JACOBI FIELDS ON THE MANIFOLD OF FREUND

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Abstract. In this paper, the geometric structures of Freund manifold are considered By defining a Riemannian metric, the curvature tensor and the scalar curvature are given. Then, the Jacobi fields on the Freund manifold have been considered to investigate the instability of the geodesics in view of differential geometry. Moreover, we take submanifold of Freund manifold as an example to illustrate our results.

Keywords: Freund manifold; Riemannian metric; α-connection; α-curvature tensor; Jacobi field.

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1. Introduction

Since people consider all the probability density functions as manifolds and treat the Fisher information matrices as the Riemannian metric, the research concerning the geometric structures of all the statistic manifolds achieves a lot of attention. Based on this fundamental idea, geometry is widely used in several fields says, information theory, computer science and radar imaging. By now the geometric structures of some important statistic manifolds have already been investigated. Further more we use these geometric structures to study other properties of statistic manifolds. Especially the study concerning the stability of Jacobi field on statistic manifold is becoming a hot research field. Cafaro [4], L. Peng [8], [9], C. Li [10] and L. Casetti [5] studied the stability of Jacobi fields on some of statistical manifolds.

In this paper, authors consider the two dimensional Freund distribution as a statistical manifold. First, we define the Riemannian metric on it, also give the corresponding Riemannian connection and curvature tensor then calculate the corresponding geometric variables. Finally, we illustrate the Jacobi field by the submanifold of the Freund manifold and analysis the stability of geodesic.

2. The geometric structure of the Freund manifold

Definition 2.1. We call the set

$$M = \left\{ p \,\middle|\, p(x,y,\zeta) = \left\{ \begin{array}{c} \alpha_1\beta_2 e^{-\beta_2 y-(\alpha_1+\alpha_2-\beta_2)x},\ 0<x<y \\ \alpha_2\beta_1 e^{-\beta_1 x-(\alpha_1+\alpha_2-\beta_1)y},\ 0<y<x \\ \zeta=(\zeta^1,\zeta^2,\zeta^3,\zeta^4)=(\alpha_1,\beta_1,\alpha_2,\beta_2)\in R^4_+ \end{array} \right., \right\}$$

as a Freund manifold, where

$$p(x,y,\zeta) = \left\{ \left\{ \begin{array}{c} \alpha_1\beta_2 e^{-\beta_2 y-(\alpha_1+\alpha_2-\beta_2)x},\ 0<x<y \\ \alpha_2\beta_1 e^{-\beta_1 x-(\alpha_1+\alpha_2-\beta_1)y},\ 0<y<x \\ \zeta=(\zeta^1,\zeta^2,\zeta^3,\zeta^4)=(\alpha_1,\beta_1,\alpha_2,\beta_2)\in R^4_+ \end{array} \right., \right\}$$

is the probability density function of 2-dimensional Freund distribution [6].

Definition 2.2. The fisher information matrix (g_{ij}) is defined as

$$(g_{ij}) = E[\partial_i l \partial_j l], \tag{1}$$

where $l(x,\zeta) = lnp(x,\zeta)$, $\partial_i l = \dfrac{\partial l(x,\zeta)}{\partial\theta^i}$, and E denotes the expectation of $p(x,\zeta)$ [1].

Proposition 2.3. *The Fisher information matrix* (g_{ij}) *is*

$$(g_{ij}) = \begin{bmatrix} \dfrac{1}{\zeta^1(\zeta^1+\zeta^3)} & 0 & 0 & 0 \\ 0 & \frac{\zeta^3}{(\zeta^2)^2(\zeta^1+\zeta^3)} & 0 & 0 \\ 0 & 0 & \dfrac{1}{\zeta^3(\zeta^1+\zeta^3)} & 0 \\ 0 & 0 & 0 & \dfrac{\zeta^1}{(\zeta^4)^2(\zeta^1+\zeta^3)} \end{bmatrix} \tag{2}$$

Definition 2.4. The Riemannian connection ∇ with respect to Riemannian metric (1) is given by

$$g(\nabla_{\partial_i}\partial_j, \partial_k) = \Gamma_{ijk} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ki} - \partial_k g_{ij}), \tag{3}$$

$\alpha-$connection is defined by

$$\Gamma^{(\alpha)}_{ijk} = \Gamma_{ijk} - \frac{\alpha}{2} T_{ijk}, \tag{4}$$

where $T_{ijk} = E[\partial_i l \partial_j l \partial_k l]$.

According to calculations, we get

Proposition 2.5. *The nonzero α-connection coefficients on the Freund manifold are obtained as follows:*

$$\begin{aligned}
\Gamma^{(\alpha)}_{111} &= \frac{2(\alpha-1)\zeta^1 - (1+\alpha)\zeta^3}{2(\zeta^1)^2(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{113} &= \frac{1+\alpha}{2\zeta^1(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{221} &= \frac{(1+\alpha)\zeta^3}{2(\zeta^2)^2(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{223} &= \frac{(1+\alpha)\zeta^1}{2(\zeta^2)^2(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{331} &= \frac{1+\alpha}{2\zeta^3(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{333} &= \frac{2(\alpha-1)\zeta^3 - (1+\alpha)\zeta^1}{2(\zeta^3)^2(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{441} &= \frac{(1+\alpha)\zeta^3}{2(\zeta^4)^2(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{443} &= \frac{(1+\alpha)\zeta^1}{2(\zeta^4)^2(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{222} &= \frac{(1-\alpha)\zeta^3}{(\zeta^2)^3(\zeta^1+\zeta^3)}, & \Gamma^{(\alpha)}_{444} &= \frac{(1-\alpha)\zeta^1}{(\zeta^4)^3(\zeta^1+\zeta^3)},\\
\Gamma^{(\alpha)}_{131} &= \Gamma^{(\alpha)}_{311} = \frac{1-\alpha}{2\zeta^1(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{133} &= \Gamma^{(\alpha)}_{313} = \frac{1-\alpha}{2\zeta^3(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{122} &= \Gamma^{(\alpha)}_{212} = \frac{(1-\alpha)\zeta^3}{2(\zeta^2)^2(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{344} &= \Gamma^{(\alpha)}_{434} = \frac{(1-\alpha)\zeta^1}{2(\zeta^4)^2(\zeta^1+\zeta^3)^2},\\
\Gamma^{(\alpha)}_{232} &= \Gamma^{(\alpha)}_{322} = \frac{(1-\alpha)\zeta^1}{2(\zeta^2)^2(\zeta^1+\zeta^3)^2}, & \Gamma^{(\alpha)}_{144} &= \Gamma^{(\alpha)}_{414} = \frac{(1-\alpha)\zeta^3}{2(\zeta^4)^2(\zeta^1+\zeta^3)^2}.
\end{aligned}$$

Definition 2.6. α-connection tensor is defined by

$$R^{(\alpha)}_{ijkm} = (\partial_i \Gamma^{(\alpha)s}_{jk} - \partial_j \Gamma^{(\alpha)s}_{ik}) g_{sm} + \Gamma^{(\alpha)}_{itm}\Gamma^{(\alpha)t}_{jk} - \Gamma^{(\alpha)}_{jtm}\Gamma^{(\alpha)t}_{ik}. \tag{5}$$

where $\Gamma^{(\alpha)k}_{ij} = \Gamma^{(\alpha)}_{ijm} g^{mk}$, α-Ricci curvature $R^{(\alpha)}_{ik}$ and α-scalar curvature $R^{(\alpha)}$ are defined as

$$R^{(\alpha)}_{ik} = R^{(\alpha)}_{ijkl} g^{ji} \tag{6}$$

and

$$R(\alpha) = R^{(\alpha)}_{ik} g^{ik} \tag{7}$$

respectively.

By calculations, we obtain the following proposition

Proposition2.7. *The non-zero α-curvature tensors on the Freund manifold are*

$$\begin{aligned}
R^{(\alpha)}_{1212} &= -\frac{(1-\alpha^2)(\zeta^3)^2}{4\zeta^1(\zeta^2)^2(\zeta^1+\zeta^3)^3},\\
R^{(\alpha)}_{3434} &= -\frac{(1-\alpha^2)(\zeta^1)^2}{4\zeta^3(\zeta^4)^2(\zeta^1+\zeta^3)^3},\\
R^{(\alpha)}_{1414} &= -\frac{(1-\alpha^2)\zeta^3}{4(\zeta^4)^2(\zeta^1+\zeta^3)^3},\\
R^{(\alpha)}_{2323} &= -\frac{(1-\alpha^2)\zeta^1}{4(\zeta^2)^2(\zeta^1+\zeta^3)^3},\\
R^{(\alpha)}_{2424} &= -\frac{(1-\alpha^2)\zeta^1\zeta^3}{4(\zeta^2)^2(\zeta^4)^2(\zeta^1+\zeta^3)^2},\\
R^{(\alpha)}_{2123} &= \frac{(1-\alpha^2)\zeta^3}{4(\zeta^2)^2(\zeta^1+\zeta^3)^3},\\
R^{(\alpha)}_{4341} &= \frac{(1-\alpha^2)\zeta^1}{4(\zeta^4)^2(\zeta^1+\zeta^3)^3}.
\end{aligned}$$

Freund manifold is ± 1 flat which means that all these $\alpha-$curvature vanish when $\alpha = \pm 1$. The non-zero α-Ricci curvature on the Freund manifold are

$$\begin{aligned}
R^{(\alpha)}_{11} &= -\frac{(1-\alpha^2)\zeta^3}{2\zeta^1(\zeta^1+\zeta^3)^2},\\
R^{(\alpha)}_{22} &= -\frac{(1-\alpha^2)\zeta^3}{2(\zeta^2)^2(\zeta^1+\zeta^3)},\\
R^{(\alpha)}_{33} &= -\frac{(1-\alpha^2)\zeta^1}{2\zeta^3(\zeta^1+\zeta^3)^2},\\
R^{(\alpha)}_{44} &= -\frac{(1-\alpha^2)\zeta^1}{2(\zeta^4)^2(\zeta^1+\zeta^3)},\\
R^{(\alpha)}_{13} &= \frac{1-\alpha^2}{2(\zeta^1+\zeta^3)^2}.
\end{aligned}$$

α-scalar curvature satisfies

$$R^{(\alpha)} = -\frac{3}{2}(1-\alpha^2)$$

Definition 2.8. [1] The geodesic on the n-dimensional Riemannian manifold can be denoted by

$$\frac{d^2\zeta^k}{dt^2} + \Gamma^k_{ij}\frac{d\zeta^i}{dt}\frac{d\zeta^j}{dt} = 0, \ k = 1, 2, ..., n. \tag{8}$$

Example 2.9. Consider the 2-dimensional submanifold of the Freund manifold

$$M_1 = \left\{ p \,\middle|\, p(x,y,\zeta) = \left\{ \begin{array}{c} \alpha_1\beta_2 e^{-\beta_2 y-(\alpha_1+\alpha_2-\beta_2)x}, \ 0<x<y, \\ \alpha_2\beta_1 e^{-\beta_1 x-(\alpha_1+\alpha_2-\beta_1)y}, \ 0<y<x, \\ \zeta = (\zeta^2,\zeta^4) = (\beta_1,\beta_2) \in R^2_+. \end{array} \right. , \right\}$$

From (2) we get

$$(g_{ij}) = \begin{bmatrix} \frac{\zeta^3}{(\zeta^2)^2(\zeta^1+\zeta^3)} & 0 \\ 0 & \frac{\zeta^1}{(\zeta^4)^2(\zeta^1+\zeta^3)} \end{bmatrix}$$

and its inverse matrix is

$$(g^{ij}) = \begin{bmatrix} \frac{(\zeta^2)^2(\zeta^1+\zeta^3)}{\zeta^3} & 0 \\ 0 & \frac{(\zeta^4)^2(\zeta^1+\zeta^3)}{\zeta^1} \end{bmatrix}$$

the corresponding non-zero α-connection coefficients are

$$\Gamma_{22}^{(\alpha)2} = -\frac{1-\alpha}{\zeta^2}, \ \Gamma_{44}^{(\alpha)4} = -\frac{1-\alpha}{\zeta^4}. \tag{9}$$

From (5) we get $R_{2424}^{(\alpha)} = 0$, so the Gaussian curvature is

$$K = -\frac{R_{2424}^{(\alpha)}}{g_{22}g_{44}-(g_{24})^2} = 0. \tag{10}$$

3. The Jacobi field on the Freund manifold

Now, we consider the Jacobi field on the Freund manifold (M_1, g). Let $\zeta^l : [a,b] \rightarrow M_1$ be the geodesic on M_1, $\zeta^l(t,\beta) : [a,b] \times (-\epsilon,\epsilon) \rightarrow M_1$ is a variation of ζ. For each fixed β, the curvature $\zeta^l(t,\beta)$ is a geodesic, which is called a geodesic variation of ζ. The Jacobi equation along the geodesic satisfies

$$\frac{D^2\mathbf{J}}{Dt^2} + R(\mathbf{J}, v)v = 0, \tag{11}$$

where t is the time, $R(\mathbf{J}, v)$ is the Riemannian curvature tensor. $\frac{D}{Dt}$ is the covariant derivative along the geodesic, $v = \frac{\partial \theta^k}{\partial t}$ is the velocity of geodesic. $\mathbf{J}$ is called the Jacobi field. The component of Jacobi equation can be denoted by

$$\frac{D^2(\delta\zeta^i)}{Dt^2} + R^i_{kml}\frac{\partial \zeta^k}{\partial t}\frac{\partial \zeta^l}{\partial t}\delta\zeta^m = 0, \tag{12}$$

where $\delta\zeta^k = J^k$ is the component of the Jacobi field. From (12) we get

$$g_{ij}\frac{D^2(\delta\zeta^i)}{Dt^2} + R_{jkml}\frac{\partial \zeta^k}{\partial t}\frac{\partial \zeta^l}{\partial t}\delta\zeta^m = 0, \tag{13}$$

the length of Jacobi field $\mathbf{J}$ is defined by

$$\mathbf{J}^2 = J^i J_j = g_{ij} j^i J^j \tag{14}$$

As an application, we calculate the Jacobi equation in the manifold M_1 given in example and study its stability. From (8) and (9) we get the geodesic equation on manifold M_1 as follows

$$\frac{d^2\beta_k}{dt^2} - \frac{1-\alpha}{\beta_k}\left(\frac{d\beta_k}{dt}\right)^2 = 0, \;\; k = 1, 2.$$

when $\alpha = 0$, we get the solution

$$\beta_k = \zeta^{2k} = C_{2k-1} e^{c_{2k} t}, \;\; k = 1, 2, \tag{15}$$

where C_i $(i = 1, ..., 4)$ are integration constants. Then we consider the stability of Jacobi field. From (13), we get the Jacobi equation on M_1 as

$$\frac{D^2(\delta\zeta^i)}{Dt^2} = 0.$$

then we get

$$\begin{aligned} &\frac{d^2\delta\zeta^{2k}}{dt^2} + 2\Gamma^k_{kk}\frac{d\delta\zeta^{2k}}{dt}\frac{d\zeta^{2k}}{dt} \\ &\quad + \left[\Gamma^k_{kk}\frac{d^2\zeta^{2k}}{dt^2} + \frac{\partial\Gamma^k_{kk}}{\partial\zeta^{2k}}\left(\frac{d\zeta^{2k}}{dt}\right)^2 + \left(\Gamma^k_{kk}\frac{d\zeta^{2k}}{dt}\right)^2\right]\delta\zeta^{2k} = 0 \end{aligned} \tag{16}$$

where $k = 1, 2$ put (9) and (15) in to (16) we get

$$\frac{d^2\delta\zeta^{2k}}{dt^2} - 2C_{2k}\frac{d\delta\zeta^{2k}}{dt} + (C_{2k})^2\delta\zeta^{2k} = 0. \tag{17}$$

Integrate (17), we obtain

$$\delta\zeta^{2k} = (C_{2k+3}t + C_{2k+4})e^{C_{2k}t}, \;\; k = 1, 2,$$

where C_i $(i = 1, ..., 8)$ are integration constants. Finally, from (14) we get the Jacobi field on M as follows

$$\begin{aligned} \mathbf{J}^2_M &= g_{22}(\delta\zeta^2)^2 + g_{44}(\delta\zeta^4)^2 \\ &= \frac{\alpha_2}{(\alpha_1+\alpha_2)(C_1)^2}(C_5 t + C_6)^2 + \frac{\alpha_1}{(\alpha_1+\alpha_2)(C_3)^2}(C_7 t + C_8)^2. \end{aligned}$$

then

$$\mathbf{J}^2_M = O(t^2). \tag{18}$$

Equation (18) shows that $\mathbf{J}^2_M$ is divergent when $t \to \infty$ which means Jacobi field is unstable.

Conclusion

We consider the probability density function of the two-dimensional Freund distribution as a statistical manifold, define the Riemannian metric give the α-connection and the $\alpha-$curvature. Moreover, we study the Jacobi field on it and obtain the convergence of the geodesic, which is the foundation of information geometry theory that play crucial role in practical applications.

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References

[1] Amari, S., *Differential geometrical methods in statistics*, Springer-Verlag, Berlin, 1985.

[2] Amari, S., Nagaoka, H., *Methods of information geometry*, Oxford University Press, Oxford, 2000.

[3] Barbaresco, F., *Interactions between Symmetric Cones and Information Geometrics: Bruhat-Tits and Siegel Spaces Models for High Resolution Autoregressive Doppler Imagery*, ETCV08 Conference, Ecole Polytechnique, Nov. 2008, published by Springer in Lecture Notes in Computer.

[4] Cafaro, C., Ali, S.A., *Jacobi fields on statistical manifolds of negative curvature*, Physica D, 234 (1) (2007), 70-80.

[5] Casetti, L., Pettini, M., Cohen, E.G.D., *Geometric approach to Hamiltonian dynamics and statistical mechanics*, Physics Reports, 337 (3) (2000), 237-341.

[6] Cao, L., Sun, H., Wang, X., *The geometric structure of the Weibull distribution manifold and the generalized exponential distribution manifold*, Tamkang Journal of Mathematics, 39 (1) (2008), 45-51.

[7] Khadiga, Arwini, Dodson, C.T.J., *Neighborhoods of independence for random processes via information geometry*, Mathematica Journal 9 (4) (2005), 863-877.

[8] Peng, L., Sun, H., Sun, D., Yi, J., *The geometric structures and instability of entropic dynamical models*, Advances in Mathematics, 227 (2011), 459-471.

[9] Peng, L., Sun, H., Xu, G., *Information geometric characterization of the complexity of fractional Brownian motions*, Journal of Mathematical Physics, 53 (2012), 123-305.

[10] Li, C., Sun, H., Zhang, S., *Characterization of the complexity of an ED model via information geometry*, The European Physical Journal Plus, 128(2013).

[11] Lenglet, C., Rousson, M., Deriche, R. et al., *Statistics on the manifold of multivariate normal distributions: Theory and application to diffusion tensor MRI processing*, Journal of Mathematical Imaging and Vision, 25 (2006), 423-444.

[12] Moakher, M., *A differential geometric approach to the geometric mean of symmetric positivedefinite matrices*, SIAM J. Matrix Anal. Appl., 26 (2005), 735-747.

[13] Zhang, Z., Sun, H., Zhong, F., *Geometric structure of stable output feedback system*, Kybernetika, 45 (3) (2009), 387-404.

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