M-FUZZY h-IDEALS IN h-SEMISIMPLE M-Γ-HEMIRINGS

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Abstract. In this paper, the concepts of M-fuzzy h-interior ideals and prime M-fuzzy h- ideals in M-Γ-hemirings are introduced. Some new properties of these kinds of M-fuzzy h-ideals are also given. Finally, some characterizations of the h-semisimple M-Γ-hemirings are investigated by these kinds of M-fuzzy h-ideals.

Keywords: M-Γ-hemiring, M-fuzzy h-interior ideal, prime M-fuzzy h-ideal, h-semisimple M-Γ-hemiring.

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1. Introduction

The concept of Γ-rings was first introduced in 1966 by Barnes [1] which is more a general concept than that of a ring. After the paper of Barnes, many researchers were engaged in studying of some special Γ-rings. Jun and Lee [6] discussed fuzzy Γ-rings, and Jun [5] investigated fuzzy prime ideals of Γ-rings. In particular, Dutta and Chanda [3] studied the structure of fuzzy ideals of a Γ-ring. The concept of Γ-semirings was then introduced by Rao [15], and some properties of such Γ-semirings have been studied by Sardar et al. [16]. Recently, Ma and Zhan [11] investigated fuzzy h-ideals in h-hemiregular and h-semisimple Γ-hemiring, and Zhan and Shum [23] discussed fuzzy h-ideals in Γ-hemirings.

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The notion of semirings was first introduced by H. S. Vandiver in 1934. In the 1980's the theory of semirings contributed to computer science, since the rapid development of computer science need additional theoretical mathematical background. We note that the ideals of semirings also play a crucial role in the structure theory. Although ideals in semirings are useful in their own way, they do not in general coincide with the role of ideals in a ring. For this reason, the usage of ideals in semirings was somewhat limited. By a hemiring, we mean a special semiring with a zero and with a commutative addition. The properties of h-ideals of hemirings were thoroughly investigated by Torre [17]. By using h-ideals in hemirings, Torre established the quotient hemirings which are an exact analog to the ring theory. Recently, Han et al. [4] investigated some characterizations of semiring orders in a semiring, In 2004, Jun [7] defined the fuzzy h-ideals in hemirings. Yin and Li [19] introduced the concepts of fuzzy h-bi-ideals and fuzzy h-quasi-ideals of hemirings. After that, Ma and Zhan [10] introduced the concepts of $(\in, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideals (resp., h-quasi-ideals) of a hemiring and investigated some of their properties. Recently, Ma et al. [8] introduced the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-(h-quasi-)ideals of hemirings. In particular, some characterizations of the h-intra-hemiregular and h-quasi-hemiregular hemirings were investigated by these kinds of fuzzy h-ideals. The general properties of fuzzy h-ideals have been considered by Dudek, Kim, Jun, Ma, Zhan, and others. The readers refer to [2], [6], [9], [12], [22] in detail.

In 2007, Zhan and Davvaz [21] gave the fuzzy h-ideals with operators in hemirings and some properties were investigated. Pan [14] gave the concept of M-Γ-hemiring, and established a new fuzzy left h-ideal with operators. The present paper is organized as follows. In Section 2, we recall some basic definitions and properties of M-Γ-hemirings and fuzzy sets. In Sections 3 and 4, we introduce the concepts of M-fuzzy h-interior ideals and prime M-fuzzy h-ideals of M-Γ-hemirings, and we give some related properties. In Section 5, we describe the characterizations of h-semisimple M-Γ-hemirings.

2. Preliminaries

First, we recall some basic notions and results concerning Γ-hemirings, M-Γ-hemirings and fuzzy sets (for more details, see [11, 23]).

2.1. Γ-hemirings

Let S and Γ be two commutative additive semigroups. Then S is said to be a Γ-*semiring* if there exists a mapping $S \times \Gamma \times S \to S$ (images are denoted by $a\alpha b$ for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:

(i) $a\alpha(b + c) = a\alpha b + a\alpha c$,

(ii) $(a + b)\alpha c = a\alpha c + b\alpha c$,

(iii) $a(\alpha+\beta)c = a\alpha c + a\beta c$,

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

By a *zero* of a Γ-semiring S, we mean an element $0 \in S$ such that $0\alpha x = x\alpha 0 = 0$ and $0 + x = x + 0 = x$, for all $x \in S$ and $\alpha \in \Gamma$. A Γ-semiring with a zero is said to be a Γ-*hemiring*.

Throughout this paper, S is a Γ-hemiring and we use the symbol 0_S to denote the zero element of S.

A *left* (resp., *right*) *ideal* of a Γ-hemiring S is a subset A of S which is closed under addition such that $S\Gamma A \subseteq A$ (resp., $A\Gamma S \subseteq A$), where $S\Gamma A = \{x\alpha y \mid x \in S, y \in A, \alpha \in \Gamma\}$. Naturally, a subset A of S is called an *ideal* of S if it is both a left and a right ideal of S. A subset A of S is called an *interior ideal* if A is closed under addition such that $A\Gamma A \subseteq A$ and $S\Gamma A\Gamma S \subseteq A$.

A left ideal (right ideal, ideal) A of S is called a *left h-ideal* (*right h-ideal, h-ideal,*) of S, respectively, if, for any x, $z \in S$ and a, $b \in A$, $x + a + z = b + z$ implies that $x \in A$.

The *h-closure* $\overline{A}$ of A in S is defined by $\overline{A} = \{x \in S \mid x + a_1 + z = a_2 + z$ for some $a_1, a_2 \in A, z \in S\}$.

Clearly, if A is a left ideal of S, then $\overline{A}$ is the smallest left h-ideal of S containing A. We also have $\overline{\overline{A}} = \overline{A}$, for each $A \subseteq S$. Moreover, $A \subseteq B \subseteq S$ implies $\overline{A} \subseteq \overline{B}$.

An interior ideal A of S is called an *h-interior ideal* of S if A is closed under addition such that $\overline{A\Gamma A} \subseteq A, \overline{S\Gamma A\Gamma S} \subseteq A$ and $x + a + z = b + z$ implies that $x \in A$, for all $x, z \in S, a, b \in A$.

Definition 2.1 ([11], [23])

(i) Let μ and ν be fuzzy subsets of S. Then the h-product of μ and ν is defined by

$$(\mu\Gamma_h\nu)(x) = \bigvee_{x+a_1\gamma_1 b_1+z=a_2\gamma_2 b_2+z} \min\{\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)\}$$

$(\mu\Gamma_h\nu)(x) = 0$ if x cannot be expressed as $x + a_1\gamma_1 b_1 + z = a_2\gamma_2 b_2 + z$.

(ii) Let μ and ν be fuzzy subsets of M-Γ-hemiring S, for any $x \in S$, there exist $a_1, a_2, z \in S, \gamma_1, \gamma_2 \in \Gamma$ and $m_1, m_2, \in M$. Then the M-h-product of μ and ν is defined by

$$(\mu\Gamma_h\nu)(x) = \bigvee_{x+m_1\gamma_1 a_1+z=m_2\gamma_2 a_2+z} \min\{\mu(m_1), \mu(m_2), \nu(a_1), \nu(a_2)\}$$

$(\mu\Gamma_h\nu)(x) = 0$ if x cannot be expressed as $x + m_1\gamma_1 a_1 + z = m_2\gamma_2 a_2 + z$.

A fuzzy set is a function μ: $S \to [0,1]$. For any $A \subseteq S$, we denote the characteristic function of A by χ_A

$$\chi_A = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Proposition 2.2 ([11], [23]) *Let $A, B \subseteq S$. Then, the following statements hold:*

(1) $A \subseteq B \Leftrightarrow \chi_A \subseteq \chi_B$,

(2) $\chi_A \cap \chi_B = \chi_{A \cap B}$,

(3) $\chi_A \Gamma_h \chi_B = \chi_{\overline{A\Gamma B}}$.

2.2. M-Γ-hemirings

Definition 2.3 ([14]) A Γ-*hemiring* S *with operators* is an algebraic system consisting of a Γ-hemiring S, a set M and a function defined in the product set $M \times \Gamma \times S$ and having values in S ($M \times \Gamma \times S \to S$) such that, if $m\alpha x$ denotes the elements in S determined by the element m of M , x of S and the elements α, β of Γ, then

$$m\alpha(x + y) = m\alpha x + m\alpha y$$

and

$$m\alpha(x\beta y) = (m\alpha x)\beta(m\alpha y)$$

hold for any $x, y \in S$, $m \in M$ and $\alpha, \beta \in \Gamma$. We usually use the phrase "S is an M-Γ-hemiring" instead of a "Γ-hemiring with operators".

Example 2.4 Let $S = \{0, a, b\}$ be a set with an addition operation $(+)$ and a multiplication operation $(\cdot)$ as follows:

$+$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

and

$\cdot$	0	a	b
0	0	0	0
a	0	a	a
b	0	a	a

Then S is an M-Γ-hemiring where $\Gamma = M = S$.

Definition 2.5 ([14]) A left h-ideal I of an M-Γ-hemiring S is called a *left* M-h-*ideal* of S if $m\alpha x \in I$ for all $m \in M, x \in I$ and $\alpha \in \Gamma$.

Definition 2.6 ([14]) Let S be an M-Γ-hemiring and μ a fuzzy h-ideal of S. If the inequality $\mu(m\alpha x) \geqslant \mu(x)$ holds for any $x \in S$, $m \in M$ and $\alpha \in \Gamma$, then μ is said to be a *fuzzy left* h-*ideal with operators* of S. We use the phrases "an M-fuzzy left h-ideal of S" instead of "a fuzzy h-ideal with operators of S".

Definition 2.7 ([14]) A fuzzy set μ over M-Γ-hemiring S is called an M-*fuzzy left* (resp., *right*) h-*ideal* over S if it satisfies:

(F1) $\mu(x + y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$,

(F2) $\mu(x\alpha y) \geq \mu(y)$ for all $x, y \in S$ and $\alpha \in \Gamma$ (resp., $\mu(x\alpha y) \geq \mu(x)$),

(F3) $x + a + z = b + z$ implies $\mu(x) \geq \min\{\mu(a), \mu(b)\}$ for all $a, b, x, z \in S$,

(F4) $\mu(m\alpha x) \geq \mu(x)$ for all $x \in S$, $m \in M$ and $\alpha \in \Gamma$.

3. M-fuzzy h-interior ideals

It is well known that ideal theory plays a fundamental role in the development of hemirings. In this section, we consider M-fuzzy h-interior ideals of M-Γ-hemirings.

Definition 3.1 An h-interior ideal I of an M-Γ-hemiring S is called a *M-h-interior ideal* of S if $m\alpha x \in I$ for all $m \in M, x \in I$ and $\alpha \in \Gamma$.

Definition 3.2 A fuzzy set μ over M-Γ-hemiring S is called an *M-fuzzy h-interior ideal* over S if it satisfies (F1), (F3), (F4) and

(F5) $\mu(x\alpha y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S, \alpha \in \Gamma$,

(F6) $\mu(x\alpha y\beta z) \geq \mu(y)$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$.

Example 3.3 Assume $S = \mathbb{Z}_3 = \{0, 1, 2\}$, non-negative positive integers module 3. Then S is an M-Γ-hemiring where $\Gamma = M = \{0, 1\}$. Let $\alpha, \beta \in [0, 1)$ be such that $\alpha \geq \beta$. Define a fuzzy set μ over S by $\mu(0) = \alpha, \mu(1) = \mu(2) = \beta$. The one can easily check that μ is an M-h-interior ideal of S.

Note that, if μ is an M-fuzzy h-interior ideal of S, then $\mu(0) \geq \mu(x)$.
For any $t \in [0, 1]$, the set

$$U(\mu; t) = \{x \in S | \mu(x) \geq t\}$$

is called a level subset of μ.

Lemma 3.4 [23] *A fuzzy set μ in a Γ-hemiring S is a fuzzy h-interior ideal of S if and only if the each nonempty level subset $U(\mu; t)$, $t \in (0, 1)$, of μ is an h-interior ideal of S.*

Theorem 3.5 *A fuzzy set μ in an M-Γ-hemiring S is an M-fuzzy h-interior ideal of S if and only if the each nonempty level subset $U(\mu; t)$, $t \in (0, 1)$, of μ is an M-h-interior ideal of S.*

Proof. Let μ be an M-fuzzy h-interior ideal of S. Assume that $U(\mu; t) \neq \emptyset$ for $t \in [0, 1]$. Then by Lemma 3.4, $U(\mu; t)$ is an h-interior ideal of S. For every $x \in U(\mu; t)$, $\alpha \in \Gamma$, $m \in M$, we have

$$\mu(m\alpha x) \geqslant \mu(x) \geqslant t,$$

and hence $m\alpha x \in U(\mu; t)$. Thus $U(\mu; t)$ is an M-fuzzy h-interior ideal of S.

Conversely, suppose that $U(\mu; t) \neq \emptyset$ is an M-h-interior ideal of S. Then μ is a fuzzy h-interior ideal of S by Lemma 3.3. Now assume that there exist $y \in S$, $\gamma \in \Gamma$ and $k \in M$ such that

$$\mu(k\gamma y) < \mu(y).$$

Taking

$$t_0 := \frac{1}{2}(\mu(k\gamma y) + \mu(y)),$$

we obtain $t_0 \in [0, 1]$ and

$$\mu(k\gamma y) < t_0 < \mu(y).$$

This implies that $k\gamma y \notin U(\mu; t_0)$ and $y \in U(\mu; t_0)$, which leads a contradiction. Therefore

$$\mu(k\gamma y) \geq \mu(y),$$

for all $y \in S$, $\gamma \in \Gamma$ and $k \in M$. This completes the proof. ■

Proposition 3.6 *Every M-fuzzy h-ideal of M-Γ-hemiring S is an M-fuzzy h-interior ideal.*

Proof. By the Definitions 2.7 and 3.2, we only prove (F6) holds. Assume μ is an M-fuzzy h-ideal of S. Let $y, z \in S$, $\alpha, \beta \in \Gamma$. Then we have $\mu(x\alpha y\beta z) \geq \mu(y\beta z) \geq \mu(y)$ since μ is an M-fuzzy h-ideal of S. Hence, $\mu(x\alpha y\beta z) \geq \mu(y)$. ■

4. Prime M-fuzzy h-ideals

In this section, we consider prime M-fuzzy h-ideals of M-Γ-hemirings. A left (right) M-h-ideal P of S is said to be *prime* if $P \neq S$ and for any two left (right) h-ideals A and B of S from $A\Gamma B \subseteq P$ it follows either $A \subseteq P$ or $B \subseteq P$.

Definition 4.1 An M-fuzzy left (resp., right) h-ideal ψ of S is said to be *prime* if ψ is a non-constant function and for any two M-fuzzy left(right) h-ideals μ and ν of S, $\mu\Gamma\nu \subseteq \psi$ implies $\mu \subseteq \psi$ or $\nu \subseteq \psi$.

Example 4.2 Let $(S, +)$ and $(\Gamma, +)$ be two semigroups, where S and Γ are the sets of all non-negative integers and the operations are the usual additive operations. Define a mapping $S \times \Gamma \times S \to S$ by $a\gamma b = a \cdot \gamma \cdot b$, for all $a, b \in S$ and $\gamma \in \Gamma$, where "$\cdot$" is the usual multiplication. Then it can be easily verified that S, under the above multiplication and the structure Γ-mapping, is a Γ-hemiring. If we let $M := \{1\}$, then it is clear that S is an M-Γ-hemiring. Let $r, s \in [0, 1)$ be such that $r \leq s$. Define a fuzzy set μ over S by

$$\mu(x) = \begin{cases} s & \textit{if } x \textit{ is even}, \\ r & \textit{otherwise}. \end{cases}$$

Then μ is a prime M-fuzzy h-ideal over S.

Proposition 4.3 *A fuzzy set χ_P in an M-Γ-hemiring S is a prime M-fuzzy left(right) h-ideal of S if and only if P is a prime left(right) M-h-ideal of S, respectively.*

Proof. Straightforward. ■

Theorem 4.4 *A fuzzy subset* ζ *of* M*-*Γ*-hemiring* S *is a prime* M*-fuzzy left(right)* h*-ideal of* S *if and only if*

(1) $\zeta^0 = \{x \in S | \zeta(x) = \zeta(0)\}$ *is a prime left(right)* M*-*h*-ideal of* S*,*

(2) $Im\zeta = \{\zeta(x) | x \in S\}$ *contains exactly two elements,*

(3) $\zeta(0) = 1$.

Proof. We prove only the case of M-fuzzy left h-ideals. The proof for the right h-ideals is similar, and we omit it.

(1) Let ζ be a prime M-fuzzy left h-ideal of S. Then it is easy to check that ζ^0 is a prime left M-h-ideal of S.

(2) Suppose that $Im\zeta$ has more than two values. Then there exist two elements $p, q \in S \setminus \zeta^0$ such that $\zeta(p) \neq \zeta(q)$. Without loss of generality, we can assume that $\zeta(p) < \zeta(q)$. Since ζ is an M-fuzzy left h-ideal and $q \notin \zeta^0$, it follows that $\zeta(p) < \zeta(q) < \zeta(0)$. Hence there exist $r, t \in [0, 1]$ such that

$$\zeta(p) < r < \zeta(q) < t < \zeta(0). \qquad (*)$$

Let ν and μ be M-fuzzy left h-ideals defined by

$$\nu(x) = \begin{cases} r & \text{if } x \in \langle p \rangle, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \mu(x) = \begin{cases} t & \text{if } x \in \langle q \rangle, \\ 0 & \text{otherwise} \end{cases}$$

where $\langle p \rangle$ and $\langle q \rangle$ are ideals generated by p and q, respectively.

Then, for any $x \in S$, which can not be expressed in the form

$$x + m_1\gamma_1 b_1 + z = m_2\gamma_2 b_2 + z,$$

where $z \in S, m_1, m_2 \in \langle p \rangle$, $b_1, b_2 \in \langle q \rangle$ and $\gamma_1, \gamma_2 \in \Gamma$, we have

$$(\nu\Gamma\mu)(x) = 0.$$

Otherwise,

$$\begin{aligned} (\nu\Gamma_h\mu)(x) &= \bigvee_{x+m_1\gamma_1 b_1+z=m_2\gamma_2 b_2+z} (\min\{\nu(m_1), \nu(m_2), \mu(b_1), \mu(b_2)\}) \\ &= \min\{r, t\} = r. \end{aligned}$$

Since ζ is an M-fuzzy left h-ideal, from $x + m_1\gamma_1 b_1 + z = m_2\gamma_2 b_2 + z$ it follows that

$$\zeta(x) \geq \zeta(m_1\gamma_1 b_1) \wedge \zeta(m_2\gamma_2 b_2) \geq \zeta(b_1) \wedge \zeta(b_2) \geq r.$$

So, $(\nu\Gamma_h\mu)(x) \leq \zeta(x)$, whence $\nu\Gamma_h\mu \subseteq \zeta$, for ζ is a prime M-fuzzy left h-ideal, we can get $\nu \subseteq \zeta$ or $\mu \subseteq \zeta$. Therefore, $\nu(p) = r \leq \zeta(p)$ or $\mu(q) = t \leq \zeta(q)$ which contradicts to $(*)$. Consequently, $Im\zeta$ contains exactly two elements.

(3) Suppose that $\zeta(0) \neq 1$. Then, according to (2), $Im\zeta = \{a, b\}$, where $0 \leq a < b < 1$. Since $\zeta(0) \geq \zeta(x)$ for all $x \in S$, we have $\zeta(0) = b$. Thus,

$$\zeta(x) = \begin{cases} b & \text{if } x \in \zeta^0, \\ a & \text{otherwise,} \end{cases}$$

Consider, for fixed $p \in \zeta^0$ and $q \in S \setminus \zeta^0$, two fuzzy subsets

$$\mu(x) = \begin{cases} t & \text{if } x \in \langle p \rangle, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \nu(x) = \begin{cases} r & \text{if } x \in \langle q \rangle, \\ 0 & \text{otherwise} \end{cases}$$

where $0 \leq a < r < b < t \leq 1$.

It is clear that μ and ν are M-fuzzy left h-ideals of S.

Then, for any $x \in S$, if x does not satisfy the equality

$$x + m_1\gamma_1 b_1 + z = m_2\gamma_2 b_2 + z,$$

where $z \in S, m_1, m_2 \in \langle p \rangle$, $b_1, b_2 \in \langle q \rangle$ and $\gamma_1, \gamma_2 \in \Gamma$, we have

$$(\nu\Gamma\mu)(x) = 0.$$

Otherwise,

$$\begin{aligned}(\mu\Gamma_h\nu)(x) &= \bigvee_{x+m_1\gamma_1 b_1+z=m_2\gamma_2 b_2+z} (\min\{\mu(m_1), \mu(m_2), \nu(b_1), \nu(b_2)\}) \\ &= \min\{t, r\} = r.\end{aligned}$$

By (1), ζ^0 is a prime left M-h-ideal. If $a_1, a_2 \in \langle p \rangle$, then $a_1, a_2 \in \zeta^0$, because $p \in \zeta^0$ and $\langle p \rangle \subseteq \zeta^0$. This implies $x \in \zeta^0$. Thus $\zeta(x) = b > r = (\mu\Gamma_h\nu)(x)$. Therefore, $\mu\Gamma_h\nu \subseteq \zeta$. But $\mu(p) = t > b = \zeta(p)$ and $\nu(q) = r > a = \zeta(q)$, which gives $\mu \nsubseteq \zeta$ and $\nu \nsubseteq \zeta$. This contradicts to the assumption that ζ is a prime M-fuzzy left h-ideal of S. Hence $\zeta(0) = 1$. ∎

5. h-semisimple M-Γ-hemirings

In this section, we describe the characterizations of h-semisimple M-Γ-hemirings.

Definition 5.1

(1) A subset A of S is said to be Γ-*idempotent* if $A = \overline{A\Gamma A}$.

(2) A fuzzy set μ over S is said to be M-*fuzzy idempotent* if $\mu = \mu\widetilde{\Gamma}_h\mu$.

(3) An M-Γ-hemiring S is said to be h-*semisimple* if every M-h-ideal is Γ-idempotent.

Now, we can give the following lemma.

Lemma 5.2 *Let S be an M-Γ-hemiring. Then the following statements are equivalent:*

(1) *S is h-semisimple,*

(2) *$x \in \overline{M\Gamma x\Gamma S\Gamma x\Gamma S}$, for all $x \in S$,*

(3) *$A \subseteq \overline{M\Gamma A\Gamma S\Gamma A\Gamma S}$, for all $A \in S$.*

Proof. (1) $\Rightarrow$ (2): Let S be an h-semisimple M-Γ-hemiring. Then, for any $x \in S$, we have

$$\overline{M\Gamma S + S\Gamma x + S\Gamma x\Gamma S + \mathbb{N}_0 x},$$

where $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$, is the principle M-h-ideal of S generated by x. Thus,

$$x \in (\overline{M\Gamma S+S\Gamma x+S\Gamma x\Gamma S+\mathbb{N}_0 x})\Gamma(\overline{M\Gamma S+S\Gamma x+S\Gamma x\Gamma S+\mathbb{N}_0 x}) \subseteq \overline{M\Gamma x\Gamma S\Gamma x\Gamma S},$$

which implies $x \in \overline{M\Gamma x\Gamma S\Gamma x\Gamma S}$, for all $x \in \overline{M\Gamma x\Gamma S\Gamma x\Gamma S}$.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Let A be any M-h-ideal of S. Then,

$$A \subseteq \overline{M\Gamma A\Gamma S\Gamma A\Gamma S} \subseteq \overline{A\Gamma S\Gamma SA} \subseteq \overline{A\Gamma A}.$$

Therefore, S is h-semisimple. ■

Next, we discuss the relationship between M-fuzzy h-ideals and M-fuzzy h-interior ideals in h-semisimple M-Γ-hemirings.

Theorem 5.3 *Let S be an h-semisimple M-Γ-hemiring and let μ be any fuzzy set of S. Then μ is an M-fuzzy h-ideal if and only if it is an M-fuzzy h-interior ideal.*

Proof. If μ is an M-fuzzy h-ideal of S. Then, by Proposition 3.4, we know that μ is an M-fuzzy h-interior ideal.

Conversely, if μ is an M-fuzzy h-interior ideal of S. For any $x, y \in S$ and $\alpha \in \Gamma$. Since S is h-semisimple, by Lemma 5.2, there exist $a_i, a'_i, z \in S (i = 1, 2, 3)$, $\beta_i, \beta'_i, \in \Gamma (i = 1, 2, 3, 4, 5)$ and $m, m' \in M$ such that

$$x + m\beta_1 x\beta_2 a_1\beta_3 a_2\beta_4 x\beta_5 a_3 + z = m'\beta'_1 x\beta'_2 a'_1\beta'_3 a'_2\beta'_4 x\beta'_5 a'_3 + z,$$

and so

$$x\alpha y + m\beta_1 x\beta_2 a_1\beta_3 a_2\beta_4 x\beta_5 a_3\alpha y + z\alpha y = m'\beta'_1 x\beta'_2 a'_1\beta'_3 a'_2\beta'_4 x\beta'_5 a'_3\alpha y + z\alpha y.$$

Thus we have

$$\begin{aligned} \mu(x\alpha y) &\geqslant \mu(m\beta_1 x\beta_2 a_1\beta_3 a_2\beta_4 x\beta_5 a_3\alpha y) \wedge \mu(m'\beta'_1 x\beta'_2 a'_1\beta'_3 a'_2\beta'_4 x\beta'_5 a'_3\alpha y) \\ &\geqslant \mu(x). \end{aligned}$$

This proves that μ is an M-fuzzy right h-ideal of S. Similarly, we can prove that μ is an M-fuzzy left h-ideal of S. Therefore μ is an M-fuzzy h-ideal of S. ∎

Finally, we give a characterization of h-semisimple M-Γ-hemirings by M-fuzzy h-interior ideals.

Theorem 5.4 *An M-Γ-hemiring S is h-semisimple if and only if $\mu \cap \nu = \mu\Gamma_h\nu$, for any M-fuzzy h-interior ideals μ and ν.*

Proof. Let S be an h-semisimple M-Γ-hemiring. If μ and ν are M-fuzzy h-interior ideals, then by Proposition 3.4, we know μ and ν are M-fuzzy h-ideals of S. Thus, we have $\mu\Gamma_h\nu \subseteq \mu\Gamma_h\chi_S \subseteq \mu$ and $\mu\Gamma_h\nu \subseteq \chi_S\Gamma_h\nu \subseteq \nu$. So $\mu\Gamma_h\nu \subseteq \mu \cap \nu$.

For any $x \in S$, since S is h-semisimple, by Lemma 5.2, there exist $a_i, a_i', z \in S$ $(i = 1, 2, 3)$, $\beta_i, \beta_i', \in \Gamma$ $(i = 1, 2, 3, 4, 5)$ and $m, m' \in M$ such that

$$x + m\beta_1 x\beta_2 a_1\beta_3 a_2\beta_4 x\beta_5 a_3 + z = m'\beta_1' x\beta_2' a_1'\beta_3' a_2'\beta_4' x\beta_5' a_3' + z,$$

Thus we have

$$\begin{aligned}
(\mu\Gamma_h\nu)(x) &= \bigvee_{x+m_1\gamma_1 b_1+z=m_2\gamma_2 b_2+z} (\min\{\mu(m_1), \mu(m_2), \nu(b_1), \nu(b_2)\}) \\
&\geq \min\{\mu(m\beta_1 x\beta_2 a_1), \mu(m'\beta_1' x\beta_2' a_1'), \nu(a_2\beta_4 x\beta_5 a_3), \nu(a_2'\beta_4' x\beta_5' a_3')\} \\
&\geq \min\{\mu(x), \nu(x)\} \\
&= (\mu \cap \nu)(x),
\end{aligned}$$

i.e., $\mu \cap \nu \subseteq \mu\Gamma_h\nu$, whence $\mu \cap \nu = \mu\Gamma_h\nu$.

Conversely, let A be any M-h-ideal of S, then it is an M-h-interior ideal. Thus, we have

$$\chi_A = \chi_A \cap \chi_A = \chi_A\Gamma_h\chi_A = \chi_{\overline{A\Gamma A}},$$

which implies, $A = \overline{A\Gamma A}$. Thus S is h-semisimple. ∎

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References

[1] Barnes, W.E., *On the Γ-rings of Nobusawa*, Pacific J. Math., 18 (1966), 411-422.

[2] Dudek, W.A., Shabir, M., Anjum, R., *Characterizations of hemirings by their h-ideals*, Comput. Math. Appl., 59 (2010), 3167-3179.

[3] Dutta, T.K., Chanda, T., *Structures of fuzzy ideals of* Γ*-rings*, Bull. Malays. Math. Sci. Soc., 28 (1) (2005), 9-15.

[4] Han, J.S., Kim, H.S., Neggers, J., *Semiring orders in a semiring*, Appl. Math. Inf. Sci., 6 (2012), 99–102.

[5] Jun, Y.B., *On fuzzy prime ideals of* Γ*-rings*, Soochow J. Math., 21 (1) (1995), 41-48.

[6] Jun, Y.B., Lee, C.Y., *Fuzzy* Γ*-rings*, Pusan Kyongnan Math. J. (presently, East Asian Math. J.), 8 (2) (1992), 163-170.

[7] Jun, Y.B., Öztürk, M.A., Song, S.Z., *On fuzzy h-ideals in hemirings*, Inform. Sci **162** (2004), 211-226.

[8] Ma, X., Yin, Y., Zhan, J., *Characterizations of h-intra- and h-quasi-hemiregular hemirings*, Comput. Math. Appl., 63 (2012), 783-793.

[9] Ma, X., Zhan, J., *On fuzzy h-ideals of hemirings*, J. Syst. Sci. Complexity, 20 (2007), 470-478.

[10] Ma, X., Zhan, J., *Generalized fuzzy h-bi-ideals and h-quasi-ideals of hemirings*, Inform. Sci., 179 (2009), 1249-1268.

[11] Ma, X., Zhan, J., *Fuzzy h-ideals in h-hemiregular and h-semisimple* Γ*-hemirings*, Neural Comput. Appl., 19 (2010), 477-485.

[12] Ma, X., Zhan, J., Shum, K.P., *Generalized fuzzy h-ideals of hemirings*, Bull. Malays. Math. Sci. Soc., (2), 34 (3) (2011), 561-574.

[13] Öztürk, M.A., Uckum, M., Jun, Y.B., *Fuzzy ideals in Gamma-rings*, Turk. J. Math., 27 (2003), 369-374.

[14] Pan, D., Zhan, J., *Fuzzy h-ideals with operarors in* Γ*-hemirings*, Ann. Fuzzy Math. Inform., 5 (2013), 183-192.

[15] Rao, M.K., Γ*-semirings 1*, Southeast Asian Bull. Math., 19 (1995), 49-54.

[16] Sardar, S.K., Mandal, D., *Fuzzy h-ideal in* Γ*-hemiring*, Int. J. Pure Appl. Math., 56 (2009), 439–450.

[17] La Torre, D.R., *On h-ideals and k-ideals in hemirings*, Publ. Math. (Debrecen), 12 (1965), 219-226.

[18] Yin, Y., Huang, X., Xu, D., Li, H., *The characterizations of h-semisimple hemirings*, Int. J. Fuzzy Systems, 11 (2009), 116-122.

[19] Yin, Y., Li, H., *The characterizations of h-hemiregular hemirings and h-intra-hemiregular hemirings*, Inform Sci., 178 (2008), 3451-3464.

[20] Zadeh, L.A., *Fuzzy sets*, Inform. Control, 8 (1965), 338-353.

[21] Zhan, J., Davvaz, B., *L-fuzzy h-ideals with operators in hemirings*, Northeast Math. J., 23 (1) (2007), 1-14.

[22] Zhan, J., Dudek, W., *Fuzzy h-ideal of hemirings*, Inform. Sci., 177 (2007), 876-886.

[23] Zhan, J., Shum, K.P., *On fuzzy h-ideals in* Γ*-hemirings*, Neural Comput. Appl., 20 (2011), 495-505.

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