AN APPROACH TO THE RELATIVE ORDER BASED GROWTH PROPERTIES OF DIFFERENTIAL MONOMIALS

Sanjib Kumar Datta

Department of Mathematics
University of Kalyani
P.O. Kalyani, Dist-Nadia
PIN: 741235, West Bengal
India
e-mail: sanjib_kr_datta@yahoo.co.in

Tanmay Biswas

Rajbari, Rabindrapalli, R.N. Tagore Road
P.O. Krishnagar, P.S. Kotwali, Dist-Nadia
PIN: 741101, West Bengal
India
e-mail: tanmaybiswas_math@rediffmail.com

Sarmila Bhattacharyya

Jhorehat F.C. High School for Girls
P.O. Jhorehat, P.S. Sankrail, Dist-Howrah
PIN: 711302, West Bengal
India
e-mail: bsarmila@gmail.com

Abstract. In this paper an attempt is taken to study the comparative growth properties of composition of entire and meromorphic functions on the basis of relative order and relative lower order of differential monomials generated by transcendental entire and transcendental meromorphic functions.

Keywords and Phrases: transcendental entire function, transcendental meromorphic function, relative order, relative lower order, differential monomial.
AMS Subject Classification (2010): 30D30, 30D35, 30D20.

1. Introduction

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The function $M_f(r)$ on $|z| = r$ known as maximum modulus function corresponding to f is defined as follows:

$$M_f(r) = \max_{|z|=r} |f(z)| \ .$$

When f is meromorphic, $M_f(r)$ cannot be defined as f is not analytic. In this situation, one may define another function $T_f(r)$ known as Nevanlinna's Characteristic function of f, playing the same role as $M_f(r)$ in the following manner:

$$T_f(r) = N_f(r) + m_f(r) \ .$$

Given two meromorphic functions f and g, the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their Nevanlinna's Characteristic functions.

When f is entire function, the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r) \ .$$

We call the function $N_f(r,a) \left(\bar{N}_f(r,a)\right)$ as counting function of a-points (distinct a-points) of f. In many occasions $N_f(r,\infty)$ and $\bar{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\bar{N}_f(r)$ respectively. We put

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + \bar{n}_f(0,a) \log r \ ,$$

where we denote by $n_f(r,a) \left(\bar{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a pole of f. Also we denote by $n_{f|=1}(r,a)$,the number of simple zeros of $f-a$ in $|z| \leq r$.

Accordingly, $N_{f|=1}(r,a)$ is defined in terms of $n_{f|=1}(r,a)$ in the usual way and we set

$$\delta_1(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f \mid= 1)}{T_f(r)} \quad \text{(cf. [8])}\,,$$

the deficiency of 'a' corresponding to the simple a- points of f i,e. simple zeros of $f-a$. In this connection Yang [7] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which

$$\delta_1(a;f) > 0 \text{ and } \sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) \leq 4.$$

On the other hand, $m\left(r, \frac{1}{f-a}\right)$ is denoted by $m_f(r,a)$ and we mean $m_f(r,\infty)$ by $m_f(r)$, which is called the proximity function of f. We also put

$$m_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right)\right| d\theta,$$

where $\log^+ x = \max(\log x, 0)$ for all $x \geqslant 0$.

Further, a meromorphic function $b \equiv b(z)$ is called small with respect to f if $T_b(r) = S_f(r)$ where $S_f(r) = o\{T_f(r)\}$ i.e., $\frac{S_f(r)}{T_f(r)} \to 0$ as $r \to \infty$. Moreover, for any transcendental meromorphic function f , we call $P[f] = bf^{n_0}(f^{(1)})^{n_1}...(f^{(k)})^{n_k}$, to be a differential monomial generated by it where $\sum_{i=0}^{k} n_i \geq 1$ (all $n_i \mid i = 0, 1, ..., k$ are non-negative integers) and the meromorphic function b is small with respect to f. In this connection the numbers $\gamma_{P[f]} = \sum_{i=0}^{k} n_i$ and $\Gamma_{P[f]} = \sum_{i=0}^{k}(i+1)n_i$ are called the degree and weight of $P[f]$ respectively {cf. [2]}.

The *order* of a meromorphic function f which is generally used in computational purpose is defined in terms of the growth of f with respect to the exponential function as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r + O(1)} .$$

Lahiri and Banerjee [5] introduced the *relative order* (respectively *relative lower order*) of a meromorphic function with respect to an entire function to avoid comparing growth just with $\exp z$. Extending the notion of relative order as cited in the reference, Datta, Biswas and Bhattacharyya [3] gave the definition of *relative order* (respectively *relative lower order*) of *differential monomials* generated by transcendental entire and transcendental meromorphic functions.

For entire and meromorphic functions, the notion of their growth indicators such as *order* and *lower order* are classical in complex analysis and during the past decades, several researchers have already been continuing their studies in the area of comparative growth properties of composite entire and meromorphic functions in different directions using the same. But at that time, the concept of *relative order* and consequently *relative lower order* of entire and meromorphic functions with respect to another entire function was mostly unknown to complex analysis and they are not aware of the technical advantages of using the relative growth indicators of the functions. Therefore the growth of composite entire and meromorphic functions needs to be modified on the basis of their *relative order* and *relative lower type* some of which has been explored in this paper. Actually in this paper we establish some newly developed results based on the growth properties of *relative order* and *relative lower order* of *monomials* generated by transcendental entire and transcendental meromorphic functions.

2. Notation and preliminary remarks

We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [4] and [6]. Henceforth, we do not explain those in details. Now, we just recall some definitions which will be needed in the sequel.

Definition 1 The *order* ρ_f and *lower order* λ_f of a meromorphic function f are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r} .$$

The notion of *order* (*lower order*) to determine the relative growth of two meromorphic functions having same non zero finite order is classical in complex analysis and is given by

Given a non-constant entire function f defined in the open complex plane $\mathbb{C}$, its Nevanlinna's Characteristic function is strictly increasing and continuous. Hence there exists its inverse function $T_g^{-1} : (T_g(0), \infty) \rightarrow (0, \infty)$ with $\lim_{s\rightarrow\infty} T_g^{-1}(s) = \infty$.

Lahiri and Banerjee [5] introduced the definition of *relative order* of a meromorphic function f with respect to an entire function g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\rightarrow\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} \ .\end{aligned}$$

The definition coincides with the classical one [5] if $g(z) = \exp z$. Similarly, one can define the *relative lower order* of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda_g(f) = \liminf_{r\rightarrow\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In this connection, the following two definitions are relevant:

Definition 2 [1] A non-constant entire function f is said to have the property (A) if for any $\delta > 1$ and for all large r, $[M_f(r)]^2 \leq M_f(r^\delta)$ holds. For examples of functions with or without the property (A), one may see [1].

Definition 3 Two entire functions g and h are said to be asymptotically equivalent if there exists l $(0 < l < \infty)$ such that

$$\frac{M_g(r)}{M_h(r)} \rightarrow l \text{ as } r \rightarrow \infty$$

and in this case we write $g \sim h$. Clearly if $g \sim h$ then $h \sim g$.

3. Some examples

In this section, we present some examples in connection with definitions given in the previous section.

Example 1 (Order (lower order)) Given any natural number n, let $f(z) = \exp z^n$. Then $M_f(r) = \exp r^n$. Therefore putting $R = 2$ in the inequality $T_f(r) \leq \log M_f(r) \leq \frac{R+r}{R-r} T_f(R)$ (cf. [4]) we get that $T_f(r) \leq r^n$ and $T_f(r) \geq \frac{1}{3}\left(\frac{r}{2}\right)^n$. Hence

$$\rho_f = \limsup_{r\rightarrow\infty} \frac{\log T_f(r)}{\log r} = n \text{ and } \lambda_f = \liminf_{r\rightarrow\infty} \frac{\log T_f(r)}{\log r} = n \ .$$

Further, if we take $g = \exp^{[2]} z$, then $T_g(r) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \ (r \rightarrow \infty)$. Therefore

$$\rho_f = \ \lambda_f = \infty \ .$$

Example 2 (Relative order (relative lower order)) Suppose $f = g = \exp^{[2]} z$ then $T_f(r) = T_g(r) \sim \frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}} \,(r \to \infty)$. Now we obtain that

$$\begin{aligned} T_g(r) &\leq \log M_g(r) \leq 3T_g(2r) \quad \text{(cf. [4])} \\ \text{i.e., } T_g(r) &\leq \exp r \leq 3T_g(2r) \ . \end{aligned}$$

Therefore

$$T_g^{-1} T_f(r) \geq \log\left(\frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\right), \quad \text{i.e., } \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} \geq 1$$

and

$$T_g^{-1} T_f(r) \leq 2\log\left(\frac{3\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\right), \quad \text{i.e., } \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} \leq 1 \ .$$

Hence

$$\rho_g(f) = \lambda_g(f) = 1 \ .$$

4. Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 1 [1] *Let g be an entire function and $\alpha > 1, 0 < \beta < \alpha$. Then*

$$M_g(\alpha r) > \beta M_g(r) \text{ for all sufficiently large } r.$$

Lemma 2 [1] *Let f be an entire function which satisfies Property* (A)*. Then for any positive integer n and for all sufficiently large r*

$$[M_f(r)]^n \leq M_f\left(r^\delta\right)$$

holds where $\delta > 1$.

Lemma 3 *Let g be an entire. Then for all sufficiently large values of r,*

$$T_g(r) \leq \log M_g(r) \leq 3T_g(2r) \ .$$

Lemma 3 follows from Theorem 1.6 (cf. [4], p.18), on putting $R = 2r$.

Lemma 4 [4] *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$. Also, let g be a transcendental entire function of regular growth having non zero finite order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; g) = 4$. Then the relative order and relative lower order of $P[f]$ with respect to $P[g]$ are same as those of f with respect to g.*

Lemma 5 *Let g and h be any two transcendental entire functions of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$ and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ respectively. Then for any transcendental meromorphic function f of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$,*

$$\rho_{P[g]}\left(P[f]\right)=\rho_{P[h]}\left(P[f]\right)$$

and

$$\lambda_{P[g]}\left(P[f]\right)=\lambda_{P[h]}\left(P[f]\right)\ .$$

if g and h have Property (A) *and $g\sim h$.*

Proof. Let $\varepsilon>0$ is arbitrary. Now, we get from Definition 3 and Lemma 1 for all sufficiently large values of r that

$$(1)\qquad M_g\left(r\right)<\left(l+\varepsilon\right)M_h\left(r\right)\leq M_h\left(\alpha r\right)\ ,$$

where $\alpha>1$ is such that $l+\varepsilon<\alpha$.

Now, from Lemma 3 and in view of definition of relative order, we obtain for all sufficiently large values of r that

$$T_f\left(r\right)\leq T_g\left[\left(r\right)^{\left(\rho_g(f)+\varepsilon\right)}\right],\quad \text{i.e., } T_f\left(r\right)\leq \log M_g\left[\left(r\right)^{\left(\rho_g(f)+\varepsilon\right)}\right]\ .$$

Therefore, in view of (1), Lemma 2 and Lemma 3, it follows from above for any $\delta>1$ that

$$\begin{aligned}
T_f\left(r\right) &\leq \frac{1}{3}\log\left[M_h\left[\left(\alpha r\right)^{\left(\rho_g(f)+\varepsilon\right)}\right]\right]^3\\
i.e.,\ T_f\left(r\right) &\leq \frac{1}{3}\log M_h\left[\left(\alpha r\right)^{\delta\left(\rho_g(f)+\varepsilon\right)}\right]\\
\text{i.e., } T_f\left(r\right) &\leq T_h\left[\left(2\alpha r\right)^{\delta\left(\rho_g(f)+\varepsilon\right)}\right]\\
i.e.,\ \frac{\log T_h^{-1}T_f\left(r\right)}{\log r} &\leq \delta\left(\rho_g\left(f\right)+\varepsilon\right)\frac{\log\left(2\alpha r\right)}{\log r}\ .
\end{aligned}$$

Letting $\delta\to 1+$, we get from above that

$$(2)\qquad \rho_h\left(f\right)\leq\rho_g\left(f\right)\ .$$

Since $h\sim g$, we also obtain that

$$(3)\qquad \rho_g\left(f\right)\leq\rho_h\left(f\right)\ .$$

Now in view of Lemma 4, we obtain from (2) and (3) that

$$\rho_{P[g]}\left(P[f]\right)=\rho_{P[h]}\left(P[f]\right)\ .$$

Similarly, we have

$$\lambda_{P[g]}\left(P[f]\right)=\lambda_{P[h]}\left(P[f]\right)\ .$$

Thus the lemma follows.

5. Theorems

In this section, we present the main results of the paper.

Theorem 1 *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Also, let h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and g be any entire function such that*

$$0<\lambda_h\left(f\circ g\right)\leq\rho_h\left(f\circ g\right)<\infty \text{ and } 0<\lambda_h\left(f\right)\leq\rho_h\left(f\right)<\infty.$$

Then

$$\begin{aligned}\frac{\lambda_h\left(f\circ g\right)}{\rho_h\left(f\right)} &\leq \liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}\left(r\right)}{\log T_{P[h]}^{-1}T_{P[f]}\left(r\right)} \leq \frac{\lambda_h\left(f\circ g\right)}{\lambda_h\left(f\right)}\\ &\leq \limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}\left(r\right)}{\log T_{P[h]}^{-1}T_{P[f]}\left(r\right)} \leq \frac{\rho_h\left(f\circ g\right)}{\lambda_h\left(f\right)}\ .\end{aligned}$$

Proof. From the definition of $\rho_h\left(f\right)$ and $\lambda_h\left(f\circ g\right)$ and Lemma 4 we have for arbitrary positive ε and for all sufficiently large values of r that

$$\log T_h^{-1}T_{f\circ g}\left(r\right)\geqslant\left(\lambda_h\left(f\circ g\right)-\varepsilon\right)\log r \tag{4}$$

and

$$\begin{aligned}\log T_{P[h]}^{-1}T_{P[f]}\left(r\right) &\leq \left(\rho_{P[h]}\left(P[f]\right)+\varepsilon\right)\log r\\ \text{i.e., } \log T_{P[h]}^{-1}T_{P[f]}\left(r\right) &\leq \left(\rho_h\left(f\right)+\varepsilon\right)\log r\ .\end{aligned} \tag{5}$$

Now, from (4), (5) it follows for all sufficiently large values of r that

$$\frac{\log T_h^{-1}T_{f\circ g}\left(r\right)}{\log T_{P[h]}^{-1}T_{P[f]}\left(r\right)}\geqslant\frac{\left(\lambda_h\left(f\circ g\right)-\varepsilon\right)\log r}{\left(\rho_h\left(f\right)+\varepsilon\right)\log r}\ .$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain that

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}\left(r\right)}{\log T_{P[h]}^{-1}T_{P[f]}\left(r\right)}\geqslant\frac{\lambda_h\left(f\circ g\right)}{\rho_h\left(f\right)}\ . \tag{6}$$

Again for a sequence of values of r tending to infinity,

$$\log T_h^{-1}T_{f\circ g}\left(r\right)\leq\left(\lambda_h\left(f\circ g\right)+\varepsilon\right)\log r \tag{7}$$

and for all sufficiently large values of r,

$$\begin{aligned}\log T_{P[h]}^{-1}T_{P[f]}\left(r\right) &\geqslant \left(\lambda_{P[h]}\left(P[f]\right)-\varepsilon\right)\log r\\ \text{i.e., } \log T_{P[h]}^{-1}T_{P[f]}\left(r\right) &\geqslant \left(\lambda_h\left(f\right)-\varepsilon\right)\log r\ .\end{aligned} \tag{8}$$

Combining (7) and (8), we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{(\lambda_h(f \circ g) + \varepsilon) \log r}{(\lambda_h(f) - \varepsilon) \log r} .$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$\liminf_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{\lambda_h(f \circ g)}{\lambda_h(f)} . \tag{9}$$

Also for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_{P[h]}^{-1} T_{P[f]}(r) &\leq \left(\lambda_{P[h]}(P[f]) + \varepsilon\right) \log r \\ \text{i.e., } \log T_{P[h]}^{-1} T_{P[f]}(r) &\leq (\lambda_h(f) + \varepsilon) \log r . \end{aligned} \tag{10}$$

Now, from (4) and (10), we obtain for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \geq \frac{(\lambda_h(f \circ g) - \varepsilon) \log r}{(\lambda_h(f) + \varepsilon) \log r} .$$

As $\varepsilon (> 0)$ is arbitrary, we get from above that

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \geq \frac{\lambda_h(f \circ g)}{\lambda_h(f)} . \tag{11}$$

Also for all sufficiently large values of r,

$$\log T_h^{-1} T_{f \circ g}(r) \leq (\rho_h(f \circ g) + \varepsilon) \log r . \tag{12}$$

Now, it follows from (8) and (12) for all sufficiently large values of r that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{(\rho_h(f \circ g) + \varepsilon) \log r}{(\lambda_h(f) - \varepsilon) \log r} .$$

Since $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\limsup_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P[h]}^{-1} T_{P[f]}(r)} \leq \frac{\rho_h(f \circ g)}{\lambda_h(f)} . \tag{13}$$

Thus the theorem follows from (6), (9), (11) and (13).

The following theorem can be proved in the line of Theorem 1 and so the proof is omitted.

Theorem 2 *Suppose g be a transcendental entire function of finite order or of non-zero lower order and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; g) = 4$. *Also let h be a transcendental entire function of regular growth having non zero finite order with* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a; h) = 4$

and f be any meromorphic function such that $0 < \lambda_h (f \circ g) \leq \rho_h (f \circ g) < \infty$ and $0 < \lambda_h (g) \leq \rho_h (g) < \infty$. Then

$$\frac{\lambda_h (f \circ g)}{\rho_h (g)} \leq \liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g} (r)}{\log T_{P[h]}^{-1} T_{P[g]} (r)} \leq \frac{\lambda_h (f \circ g)}{\lambda_h (g)}$$

$$\leq \limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g} (r)}{\log T_{P[h]}^{-1} T_{P[g]} (r)} \leq \frac{\rho_h (f \circ g)}{\lambda_h (g)} .$$

Theorem 3 *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; f) = 4$. Also, let h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a; h) = 4$ and g be any entire function with $0 < \rho_h (f \circ g) < \infty$ and $0 < \rho_h (f) < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g} (r)}{\log T_{P[h]}^{-1} T_{P[f]} (r)} \leq \frac{\rho_h (f \circ g)}{\rho_h (f)} \leq \limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g} (r)}{\log T_{P[h]}^{-1} T_{P[f]} (r)} .$$

Proof. From the definition of $\rho_{P[h]} (P[f])$ and in view of Lemma 4, we get for a sequence of values of r tending to infinity that

$$\log T_{P[h]}^{-1} T_{P[f]} (r) \geqslant \left(\rho_{P[h]} (P[f]) - \varepsilon\right) \log r$$

$$\text{(14)} \qquad \text{i.e., } \log T_{P[h]}^{-1} T_{P[f]} (r) \geqslant (\rho_h (f) - \varepsilon) \log r .$$

Now, from (12) and (14), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g} (r)}{\log T_{P[h]}^{-1} T_{P[f]} (r)} \leq \frac{(\rho_h (f \circ g) + \varepsilon) \log r}{(\rho_h (f) - \varepsilon) \log r} .$$

As $\varepsilon (> 0)$ is arbitrary, we obtain that

$$\text{(15)} \qquad \liminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g} (r)}{\log T_{P[h]}^{-1} T_{P[f]} (r)} \leq \frac{\rho_h (f \circ g)}{\rho_h (f)} .$$

Again for a sequence of values of r tending to infinity,

$$\text{(16)} \qquad \log T_h^{-1} T_{f\circ g} (r) \geqslant (\rho_h (f \circ g) - \varepsilon) \log r .$$

So combining (5) and 16, we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g} (r)}{\log T_{P[h]}^{-1} T_{P[f]} (r)} \geqslant \frac{(\rho_h (f \circ g) - \varepsilon) \log r}{(\rho_h (f) + \varepsilon) \log r} .$$

Since $\varepsilon (> 0)$ is arbitrary, it follows that

$$\text{(17)} \qquad \limsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g} (r)}{\log T_{P[h]}^{-1} T_{P[f]} (r)} \geqslant \frac{\rho_h (f \circ g)}{\rho_h (f)} .$$

Thus the theorem follows from (15) and (17).

The following theorem can be carried out in the line of Theorem 3 and therefore we omit its proof.

Theorem 4 *Suppose g be a transcendental entire function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and f be any meromorphic function such that $0<\rho_h(f\circ g)<\infty$ and $0<\rho_h(g)<\infty$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[g]}(r)}\leq\frac{\rho_h(f\circ g)}{\rho_h(g)}\leq\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[g]}(r)}\ .$$

The following theorem is a natural consequence of Theorem 1 and Theorem 3:

Theorem 5 *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and g be any entire function with $0<\lambda_h(f\circ g)\leq\rho_h(f\circ g)<\infty$ and $0<\lambda_h(f)\leq\rho_h(f)<\infty$. Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)}\leq\min\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(f)},\frac{\rho_h(f\circ g)}{\rho_h(f)}\right\}$$

$$\leq\max\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(f)},\frac{\rho_h(f\circ g)}{\rho_h(f)}\right\}\leq\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)}\ .$$

The proof is omitted.

Analogously, one may state the following theorem without its proof.

Theorem 6 *Suppose g be a transcendental entire function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;g)=4$. Also let h be a transcendental entire function of regular growth having non zero finite order with $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$ and f be any meromorphic function such that $0<\lambda_h(f\circ g)\leq\rho_h(f\circ g)<\infty$ and $0<\lambda_h(g)\leq\rho_h(g)<\infty$ .Then*

$$\liminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[g]}(r)}\leq\min\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(g)},\frac{\rho_h(f\circ g)}{\rho_h(g)}\right\}$$

$$\leq\max\left\{\frac{\lambda_h(f\circ g)}{\lambda_h(g)},\frac{\rho_h(f\circ g)}{\rho_h(g)}\right\}\leq\limsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P[h]}^{-1}T_{P[g]}(r)}\ .$$

Theorem 7 *Suppose f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$. Also let h be a transcendental entire function of regular growth having non zero finite order with*

$\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h) = 4$ *and* g *be any entire function with* $0 < \rho_h(f\circ g) < \infty$ *and* $0 < \rho_h(f) < \infty$ *and* $g \sim h$. *Then*

$$\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)} \leq 1 \leq \limsup_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)} .$$

Proof. From the definition of $\rho_g(f)$, we get for all sufficiently large values of r that

$$\log T_g^{-1}T_f(r) \leq (\rho_g(f)+\varepsilon)\log r \tag{18}$$

and for a sequence of values of r tending to infinity that

$$\log T_g^{-1}T_f(r) \geq (\rho_g(f)-\varepsilon)\log r . \tag{19}$$

Now, from (14) and (18), it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1}T_f(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)} \leq \frac{(\rho_g(f)+\varepsilon)\log r}{(\rho_h(f)-\varepsilon)\log r} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)} \leq \frac{\rho_g(f)}{\rho_h(f)} . \tag{20}$$

Now, as $g\sim h$, in view of Lemma 4 and Lemma 5 we obtain from (20) that

$$\liminf_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)} \leq 1 . \tag{21}$$

Again combining (5) and (19), we get for a sequence of values of r tending to infinity that

$$\frac{\log T_g^{-1}T_f(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)} \geqslant \frac{(\rho_g(f)-\varepsilon)\log r}{(\rho_h(f)+\varepsilon)\log r} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)} \geqslant \frac{\rho_g(f)}{\rho_h(f)} . \tag{22}$$

Now as $g\sim h$, in view of Lemma 4 and Lemma 5 we obtain from (22) that

$$\limsup_{r\to\infty}\frac{\log T_g^{-1}T_f(r)}{\log T_{P[h]}^{-1}T_{P[f]}(r)} \geqslant 1 . \tag{23}$$

Thus the theorem follows from (21) and (23).

6. Conclusion

Actually this paper deals with the extension of the works on the growth properties concerning *differential monomials* generated by transcendental entire and transcendental meromorphic functions on the basis of their *relative orders* and *relative lower orders*. These theories can also be modified by the treatment of the notions of *generalized relative orders* (*generalized relative lower orders*) and (p,q)-*th relative orders* ((p,q)-*th relative lower orders*). In addition, some extensions of the same may be done in the light of slowly changing functions. Moreover, the notion of *relative order* and *relative lower order* of *differential monomials* generated by transcendental entire and transcendental meromorphic functions may have a wide range of applications in Complex Dynamics, Factorization Theory of entire functions of single complex variable, the solution of complex differential equations etc. which must be an active area of research.

References

[1] Bernal, L., *Orden relative de crecimiento de funciones enteras*, Collect. Math., 39 (1988), 209-229.

[2] Doeringer, W., *Exceptional values of differential polynomials,* Pacific J. Math., 98 (1)(1982), 55-62.

[3] Datta, S.K., Biswas, T., Bhattacharyya, S., *On relative order and relative type based growth properties of differential monomials*, J. of Indian Math. Soc. (to appear).

[4] Hayman, W.K., *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[5] Lahiri, B.K., Banerjee, D., *Relative order of entire and meromorphic functions*, Proc. Nat. Acad. Sci. India, 69(A) III (1999), 339-354.

[6] Valiron, G., *Lectures on the General Theory of Integral Functions*, Chelsea Publishing Company, 1949.

[7] Yang, L., *Value distribution theory and new research on it*, Science Press, Beijing (1982).

[8] Yi, H.X., *On a result of Singh*, Bull. Austral. Math. Soc., 41 (1990), 417-420.

Accepted: 28.10.2014