma}_{G(\varepsilon)}(x))^n-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n\cdot(1-\underline{\gamma}_{G(\varepsilon)}(x))^n+1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n\\
&= 1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n\cdot(1-\underline{\gamma}_{G(\varepsilon)}(x))^n\\
&= 1-(1-(\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x)))^n.
\end{aligned}
$$

Consequently, $\langle H,C\rangle^n$ and $\langle O,C\rangle$ are the same interval-valued intuitionistic fuzzy soft set. Therefore, $(\langle F,A\rangle\cdot\langle G,B\rangle)^n=\langle F,A\rangle^n\cdot\langle G,B\rangle^n$.

(3) From Definition 3.1, it follows that

$$
\begin{aligned}
\langle F,A\rangle^m\cdot\langle F,A\rangle^n &= \{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^m\cdot(\underline{\mu}_{F(\varepsilon)}(x))^n,(\overline{\mu}_{F(\varepsilon)}(x))^m\cdot(\overline{\mu}_{F(\varepsilon)}(x))^n],\\
&\quad[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^m+1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n\\
&\quad-(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^m)\cdot(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n),\\
&\quad 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^m+1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n\\
&\quad-(1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^m)\cdot(1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n)]\rangle|x\in U \text{ and } \varepsilon\in A\}\\
&= \{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^{m+n},(\overline{\mu}_{F(\varepsilon)}(x))^{m+n}],[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^m\cdot(1-\underline{\gamma}_{F(\varepsilon)}(x))^n),\\
&\quad 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^m\cdot(1-\overline{\gamma}_{F(\varepsilon)}(x))^n]\rangle|x\in U \text{ and } \varepsilon\in A\}\\
&= \{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^{m+n},(\overline{\mu}_{F(\varepsilon)}(x))^{m+n}],[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^{m+n},\\
&\quad 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^{m+n}]\rangle|x\in U \text{ and } \varepsilon\in A\}\\
&= \langle F,A\rangle^{m+n}. \quad\blacksquare
\end{aligned}
$$

Theorem 3.4. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer n, we have the following properties:*

(1) $\Box\langle F,A\rangle^n=(\Box\langle F,A\rangle)^n$;

(2) $\Diamond\langle F,A\rangle^n=(\Diamond\langle F,A\rangle)^n$.

Proof. (1) Since $\langle F,A\rangle^n=\{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^n,(\overline{\mu}_{F(\varepsilon)}(x))^n],[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n]\rangle|x\in U \text{ and } \varepsilon\in A\}$, we have

$$
\Box\langle F,A\rangle^n=\{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^n,(\overline{\mu}_{F(\varepsilon)}(x))^n],[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n,1-(\overline{\mu}_{F(\varepsilon)}(x))^n]\rangle| x\in U \text{ and } \varepsilon\in A\}.
$$

Since $\Box\langle F,A\rangle=\{\langle x,[\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)],[\underline{\gamma}_{F(\varepsilon)}(x),1-\overline{\mu}_{F(\varepsilon)}(x)]\rangle|x\in U \text{ and } \varepsilon\in A\}$, we have

$$\begin{aligned}(\Box\langle F,A\rangle)^n &= \{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^n,(\overline{\mu}_{F(\varepsilon)}(x))^n],[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n,\\ &\quad 1-(1-(1-\overline{\mu}_{F(\varepsilon)}(x)))^n]\rangle|x\in U \text{ and } \varepsilon\in A\}\\ &= \{\langle x,[(\underline{\mu}_{F(\varepsilon)}(x))^n,(\overline{\mu}_{F(\varepsilon)}(x))^n],[1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n,\\ &\quad 1-(\overline{\mu}_{F(\varepsilon)}(x))^n]\rangle|x\in U \text{ and } \varepsilon\in A\}\\ &= \Box\langle F,A\rangle^n.\end{aligned}$$

(2) The proof is similar to that of (1). ∎

Theorem 3.5. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer m,n, we have the following properties:*

(1) *if $m\geq n$, then $\langle F,A\rangle^m\subseteq\langle F,A\rangle^n$;*

(2) *if $\langle F,A\rangle\Subset\langle G,B\rangle$, then $\langle F,A\rangle^n\Subset\langle G,B\rangle^n$;*

(3) *$(\langle F,A\rangle\Cup\langle G,B\rangle)^n=\langle F,A\rangle^n\Cup\langle G,B\rangle^n$;*

(4) *$(\langle F,A\rangle\Cap\langle G,B\rangle)^n=\langle F,A\rangle^n\Cap\langle G,B\rangle^n$;*

(5) *$(\langle F,A\rangle\vee\langle G,B\rangle)^n=\langle F,A\rangle^n\vee\langle G,B\rangle^n$;*

(6) *$(\langle F,A\rangle\wedge\langle G,B\rangle)^n=\langle F,A\rangle^n\wedge\langle G,B\rangle^n$.*

Proof. (1) Since $0\leq\underline{\mu}_{F(\varepsilon)}(x)\leq 1$, $0\leq\overline{\mu}_{F(\varepsilon)}(x)\leq 1$, $0\leq 1-\underline{\gamma}_{F(\varepsilon)}(x)\leq 1$ and $0\leq 1-\overline{\gamma}_{F(\varepsilon)}(x)\leq 1$, we have $(\underline{\mu}_{F(\varepsilon)}(x))^m\leq(\underline{\mu}_{F(\varepsilon)}(x))^n$, $(\overline{\mu}_{F(\varepsilon)}(x))^m\leq(\overline{\mu}_{F(\varepsilon)}(x))^n$, $1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^m\geq 1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n$ and $1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^m\geq 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n$. Thus, we deduce that $\langle F,A\rangle^m\subseteq\langle F,A\rangle^n$.

(2) Since $\langle F,A\rangle\Subset\langle G,B\rangle$, we have $\underline{\mu}_{F(\varepsilon)}(x)\leq\underline{\mu}_{G(\varepsilon)}(x)$, $\overline{\mu}_{F(\varepsilon)}(x)\leq\overline{\mu}_{G(\varepsilon)}(x)$, $\underline{\gamma}_{F(\varepsilon)}(x)\geq\underline{\gamma}_{G(\varepsilon)}(x)$ and $\overline{\gamma}_{F(\varepsilon)}(x)\geq\overline{\mu}_{G(\varepsilon)}(x)$ for all $x\in U$ and $\varepsilon\in A$. It follows that $(\underline{\mu}_{F(\varepsilon)}(x))^n\leq(\underline{\mu}_{G(\varepsilon)}(x))^n$, $(\overline{\mu}_{F(\varepsilon)}(x))^n\leq(\overline{\mu}_{G(\varepsilon)}(x))^n$, $1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n\geq 1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n$ and $1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n\geq 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n$ for all $x\in U$ and $\varepsilon\in A$. Therefore, we have $\langle F,A\rangle^n\Subset\langle G,B\rangle^n$.

(3) From Definition 2.6, we can write $\langle F,A\rangle\Cup\langle G,B\rangle=\langle H,C\rangle$, where $C=A\cup B$, and for all $\varepsilon\in C$,

$$H(\varepsilon)=\begin{cases}\{\langle x,\mu_{F(\varepsilon)}(x),\gamma_{F(\varepsilon)}(x)\rangle|x\in U\} & \text{if } \varepsilon\in A-B,\\ \{\langle x,\mu_{G(\varepsilon)}(x),\gamma_{G(\varepsilon)}(x)\rangle|x\in U\} & \text{if } \varepsilon\in B-A,\\ \{\langle x,[\sup(\underline{\mu}_{F(\varepsilon)}(x),\underline{\mu}_{G(\varepsilon)}(x)),\sup(\overline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x))], & \\ \quad[\inf(\underline{\gamma}_{F(\varepsilon)}(x),\underline{\gamma}_{G(\varepsilon)}(x)),\inf(\overline{\gamma}_{F(\varepsilon)}(x),\overline{\gamma}_{G(\varepsilon)}(x))]\rangle|x\in U\} & \text{if } \varepsilon\in A\cap B.\end{cases}$$

Then $(\langle F, A\rangle \Cup \langle G, B\rangle)^n = \langle H, C\rangle^n$, where $C = A \cup B$, and for all $\varepsilon \in C$,

$$H(\varepsilon)^n = \begin{cases} \{\langle x, [(\underline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{F(\varepsilon)}(x))^n], \\ \quad [1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n]\rangle | x \in U\} & \text{if } \varepsilon \in A - B, \\ \{\langle x, [(\underline{\mu}_{G(\varepsilon)}(x))^n, (\overline{\mu}_{G(\varepsilon)}(x))^n], \\ \quad [1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n]\rangle | x \in U\} & \text{if } \varepsilon \in B - A, \\ \{\langle x, [(\sup(\underline{\mu}_{F(\varepsilon)}(x), \underline{\mu}_{G(\varepsilon)}(x)))^n, (\sup(\overline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{G(\varepsilon)}(x)))^n], \\ \quad [1-(1-\inf(\underline{\gamma}_{F(\varepsilon)}(x), \underline{\gamma}_{G(\varepsilon)}(x)))^n, \\ \quad 1-(1-\inf(\overline{\gamma}_{F(\varepsilon)}(x), \overline{\gamma}_{G(\varepsilon)}(x)))^n]\rangle | x \in U\} \\ = \{\langle x, [\sup((\underline{\mu}_{F(\varepsilon)}(x))^n, (\underline{\mu}_{G(\varepsilon)}(x))^n), \sup((\overline{\mu}_{F(\varepsilon)}(x))^n, (\overline{\mu}_{G(\varepsilon)}(x))^n)], \\ \quad [\inf(1-(1-\underline{\gamma}_{F(\varepsilon)}(x))^n, 1-(1-\underline{\gamma}_{G(\varepsilon)}(x))^n), \\ \quad \inf(1-(1-\overline{\gamma}_{F(\varepsilon)}(x))^n, 1-(1-\overline{\gamma}_{G(\varepsilon)}(x))^n)]\rangle | x \in U\} & \text{if } \varepsilon \in A \cap B. \end{cases}$$

Consequently, $(\langle F, A\rangle \Cup \langle G, B\rangle)^n = \langle F, A\rangle^n \Cup \langle G, B\rangle^n$.

(4) The proof is similar to that of (3).

(5) From Definition 2.8, we can write $\langle F, A\rangle \vee \langle G, B\rangle = \langle O, A \times B\rangle$, where

$$O(\alpha,\beta) = \{\langle x, [\sup(\underline{\mu}_{F(\alpha)}(x), \underline{\mu}_{G(\beta)}(x)), \sup(\overline{\mu}_{F(\alpha)}(x), \overline{\mu}_{G(\beta)}(x))], \\ [\inf(\underline{\gamma}_{F(\alpha)}(x), \underline{\gamma}_{G(\beta)}(x)), \inf(\overline{\gamma}_{F(\alpha)}(x), \overline{\gamma}_{G(\beta)}(x))]\rangle | x \in U\}$$

for all $(\alpha, \beta) \in A \times B$. Thus, $(\langle F, A\rangle \vee \langle G, B\rangle)^n = \langle O, A \times B\rangle^n$, where for all $(\alpha, \beta) \in A \times B$,

$$O(\alpha,\beta)^n = \{\langle x, [(\sup(\underline{\mu}_{F(\alpha)}(x), \underline{\mu}_{G(\beta)}(x)))^n, (\sup(\overline{\mu}_{F(\alpha)}(x), \overline{\mu}_{G(\beta)}(x)))^n], \\ [1-(1-\inf(\underline{\gamma}_{F(\alpha)}(x), \underline{\gamma}_{G(\beta)}(x)))^n, 1-(1-\inf(\overline{\gamma}_{F(\alpha)}(x), \overline{\gamma}_{G(\beta)}(x)))^n]\rangle | x \in U\}\cdot \\ = \{\langle x, [\sup((\underline{\mu}_{F(\alpha)}(x))^n, (\underline{\mu}_{G(\beta)}(x))^n), \sup((\overline{\mu}_{F(\alpha)}(x))^n, (\overline{\mu}_{G(\beta)}(x))^n)], \\ [\inf(1-(1-\underline{\gamma}_{F(\alpha)}(x))^n, 1-(1-\underline{\gamma}_{G(\beta)}(x))^n), \inf(1-(1-\overline{\gamma}_{F(\alpha)}(x))^n, \\ 1-(1-\overline{\gamma}_{G(\beta)}(x))^n)]\rangle | x \in U\}$$

Hence, we have $(\langle F, A\rangle \vee \langle G, B\rangle)^n = \langle F, A\rangle^n \vee \langle G, B\rangle^n$.

(6) The proof is similar to that of (5). ∎

Definition 3.6. The operation "+" of two interval-valued intuitionistic fuzzy soft sets $\langle F, A\rangle$ and $\langle G, B\rangle$ over U is the intuitionistic fuzzy soft set $\langle H, C\rangle = \langle F, A\rangle + \langle G, B\rangle$, where $C = A \cup B$ and for all $\varepsilon \in C$,

$$H(\varepsilon)=\begin{cases}\{\langle x,[\underline{\mu}_{F(\varepsilon)}(x)+\underline{\mu}_{F(\varepsilon)}(x)-\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{F(\varepsilon)}(x),\\ \quad\overline{\mu}_{F(\varepsilon)}(x)+\overline{\mu}_{F(\varepsilon)}(x)-\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)],\\ \quad[\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x),\overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{F(\varepsilon)}(x)],\rangle|x\in U\} & \text{if } \varepsilon\in A-B,\\ \{\langle x,[\underline{\mu}_{G(\varepsilon)}(x)+\underline{\mu}_{G(\varepsilon)}(x)-\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\\ \quad\overline{\mu}_{G(\varepsilon)}(x)+\overline{\mu}_{G(\varepsilon)}(x)-\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)],\\ \quad[\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\overline{\gamma}_{G(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \varepsilon\in B-A,\\ \{\langle x,[\underline{\mu}_{F(\varepsilon)}(x)+\underline{\mu}_{G(\varepsilon)}(x)-\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\\ \quad\overline{\mu}_{F(\varepsilon)}(x)+\overline{\mu}_{G(\varepsilon)}(x)-\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)],\\ \quad[\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \varepsilon\in A\cap B.\end{cases}$$

We can write $\langle F,A\rangle+\langle F,A\rangle=2\langle F,A\rangle$. For any positive integer n, $n\langle F,A\rangle=\{\langle x,\mu_{nF(\varepsilon)}(x),\gamma_{nF(\varepsilon)}(x)\rangle|x\in U$ and $\varepsilon\in A\}$, where $\mu_{nF(\varepsilon)}(x)=[1-(1-\underline{\mu}_{F(\varepsilon)}(x))^n$, $1-(1-\overline{\mu}_{F(\varepsilon)}(x))^n]$ and $\gamma_{nF(\varepsilon)}(x)=[(\underline{\gamma}_{F(\varepsilon)}(x))^n,(\overline{\gamma}_{F(\varepsilon)}(x))^n]$.

Example 3.7. Let $\langle F,A\rangle$, $\langle G,B\rangle$ be the interval-valued intuitionistic fuzzy soft sets defined in Example 3.2. According to Definition 3.6, $\langle H,C\rangle=\langle F,A\rangle+\langle G,B\rangle$, where $C=\{$beautiful, convenient traffic, wooden, in good repair$\}$, and

H(convenient traffic) $=\{\langle h_1,[0.75,0.91],[0.0625,0.09]\rangle,\langle h_2,[0.84,0.9375],[0.0225,0.0625]\rangle,\langle h_3,[0.9775,0.99],[0.0009,0.01]\rangle\}$;

H(beautiful) $=\{\langle h_1,[0.96,0.99],[0.0025,0.01]\rangle,\langle h_2,[0.8775,0.96],[0.01,0.04]\rangle,\langle h_3,[0.91,0.9375],[0.04,0.0625]\rangle\}$;

H(wooden) $=\{\langle h_1,[0.888,0.95],[0.03,0.05]\rangle,\langle h_2,[0.892,0.946],[0.02,0.045]\rangle,\langle h_3,[0.8425,0.93],[0.039,0.07]\rangle\}$;

H(in good repair) $=\{\langle h_1,[0.928,0.97],[0.008,0.03]\rangle,\langle h_2,[0.82,0.9125],[0.04,0.075]\rangle,\langle h_3,[0.895,0.97],[0.015,0.03]\rangle\}$;

H(in the green surroundings) $=\{\langle h_1,[0.9375,0.9775],[0.01,0.0225]\rangle,\langle h_2,[0.91,0.96],[0.0025,0.04]\rangle,\langle h_3,[0.75,0.84],[0.0625,0.1225]\rangle\}$.

Theorem 3.8. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer m,n, we have the following properties:*

(1) $\langle F,A\rangle+\langle G,B\rangle=\langle G,B\rangle+\langle F,A\rangle$;

(2) $n(\langle F,A\rangle+\langle G,B\rangle)=n\langle F,A\rangle+n\langle G,B\rangle$;

(3) $m\langle F,A\rangle+n\langle F,A\rangle=(m+n)\langle F,A\rangle$.

Proof. The proof is similar to that of Theorem 3.3. ∎

Theorem 3.9. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer n, we have the following properties:*

(1) $\Box n\langle F,A\rangle = n\Box\langle F,A\rangle$;

(2) $\Diamond n\langle F,A\rangle = n\Diamond\langle F,A\rangle$.

Proof. The proof is similar to that of Theorem 3.4. ∎

Theorem 3.10. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U. For any positive integer m,n, we have the following properties:*

(1) *if $m \le n$, then $m\langle F,A\rangle \subseteq n\langle F,A\rangle$;*

(2) *if $\langle F,A\rangle \Subset \langle G,B\rangle$, then $n\langle F,A\rangle \Subset n\langle G,B\rangle$;*

(3) $n(\langle F,A\rangle \Cup \langle G,B\rangle) = n\langle F,A\rangle \Cup n\langle G,B\rangle$;

(4) $n(\langle F,A\rangle \Cap \langle G,B\rangle) = n\langle F,A\rangle \Cap n\langle G,B\rangle$;

(5) $n(\langle F,A\rangle \vee \langle G,B\rangle) = n\langle F,A\rangle \vee n\langle G,B\rangle$;

(6) $n(\langle F,A\rangle \wedge \langle G,B\rangle) = n\langle F,A\rangle \wedge n\langle G,B\rangle$.

Proof. The proof is similar to that of Theorem 3.5. ∎

Theorem 3.11. *Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U, then we have the following properties:*

(1) $\Box(\langle F,A\rangle \cdot \langle G,B\rangle) = \Box\langle F,A\rangle \cdot \Box\langle G,B\rangle$;

(2) $\Box(\langle F,A\rangle + \langle G,B\rangle) = \Box\langle F,A\rangle + \Box\langle G,B\rangle$;

(3) $\Diamond(\langle F,A\rangle \cdot \langle G,B\rangle) = \Diamond\langle F,A\rangle \cdot \Diamond\langle G,B\rangle$;

(4) $\Diamond(\langle F,A\rangle + \langle G,B\rangle) = \Diamond\langle F,A\rangle + \Diamond\langle G,B\rangle$;

(5) $(\langle F,A\rangle \cdot \langle G,B\rangle)^C = \langle F,A\rangle^C + \langle G,B\rangle^C$;

(6) $(\langle F,A\rangle + \langle G,B\rangle)^C = \langle F,A\rangle^C \cdot \langle G,B\rangle^C$.

Proof. (1) From Definition 3.1 and Definition 2.10, we have $\Box(\langle F,A\rangle \cdot \langle G,B\rangle) = \Box\langle H,C\rangle$, where $C = A \cup B$ and for all $\varepsilon \in C$,

$$\Box H(\varepsilon) = \begin{cases} \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{F(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x), \\ \quad 1-\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)]\rangle | x\in U\} & \text{if } \varepsilon \in A-B, \\ \{\langle x, [\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x), \overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x), \\ \quad 1-\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle | x\in U\} & \text{if } \varepsilon \in B-A, \\ \{\langle x, [\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)], \\ \quad [\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x), \\ \quad 1-\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle | x\in U\} & \text{if } \varepsilon \in A\cap B, \end{cases}$$

Since $\Box\langle F,A\rangle=\{\langle x,[\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)],[\underline{\gamma}_{F(\varepsilon)}(x),1-\overline{\mu}_{F(\varepsilon)}(x)]\rangle|x\in U \text{ and } \varepsilon\in A\}$ and $\Box\langle G,B\rangle=\{\langle x,[\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)],[\underline{\gamma}_{G(\varepsilon)}(x),1-\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U \text{ and } \varepsilon\in B\}$, it follows that $\Box\langle F,A\rangle\cdot\Box\langle G,B\rangle=(O,C)$, where $C=A\cup B$, for all $\varepsilon\in C$,

$$O(\varepsilon)=\begin{cases}\{\langle x,[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)],\\ \quad[\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{F(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x),\\ \quad 1-\overline{\mu}_{F(\varepsilon)}(x)+1-\overline{\mu}_{F(\varepsilon)}(x)-(1-\overline{\mu}_{F(\varepsilon)}(x))\cdot(1-\overline{\mu}_{F(\varepsilon)}(x))]\rangle|x\in U\}\\ \quad=\{\langle x,[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)],\\ \quad\ [\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{F(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x),\\ \quad\ 1-\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \varepsilon\in A-B,\\ \{\langle x,[\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)],\\ \quad[\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad 1-\overline{\mu}_{G(\varepsilon)}(x)+1-\overline{\mu}_{G(\varepsilon)}(x)-(1-\overline{\mu}_{G(\varepsilon)}(x))\cdot(1-\overline{\mu}_{G(\varepsilon)}(x))]\rangle|x\in U\}\\ \quad=\{\langle x,[\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)],\\ \quad\ [\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad\ 1-\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \varepsilon\in B-A,\\ \{\langle x,[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)],\\ \quad[\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad 1-\overline{\mu}_{F(\varepsilon)}(x)+1-\overline{\mu}_{G(\varepsilon)}(x)-(1-\overline{\mu}_{F(\varepsilon)}(x))\cdot(1-\overline{\mu}_{G(\varepsilon)}(x))]\rangle|x\in U\}\\ \quad=\{\langle x,[\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)],\\ \quad\ [\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad\ 1-\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \varepsilon\in A\cap B.\end{cases}$$

Therefore, we have $\Box(\langle F,A\rangle\cdot\langle G,B\rangle)=\Box\langle F,A\rangle\cdot\Box\langle G,B\rangle$.

The proofs of (2)-(4) are similar to that of (1).

(5) According to Definition 3.1 and Definition 2.5, we have $(\langle F,A\rangle\cdot\langle G,B\rangle)^C=\langle H,A\cup B\rangle^C=(H^C,\rceil(A\cup B))=(H^C,\rceil A\cup\rceil B)$, where for all $\rceil\varepsilon\in\rceil A\cup\rceil B$,

$$H^C(\rceil\varepsilon)=\begin{cases}\{\langle x,[\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{F(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x),\\ \quad\overline{\gamma}_{F(\varepsilon)}(x)+\overline{\gamma}_{F(\varepsilon)}(x)-\overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{F(\varepsilon)}(x)],\\ \quad[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \rceil\varepsilon\in\rceil A-\rceil B,\\ \{\langle x,[\underline{\gamma}_{G(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad\overline{\gamma}_{G(\varepsilon)}(x)+\overline{\gamma}_{G(\varepsilon)}(x)-\overline{\gamma}_{G(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)],\\ \quad[\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \rceil\varepsilon\in\rceil B-\rceil A,\\ \{\langle x,[\underline{\gamma}_{F(\varepsilon)}(x)+\underline{\gamma}_{G(\varepsilon)}(x)-\underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x),\\ \quad\overline{\gamma}_{F(\varepsilon)}(x)+\overline{\gamma}_{G(\varepsilon)}(x)-\overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)],\\ \quad[\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\} & \text{if } \rceil\varepsilon\in\rceil A\cap\rceil B.\end{cases}$$

Since $\langle F,A\rangle^C=\langle F^C,\rceil A\rangle$, where $F^C(\rceil\varepsilon)=\{\langle x,[\underline{\gamma}_{F(\varepsilon)}(x),\overline{\gamma}_{F(\varepsilon)}(x)],[\underline{\mu}_{F(\varepsilon)}(x),\overline{\mu}_{F(\varepsilon)}(x)]\rangle| x\in U\}$ for all $\rceil\varepsilon\in\rceil A$, and $\langle G,B\rangle^C=\langle G^C,\rceil B\rangle$, where $G^C(\rceil\varepsilon)=\{\langle x,[\underline{\gamma}_{G(\varepsilon)}(x),\overline{\gamma}_{G(\varepsilon)}(x)],[\underline{\mu}_{G(\varepsilon)}(x),\overline{\mu}_{G(\varepsilon)}(x)]\rangle|x\in U\}$ for all $\rceil\varepsilon\in\rceil B$, we have

$$\langle F,A\rangle^C + \langle G,B\rangle^C = \langle F^C, \rceil A\rangle + \langle G^C, \rceil B\rangle.$$

According to Definition 3.5, we can write $\langle F^C, \rceil A\rangle + \langle G^C, \rceil B\rangle = (O, \rceil A \cup \rceil B)$, where

$$O(\rceil\varepsilon) = \begin{cases} \{\langle x, [\underline{\gamma}_{F(\varepsilon)}(x) + \underline{\gamma}_{F(\varepsilon)}(x) - \underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{F(\varepsilon)}(x), \\ \quad \overline{\gamma}_{F(\varepsilon)}(x) + \overline{\gamma}_{F(\varepsilon)}(x) - \overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{F(\varepsilon)}(x)], \\ \quad [\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{F(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{F(\varepsilon)}(x)]\rangle | x\in U\} & \text{if } \rceil\varepsilon \in \rceil A - \rceil B, \\ \{\langle x, [\underline{\gamma}_{G(\varepsilon)}(x) + \underline{\gamma}_{G(\varepsilon)}(x) - \underline{\gamma}_{G(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x), \\ \quad \overline{\gamma}_{G(\varepsilon)}(x) + \overline{\gamma}_{G(\varepsilon)}(x) - \overline{\gamma}_{G(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)], \\ \quad [\underline{\mu}_{G(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x), \overline{\mu}_{G(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle | x\in U\} & \text{if } \rceil\varepsilon \in \rceil B - \rceil A, \\ \{\langle x, [\underline{\gamma}_{F(\varepsilon)}(x) + \underline{\gamma}_{G(\varepsilon)}(x) - \underline{\gamma}_{F(\varepsilon)}(x)\cdot\underline{\gamma}_{G(\varepsilon)}(x), \\ \quad \overline{\gamma}_{F(\varepsilon)}(x) + \overline{\gamma}_{G(\varepsilon)}(x) - \overline{\gamma}_{F(\varepsilon)}(x)\cdot\overline{\gamma}_{G(\varepsilon)}(x)], \\ \quad [\underline{\mu}_{F(\varepsilon)}(x)\cdot\underline{\mu}_{G(\varepsilon)}(x), \overline{\mu}_{F(\varepsilon)}(x)\cdot\overline{\mu}_{G(\varepsilon)}(x)]\rangle | x\in U\} & \text{if } \rceil\varepsilon \in \rceil A \cap \rceil B. \end{cases}$$

Hence, we have $(\langle F,A\rangle \cdot \langle G,B\rangle)^C = \langle F,A\rangle^C + \langle G,B\rangle^C$.

(6) The proof is similar to that of (5). ∎

Definition 3.12. Let $\langle F,A\rangle$ and $\langle G,B\rangle$ be two interval-valued intuitionistic fuzzy soft sets over U_1 and U_2, respectively. The Cartesian product of $\langle F,A\rangle$ and $\langle G,B\rangle$ is the intuitionistic fuzzy soft set $\langle H, A\times B\rangle = \langle F,A\rangle \times \langle G,B\rangle$, where $H(\alpha,\beta) = \{\langle\langle x,y\rangle, [\underline{\mu}_{F(\alpha)}(x)\cdot\underline{\mu}_{G(\beta)}(y), \overline{\mu}_{F(\alpha)}(x)\cdot\overline{\mu}_{G(\beta)}(y)], [\underline{\gamma}_{F(\alpha)}(x)\cdot\underline{\gamma}_{G(\beta)}(y), \overline{\gamma}_{F(\alpha)}(x)\cdot\overline{\gamma}_{G(\beta)}(y)]\rangle | x\in U_1, y\in U_2\}$, for all $\alpha\in A$ and $\beta\in B$.

Example 3.13. Consider the interval-valued intuitionistic fuzzy soft set $\langle F,A\rangle$ over U, defined in Example 3.2, which describes the "attractiveness of the houses", and the interval-valued intuitionistic fuzzy soft set $\langle G,B\rangle$ over V, which describes the "capacity of the building companies", where V is a set of three building companies, denoted by $V = \{c_1, c_2, c_3\}$, $B = \{$high quality, good service$\}$ is a parameter set, and

$$G(\text{high quality}) = \begin{matrix}\{\langle c_1, [0.73, 0.8], [0.1, 0.18]\rangle, \langle c_2, [0.55, 0.6], [0.2, 0.35]\rangle, \\ \langle c_3, [0.65, 0.75], [0.16, 0.23]\rangle\}\end{matrix};$$
$$G(\text{good service}) = \begin{matrix}\{\langle c_1, [0.5, 0.6], [0.28, 0.35]\rangle, \langle c_2, [0.75, 0.85], [0.1, 0.15]\rangle, \\ \langle c_3, [0.63, 0.78], [0.1, 0.2]\rangle\}\end{matrix}.$$

According to Definition 3.12, $\langle H, A\times B\rangle = \langle F,A\rangle \times \langle G,B\rangle$, where

$$\begin{aligned} &H(\text{convenient traffic, high quality}) = \{\langle\langle h_1, c_1\rangle, [0.365, 0.56], [0.025, 0.054]\rangle, \\ &\quad \langle\langle h_1, c_2\rangle, [0.275, 0.42], [0.05, 0.105]\rangle, \langle\langle h_1, c_3\rangle, [0.325, 0.525], [0.04, 0.069]\rangle, \\ &\quad \langle\langle h_2, c_1\rangle, [0.438, 0.6], [0.015, 0.045]\rangle, \langle\langle h_2, c_2\rangle, [0.33, 0.45], [0.03, 0.0875]\rangle, \\ &\quad \langle\langle h_2, c_3\rangle, [0.39, 0.5625], [0.024, 0.0575]\rangle, \\ &\quad \langle\langle h_3, c_1\rangle, [0.6205, 0.72], [0.003, 0.018]\rangle, \\ &\quad \langle\langle h_3, c_2\rangle, [0.4675, 0.54], [0.006, 0.035]\rangle, \\ &\quad \langle\langle h_3, c_3\rangle, [0.5525, 0.675], [0.0048, 0.023]\rangle\}; \end{aligned}$$

$H(\text{convenient traffic, good service}) = \{\langle\langle h_1, c_1\rangle, [0.25, 0.42], [0.07, 0.105]\rangle,$
$\langle\langle h_1, c_2\rangle, [0.375, 0.595], [0.025, 0.045]\rangle, \langle\langle h_1, c_3\rangle, [0.315, 0.546], [0.025, 0.06]\rangle,$
$\langle\langle h_2, c_1\rangle, [0.3, 0.45], [0.042, 0.0875]\rangle, \langle\langle h_2, c_2\rangle, [0.45, 0.6375], [0.015, 0.0375]\rangle,$
$\langle\langle h_2, c_3\rangle, [0.378, 0.585], [0.015, 0.05]\rangle,$
$\langle\langle h_3, c_1\rangle, [0.425, 0.54], [0.0084, 0.035]\rangle,$
$\langle\langle h_3, c_2\rangle, [0.6375, 0.765], [0.003, 0.015]\rangle,$
$\langle\langle h_3, c_3\rangle, [0.5355, 0.702], [0.003, 0.02]\rangle\};$

$H(\text{wooden, high quality}) = \{\langle\langle h_1, c_1\rangle, [0.438, 0.6], [0.02, 0.045]\rangle,$
$\langle\langle h_1, c_2\rangle, [0.33, 0.45], [0.04, 0.0875]\rangle,$
$\langle\langle h_1, c_3\rangle, [0.39, 0.5625], [0.032, 0.0575]\rangle,$
$\langle\langle h_2, c_1\rangle, [0.5329, 0.656], [0.01, 0.027]\rangle,$
$\langle\langle h_2, c_2\rangle, [0.4015, 0.492], [0.02, 0.0525]\rangle,$
$\langle\langle h_2, c_3\rangle, [0.4745, 0.615], [0.016, 0.0345]\rangle,$
$\langle\langle h_3, c_1\rangle, [0.4015, 0.52], [0.026, 0.063]\rangle,$
$\langle\langle h_3, c_2\rangle, [0.3025, 0.39], [0.052, 0.1225]\rangle,$
$\langle\langle h_3, c_3\rangle, [0.3575, 0.4875], [0.0416, 0.0805]\rangle\};$

$H(\text{wooden, good service}) = \{\langle\langle h_1, c_1\rangle, [0.3, 0.45], [0.056, 0.0875]\rangle,$
$\langle\langle h_1, c_2\rangle, [0.45, 0.6375], [0.02, 0.0375]\rangle,$
$\langle\langle h_1, c_3\rangle, [0.378, 0.585], [0.02, 0.05]\rangle,$
$\langle\langle h_2, c_1\rangle, [0.365, 0.492], [0.028, 0.0525]\rangle,$
$\langle\langle h_2, c_2\rangle, [0.5475, 0.697], [0.01, 0.0225]\rangle,$
$\langle\langle h_2, c_3\rangle, [0.4599, 0.6396], [0.01, 0.03]\rangle,$
$\langle\langle h_3, c_1\rangle, [0.275, 0.39], [0.0728, 0.1225]\rangle,$
$\langle\langle h_3, c_2\rangle, [0.4125, 0.5525], [0.026, 0.0525]\rangle,$
$\langle\langle h_3, c_3\rangle, [0.3465, 0.507], [0.026, 0.07]\rangle\};$

$H(\text{in good repair, high quality}) = \{\langle\langle h_1, c_1\rangle, [0.5548, 0.68], [0.008, 0.027]\rangle,$
$\langle\langle h_1, c_2\rangle, [0.418, 0.51], [0.016, 0.0525]\rangle,$
$\langle\langle h_1, c_3\rangle, [0.494, 0.6375], [0.0128, 0.0345]\rangle,$
$\langle\langle h_2, c_1\rangle, [0.4015, 0.52], [0.02, 0.054]\rangle,$
$\langle\langle h_2, c_2\rangle, [0.3025, 0.39], [0.04, 0.105]\rangle,$
$\langle\langle h_2, c_3\rangle, [0.3575, 0.4875], [0.032, 0.069]\rangle,$
$\langle\langle h_3, c_1\rangle, [0.511, 0.64], [0.015, 0.036]\rangle,$
$\langle\langle h_3, c_2\rangle, [0.385, 0.48], [0.03, 0.07]\rangle,$
$\langle\langle h_3, c_3\rangle, [0.455, 0.6], [0.024, 0.046]\rangle\};$

$H(\text{in good repair, good service}) = \{\langle\langle h_1, c_1\rangle, [0.38, 0.51], [0.0224, 0.0525]\rangle,$
$\langle\langle h_1, c_2\rangle, [0.57, 0.7225], [0.008, 0.0225]\rangle,$
$\langle\langle h_1, c_3\rangle, [0.4788, 0.663], [0.008, 0.03]\rangle,$
$\langle\langle h_2, c_1\rangle, [0.275, 0.39], [0.056, 0.105]\rangle,$
$\langle\langle h_2, c_2\rangle, [0.4125, 0.5525], [0.02, 0.045]\rangle,$
$\langle\langle h_2, c_3\rangle, [0.3465, 0.507], [0.02, 0.06]\rangle,$
$\langle\langle h_3, c_1\rangle, [0.35, 0.48], [0.042, 0.07]\rangle,$
$\langle\langle h_3, c_2\rangle, [0.525, 0.68], [0.015, 0.03]\rangle,$
$\langle\langle h_3, c_3\rangle, [0.441, 0.624], [0.015, 0.04]\rangle\}.$

4. Conclusions

In this paper, some new operations on interval-valued intuitionistic fuzzy soft sets, i.e., "·", "+"and Cartesian product, are introduced, and some basic properties are investigated. In the following work, we will consider the entropy measure, similarity measure and inclusion measure of interval-valued intuitionistic fuzzy soft sets and their relations.

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References

[1] Acar, U., Koyuncu, F., Tanay,B., *Soft sets and soft rings,*, Computers and Mathematics with Applications, 59 (2010), 3458-3463.

[2] Aktaş, H., Çağman, N., *Soft sets and soft groups,* Information Sciences, 177 (13) (2007), 2726-2735.

[3] Atanassov, K.T., *Interval valued intuitionistic fuzzy sets,* Fuzzy Sets and Systems, 31 (3) (1989), 343-349.

[4] Atanassov, K.T., *Operators over interval valued intuitionistic fuzzy sets,* Fuzzy Sets and Systems, 64 (1994), 159-174.

[5] Ali, M.I., Feng, F., Liu, X., Min, W.K., Shabira, M., *On some new operations in soft set theory,* Computers and Mathematics with Applications, 57, (9) (2009), 1547-1553.

[6] Çağman, N., Enginoğlu, S., *Soft matrix theory and its decision making,* Computers and Mathematics with Applications, 59 (2010), 3308-3314.

[7] Das, S., Kar, S., *Group decision making in medical system: An intuitionistic fuzzy soft set approach,* Applied Soft Computing, 24 (2014), 196-211.

[8] Deng, T., Wang, X., *An object-parameter approach to predicting unknown data in incomplete fuzzy soft sets,* Applied Mathematical Modelling, 37 (6) (2013), 4139-4146.

[9] Feng, F., Jun, Y.B., Zhao, X., *Soft semirings,* Computers and Mathematics with Applications, 56 (10) (2008), 2621-2628.

[10] Gong, K., Xiao, Z., Zhang, X., *The bijective soft set with its operations,* Computers and Mathematics with Applications, 60 (2010), 2270-2278.

[11] Gorzalzany, M.B., *A method of inference in approximate reasoning based on interval-valued fuzzy sets,* Fuzzy Sets and Systems, 21 (1987), 1-17.

[12] Herawan, T., Deris, M.M., *A soft set approach for association rules mining,* Knowledge-Based Systems, 24 (2011), 186-195.

[13] Jiang, Y., Tang, Y., Chen, Q., Liu, H., Tang, J., *Interval-valued intuitionistic fuzzy soft sets and their properties,* Computers and Mathematics with Applications, 60 (2010), 906-918.

[14] Jiang, Y., Tang, Y., Chen, Q., Wang, J., Tang, S., *Extending soft sets with description logics,* Computers and Mathematics with Applications, 59 (2010), 2087-2096.

[15] Jun, Y.B., Lee, K.J., Khan, A., *Soft ordered semigroups,* Mathematical Logic Quarterly, 56 (1) (2010), 42-50.

[16] Kong, Z., Zhang, G., Wang, L., Wu, Z., et al., *An efficient decision making approach in incomplete soft set,* Applied Mathematical Modelling, 38 (2014), 2141-2150.

[17] Maji, P.K., *More on intuitionistic fuzzy soft sets,* in *Proceedings of the 12th International Conference on Rough Sets, Fuzzy Sets, Data Mining and Granular Computing, Lecture Notes in Computer Science* 5908, Springer, 2009, 231-240.

[18] Maji, P.K., Biswas, R., Roy, A.R., *Fuzzy soft sets,* The Journal of Fuzzy Mathematics, 9 (3) (2001), 589-602, 2001.

[19] Maji, P.K., Biswas, R., Roy, A.R., *Intuitionistic fuzzy soft sets,* The Journal of Fuzzy Mathematics, 9 (3) (2001), 677-692.

[20] Maji, P.K., Biswas, R., Roy, A.R., *Soft set theory,* Computers and Mathematics with Applications, 45 (2003), 555-562.

[21] Maji, P.K., Roy, A.R., Biswas, R., *On intuitionistic fuzzy soft sets,* The Journal of Fuzzy Mathematics, 12 (3) (2004), 669-683.

[22] Majumdar, P., Samanta, S.K., *Generalised fuzzy soft sets,* Computers and Mathematics with Applications, 59 (2010), 1425-1432.

[23] Molodtsov, D., *Soft set theory-First results,* Computers and Mathematics with Applications, 37 (4-5) (1999), 19-31.

[24] Wang, C., Qu, A., *Entropy, similarity measure and distance measure of vague soft sets and their relations,* Information Sciences, 244 (2013), 92-106.

[25] Wang, Y., Liu, Z., Du, Z., Huang, Y., *Mobile cloud computing network attack and defense learning system based on fuzzy soft sets,* Procedia Computer Science, 17 (2013), 214-221.

[26] Wang, J., Yin, M., Gu, W., *Soft polygroups,* Computers and Mathematics with Applications, 62, no. 9, 3529-3537, 2011.

[27] Wang, J., Yin, M., Gu, W., *Notes on: 'Interval-valued intuitionistic fuzzy soft sets and their properties'* [Comput. Math. Appl. 60 (2010) 906-918], Computers and Mathematics with Applications, 64 (2012), 2954-2960.

[28] Wang, J., Yin, M., Gu, W., *Some new results on soft hypermodules,* Ars Combinatoria, 104, 289-306, 2012.

[29] Xiao, Z., Gong, K., Xia, S.S., Zou, Y., *Exclusive disjunctive soft sets,* Computers and Mathematics with Applications, 59 (2010), 2128-2137.

[30] Xiao, Z., Gong, K., Zou, Y., *A combined forecasting approach based on fuzzy soft sets,* Journal of Computational and Applied Mathematics, 228 (1) (2009), 326-333.

[31] Xu, W., Xiao, Z., Dang, X. et al., *Financial ratio selection for business failure prediction using soft set theory,* Knowledge-Based Systems, 63 (2014), 59-67.

[32] Yang, X., Lin, T.Y., Yang, J., Li, Y., Yu, D., *Combination of interval-valued fuzzy set and soft set,* Journal of Computational and Applied Mathematics, 58 (3) (2009), 521-527.

[33] Yamak, S., Kazanci, O., Davvaz, B., *Soft hyperstructure,* Computers and Mathematics with Applications, 62 (2011), 797-803.

[34] Yang, X., Yu, D., Yang, J., Wu, C., *Generalization of soft set theory: from crisp to fuzzy case,* in *Proceedings of the Second International Conference of Fuzzy Information and Engineering*, 345-355, 2007.

[35] A. Zadeh, L.A., *Fuzzy sets,* Information and Control, 8 (1965), 338-353.

[36] Zhang, Z., Wang, C., Tian, D., Li, K., *A novel approach to interval-valued intuitionistic fuzzy soft set based decision making,* Applied Mathematical Modelling, 38 (4) (2014) 1255-1270.

[37] Zou, Y., Xiao, Z., *Data analysis approaches of soft sets under incomplete information,* Knowledge-Based Systems, 21 (8) (2008), 941-945.

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