ON THE CONJUGATION INVARIANT PROBLEM IN THE MOD p DUAL STEENROD ALGEBRA

Neşet Deniz Turgay

Bornova-Izmir 35050
Turkey
e-mail: Deniz_Turgay@yahoo.com

Abstract. The Leibniz–Hopf algebra $\mathcal{F}$ is the free associative $\mathbf{Z}$-algebra on one generator in each positive degree, with coproduct given by the Cartan formula. Fix an odd prime p, and let $\mathcal{A}$ denote the Bockstein–free part of the mod p Steenrod algebra. We investigate an alternative approach to the conjugation invariant problem in the dual Steenrod algebra $\mathcal{A}^*$ using the conjugation invariants in $\mathcal{F}^* \otimes \mathbf{Z}/p$.

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1. Introduction

Let p be a fixed odd prime and $F(p) = \mathbb{F}_p\{S^0, S^1, \ldots, S^i, \ldots,\}$ the free associative graded algebra over a field of characteristic p, $\mathbb{F}_p$ on generators S^i of degree i where S^0 is the unit. We may extend $F(p)$ to more rich algebraic structures. Particulary, omitting the above grading and setting $S^i = \mathcal{P}^i$, where $\mathcal{P}^i$, $i \geq 0$, represent the Steenrod reduced powers [19] of degree $2i(p-1)$, we see that the *Bockstein–free part of the* mod p *Steenrod algebra*, which we denote by $\mathcal{A}$ is naturally defined as the quotient of $F(p)$ by the Adem relations [19] and $\mathcal{P}^0 = 1$, the identity element. Topologically, $\mathcal{A}$ is also known as the algebra of stable cohomology operations for ordinary cohomology H^* over $\mathbb{F}_p$. Furthermore, Milnor [17] has showed that $\mathcal{A}$ is a graded connected Hopf algebra.

We now investigate a relationship between $F(p)$ and the Leibniz–Hopf algebra. Precisely, $F(p)$ can be turned into a graded connected Hopf algebra by defining a coproduct to be that given by the *Cartan* formula $\Delta(S^n) = \sum_{i=0}^{n} S^i \otimes S^{n-i}$. This Hopf algebra is cocommutative in the strict (i.e., non-graded) sense. Now let $\mathcal{F}$ denote the *Leibniz–Hopf algebra* [11, Section 1]. In particular, setting $\mathbb{F}_p = \mathbf{Z}/p$, we may see that $F(p)$ is the mod p reduction of the Leibniz–Hopf algebra, $\mathcal{F} \otimes \mathbf{Z}/p$. $\mathcal{F}$ is also known as the *algebra of non-commutative symmetric functions* [10] and has been also studied in [12–16, 18] because of its various connections to other algebraic structures. The graded dual Hopf algebra $\mathcal{F}^*$ is the Hopf algebra of *qua-*

sisymmetric functions and also known as the *overlapping shuffle algebra* [11, Section 1]. $\mathcal{F}^*$ has been of interest to combinatorialists, topologists, algebraists, and studied in [2–4, 13].

We now investigate $\mathcal{F}$ and $\mathcal{F}^*$ in connection with the Steenrod algebra. Recalling the preceding paragraphs, we may see that $\mathcal{A}$ is the quotient algebra of $\mathcal{F} \otimes \mathbf{Z}/p$ by the Adem relations. This quotient structure gives us the surjective Hopf algebra morphism $\pi\colon \mathcal{F} \otimes \mathbf{Z}/p \to \mathcal{A}$, where $\pi(S^i) = \mathcal{P}^i$. Moreover, giving a new grading to $\mathcal{F}$ on generators S^i of degree $2i(p-1)$, $i \geq 0$, π extends to a graded homomorphism (from now on, we use this grading). Following this, we arrive at the graded Hopf algebra inclusion $\pi^*\colon \mathcal{A}^* \to \mathcal{F}^* \otimes \mathbf{Z}/p$ [4, Section 5] (where $\mathcal{A}^*$ denotes the graded dual of $\mathcal{A}$) dual to the homomorphism π. This will be the heart of this paper. Note that this homomorphism is also considered in [22]. The Hopf algebra structure of $\mathcal{A}^*$ admits a unique Hopf algebra conjugation (or "antipode"), $\chi_{\mathcal{A}^*}$. Invariant problem under $\chi_{\mathcal{A}^*}$ has been studied in [8], since it is relevant for the commutativity of ring spectra [1, Lecture 3].

In this paper we reconsider this problem. Let us explain briefly. The conjugation invariants in $\mathcal{A}^*$ form a subvector space, $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$ (where 1 denotes the identity homomorphism). Crossley and Whitehouse [8, Section 1] have given a description of $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$ in some generality. Particularly, it has been showed that how the Poincaré series for $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$ can be determined using Molien's theorem. In Section 3, we give an alternative approach for this (Theorem 3.3). We refer reader to [1, Lecture 3] and [7] for more detailed motivation.

The arguments used to obtain the results in this present paper are similar to those of their mod 2 corresponding parts in [21, Section 5]. However, there are two notable differences that appear in the odd primary case. Let us explain briefly. Firstly, we have to deal with making the homomorphism, π graded, and also modify some of the early results according to this (see Remark 3.2). Secondly, conjugation formula (1) is sign involved. This fact together with mod p binomial coefficients necessitate more careful analysis in Section 3. These difficulties cause the results obtained in this work to be not quite straightforward.

2. Preliminaries

As a vector space, $\mathcal{F} \otimes \mathbf{Z}/p$ has a basis of words $S^{j_1}S^{j_2}\cdots S^{j_r}$ (of finite length) in the letters $S^{j_1}, S^{j_2}, \ldots$, which we denote by $S^{j_1,j_2,\ldots,j_r}$. The *degree* of an element $S^{j_1,j_2,\ldots,j_s}$ is defined to be $2(p-1)\sum_{i=1}^{s} j_i$. We denote the dual basis for $\mathcal{F}^* \otimes \mathbf{Z}/p$ by $\{S_{j_1,j_2,\ldots,j_r}\}$. A conjugation formula for this Hopf algebra is given by the mod p reduction of formula [9, Proposition 3.4] as follows.

$$\chi(S_{j_1,\ldots,j_r}) = (-1)^r \sum S_{b_1,\ldots,b_n} \tag{1}$$

summed over all coarsenings $b_1, \ldots, b_n$ of the *reversed* word $j_r, \ldots, j_1$, i.e., all words $b_1, \ldots, b_n$ that admit $j_r, \ldots, j_1$ as a refinement [6]. For instance,

$$\chi(S_{5,3,2}) = -S_{2,3,5} - S_{5,5} - S_{2,8} - S_{10}.$$

We now recall the *overlapping shuffle product* from [21, Section 2]: the overlapping shuffle product of $S_{a_1,\ldots,a_t}$ and $S_{b_1,\ldots,b_y}$ is defined by

$$\mu(S_{a_1,\ldots,a_t} \otimes S_{b_1,\ldots,b_y}) = \sum_h h(S_{a_1,\ldots,a_t,b_1,\ldots,b_y}),$$

where h first inserts a certain number ℓ of 0s into $a_1, \ldots, a_t$, and inserts a number of ℓ' of 0s into $b_1, \ldots, b_y$, where

$$0 \leq \ell \leq y, \quad 0 \leq \ell' \leq t, \quad t + \ell = y + \ell',$$

then it adds the first indices together, then the second, and so on. The sum is over all such h for which the result contains no 0s. For instance,

$$\mu(S_{4,3} \otimes S_2) = S_{4,3,2} + S_{4,2,3} + S_{2,4,3} + S_{6,3} + S_{4,5}.$$

We refer to [11, Section 2] for an alternative definition of this product. We now give some background for $\mathcal{A}^*$. Beside its Hopf algebra structure, Milnor [17] also showed that $\mathcal{A}^*$ is the polynomial part of the mod p dual Steenrod algebra on generators ξ_i $(i \geq 1)$ of degree $2(p^i - 1)$ (see Section 3 of [19, Chapter 6]). Turning to the Hopf algebra homomorphism π^* in Section 1. We now recall the following formulas from [4, Section 5] as follows:

$$\pi^\star(\xi_n) = S_{p^{n-1},p^{n-2},\ldots,p,1}, \tag{2}$$

$$\pi^\star(\xi_n^{p^m}) = S_{p^{m+n-1},p^{m+n-2},\ldots,p^{m+1},p^m}. \tag{3}$$

It is worth pointing out that we obtain calculations in $\mathcal{F}^*$, and π^* is an algebra morphism on the target overlapping shuffle product [11, Section 6].

Now we recall some of the terminology from [6]. A word $S_{j_1,j_2,\ldots,j_n}$ is a *palindrome* if $j_1 = j_n$, $j_2 = j_{n-1}$, and so on. A palindrome is referred to as an *even-length palindrome*, which we denote by ELP, if its length is even. For example, $S_{8,3,3,8}$ is an ELP. A non-palindrome $S_{j_1,\ldots,j_r}$ is referred to as a *higher non-palindrome*, which we denote by HNP if $j_1, \ldots, j_r$ is lexicographically bigger than its reverse $j_r, \ldots, j_1$. For instance, $S_{8,5,4,8}$ is an HNP.

3. A different approach on the conjugation invariant problem in $\mathcal{A}^*$

We now introduce a different approach to determine a basis for $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$.

Theorem 3.1 [5, Theorem 2.5] *In the dual Leibniz–Hopf algebra, $\mathcal{F}^*$, and in the mod p dual $\mathcal{F}^* \otimes Z/p$ for any prime $p > 2$, the submodule* $\mathrm{Ker}(\chi - 1)$ *is equal to* $\mathrm{Im}(\chi + 1)$ *and is free on a basis consisting of: the $(\chi + 1)$-images of all ELPs and HNPs. Thus, in degree n, this module has rank 2^{n-2} if n is even, and $2^{n-2} - 2^{(n-3)/2}$ if n is odd.*

Remark 3.2 In [5, Theorem 2.5], graded algebra structure of $\mathcal{F}$ is obtained by giving S^n, $n \geq 1$, degree n. Thus, when $p > 2$, recalling the modified grading from the Section 1, we give the adapted version of the dimension formula of $\mathrm{Ker}(\chi - 1)$ in Theorem 3.1 as: in $2(p-1)n$, degrees $\mathrm{Ker}(\chi - 1)$ has dimension 2^{n-2} if n is even, and $2^{n-2} - 2^{(n-3)/2}$ if n is odd.

For simplicity, from now on, we denote $\mathcal{F}^* \otimes Z/p$ by $\mathcal{F}^*$. To have a connection between conjugation invariants in $\mathcal{F}^*$ and $\mathcal{A}^*$, we reconsider the graded Hopf algebra inclusion. In particular, π^*, being a Hopf algebra morphism, we see that the following diagram commutes

(4)
$$\begin{array}{ccc} \mathcal{A}^* & \xhookrightarrow{\pi^*} & \mathcal{F}^* \\ \downarrow{\scriptstyle \chi_{\mathcal{A}^*}} & & \downarrow{\scriptstyle \chi} \\ \mathcal{A}^* & \xhookrightarrow{\pi^*} & \mathcal{F}^* \end{array} .$$

Moreover, being an injective morphism, in each fixed degree, this gives the following relationship between the conjugation invariants in $\mathcal{A}^*$ and $\mathcal{F}^*$.

Theorem 3.3 $\pi^*(\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)) = \mathrm{Ker}(\chi - 1) \cap \mathrm{Im}(\pi^*)$. ∎

Proposition 3.4 *Let $S_{p^a,p^b} \in \mathcal{F}^*$. Then*

$$\pi^\star(\xi_1^{p^a}\xi_1^{p^b}) = (\chi + 1)(S_{p^a,p^b}).$$

Proof. Let $S_{p^a,p^b} \in \mathcal{F}^*$. Then, by formula (1), we obtain

$$(\chi + 1)(S_{p^a,p^b}) = S_{p^a,p^b} + S_{p^b,p^a} + S_{p^b+p^a}.$$

On the other hand, as π^* is an algebra morphism, formula (3) gives

$$\pi^\star(\xi_1^{p^a}\xi_1^{p^b}) = (\chi + 1)(S_{p^a,p^b}).$$ ∎

Corollary 3.5 *Let S_{p^a,p^b} be an HNP or an ELP. Then in $p^a + p^b$ degrees*

$$(\chi + 1)(S_{p^a,p^b}) \in \mathrm{Ker}(\chi - 1) \cap \mathrm{Im}(\pi^*).$$

We demonstrate Theorem 3.3 in the following examples at the prime 3.

Example 3.6 In degree 8, $\mathcal{F}^*$ has a basis: $\{S_2, S_{1,1}\}$. By Theorem 3.1, $(\chi + 1)$-images of HNPs and ELPs form a basis for $\mathrm{Ker}(\chi - 1)$, that is $(\chi + 1)(S_{1,1}) = S_2 + 2S_{1,1}$. On the other hand, in the same degree, $\{\xi_1^2\}$ is a basis for $\mathcal{A}^*$. Hence, $\mathrm{Im}(\pi^*)$ has $\pi^*(\xi_1^2)$ as a basis, since π^* is a monomorphism. Following this, by formula (2), we have $\pi^*(\xi_1^2) = S_2 + 2S_{1,1}$ from which we conclude that

$$\mathrm{Ker}(\chi - 1) = \mathrm{Im}(\pi^*).$$

It follows that, by Theorem 3.3, we see that $\{\pi^*(\xi_1^2)\}$ is a basis for $\pi^*(\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1))$, and hence $\{\xi_1^2\}$ is a basis for $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$, since π^* is a monomorphism.

Example 3.7 In degree 12, $\mathcal{F}^*$ has a basis $\{S_3, S_{2,1}, S_{1,2}, S_{1,1,1}\}$. By Theorem 3.1 and Table 2, we see that $\mathrm{Ker}(\chi - 1) \cap \pi^*(\mathcal{A}_p^*) = \emptyset$, and hence $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1) = \emptyset$.

Example 3.8 We now recall a more efficient method from [21, Section 5]. In degree 16, we first give an order to the monomial basis of $\mathcal{F}^*$ with respect to lexicographic order. We denote this ordered basis by U which is given as follows:

$$U = \{S_4, S_{3,1}, S_{2,2}, S_{2,1,1}, S_{1,3}, S_{1,2,1}, S_{1,1,2}, S_{1,1,1,1}\}.$$

For example, the basis U tells us that $S_{2,2}$ is lexicographically bigger than $S_{1,2,1}$. We now recall linear algebra from [20, pp. 199-200]: if V is the column space of a matrix A, and W is the column space of a matrix B, then $V + W$ is the column space of the matrix $D = [A \;\; B]$ and $\dim(V + W) = \mathrm{rank}\ (D)$ and $\dim(V \cap W) =$ nullity of D which leads the following formula:

$$\dim(V + W) + \dim(V \cap W) = \dim(V) + \dim(W). \tag{5}$$

Table 1: Bases of $\mathrm{Im}(\pi^*)$ in degrees $12, 16, 20$.

Degree 12	$\pi^*(\xi_1^3) = S_3$
Degree 16	$n_1 = \pi^*(\xi_1^4) = S_4 + S_{3,1} + S_{1,3}$ $n_2 = \pi^*(\xi_2) = S_{3,1}$
Degree 20	$n_1' = \pi^*(\xi_1^5) = S_5 + 2S_{4,1} + S_{3,2} + 2S_{3,1,1} + S_{2,3}+$ $2S_{1,4} + 2S_{1,3,1} + 2S_{1,1,3}$ $n_2' = \pi^*(\xi_2\xi_1) = S_{4,1} + S_{3,2} + 2S_{3,1,1} + S_{1,3,1}$

Note that Table 1 is also partially used in another point of view in [22].

Table 2: Bases of $\mathrm{Ker}(\chi - 1)$ in degrees $12, 16, 20$.

Degree 12	$(\chi + 1)(S_{2,1}) = S_3 + S_{2,1} + S_{1,2}$
Degree 16	$t_1 = (\chi + 1)(S_{3,1}) = S_4 + S_{3,1} + S_{1,3}$ $t_2 = (\chi + 1)(S_{2,2}) = S_4 + 2S_{2,2}$ $t_3 = (\chi + 1)(S_{2,1,1}) = -S_4 - S_{2,2} + S_{2,1,1} - S_{1,3} - S_{1,1,2}$ $t_4 = (\chi + 1)(S_{1,1,1,1}) = S_4 + S_{3,1} + S_{2,2} + S_{2,1,1} + S_{1,3}+$ $S_{1,2,1} + S_{1,1,2} + 2S_{1,1,1,1}$
Degree 20	$t_1' = (\chi + 1)(S_{4,1}) = S_5 + S_{4,1} + S_{1,4}$ $t_2' = (\chi + 1)(S_{3,2}) = S_5 + S_{3,2} + S_{2,3}$ $t_3' = (\chi + 1)(S_{3,1,1}) = -S_5 + S_{3,1,1} - S_{2,3} - S_{1,4} - S_{1,1,3}$ $t_4' = (\chi + 1)(S_{2,2,1}) = -S_5 - S_{3,2} + S_{2,2,1} - S_{1,4} - S_{1,2,2}$ $t_5' = (\chi + 1)(S_{2,1,1,1}) = S_5 + S_{3,2} + S_{2,3} + S_{2,1,2} + S_{2,1,1,1}+$ $S_{1,4} + S_{1,2,2} + S_{1,1,3} + S_{1,1,1,2}$ $t_6' = (\chi + 1)(S_{1,2,1,1}) = S_5 + S_{4,1} + S_{2,3} + S_{2,2,1} + S_{1,4}+$ $S_{1,3,1} + S_{1,2,1,1} + S_{1,1,3} + S_{1,1,2,1}$

To use the above argument, using Tables 1 and 2, we first write out the basis matrix of $\mathrm{Im}(\pi^*)$, denoted by $[N]_Y$, and of $\mathrm{Ker}(\chi - 1)$, denoted by $[T]_U$, relative to the basis U in the following:

$$[N]_U = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad [T]_U = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Let us be precise. The first column of $[N]_U$ represents the coordinate vector of basis element n_1 in Table 1, relative to the basis U. On the other hand, the first column of $[T]_U$ represents the coordinate vector of basis element t_1 in Table 2, relative to the basis U, the second column of $[T]_U$ represents the coordinate vector of basis element t_2 in Table 2, relative to the basis Y, and so on.

It is now clear to see the rank of $D = [[N]_U \ [T]_U]$ is 5. Thus, by formula (5), we obtain: $5 + \dim([N]_U \cap [T]_U) = 6$, from which we can deduce $\dim([M]_Y \cap [N]_Y) = 1$. By Tables 1 and 2, $\mathrm{Im}(\pi^*)$ and $\mathrm{Ker}(\chi - 1)$ have $\pi^*(\xi_1^4)$ as a common basis element. Therefore, by dimension reason, $\{\pi^*(\xi_1^4)\}$ has to be a basis for $\mathrm{Ker}(\chi - 1) \cap \mathrm{Im}(\pi^*)$, and hence $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1)$ has a basis $\{\xi_1^4\}$ in degree 16.

Example 3.9 In degree 20 we briefly give details of the calculations. We again give lexicographical order to the monomial basis of $\mathcal{F}^*$, which we denote by U' and given in the following:

$$U' = \{S_5, S_{4,1}, S_{3,2}, S_{3,1,1}, S_{2,3}, S_{2,2,1}, S_{2,1,2}, S_{2,1,1,1}, S_{1,4}, S_{1,3,1}, S_{1,2,2}, \\ S_{1,2,1,1}, S_{1,1,3}, S_{1,1,2,1}, S_{1,1,1,2}, S_{1,1,1,1,1}\}.$$

By Tables 1 and 2, writing the basis matrix of $\mathrm{Im}(\pi^*)$, denoted by $[N']_{U'}$, and of $\mathrm{Ker}(\chi - 1)$, denoted by $[T']_{U'}$, relative to the basis U', we see that the rank of $D = \left[[N']_{U'} \ [T']_{U'} \right]$ is 8. Precisely, this is because, both n_1' and n_2' in Table 1 have a summand with a coefficient 2 and these do not allow n_1' and n_2' to be written as linear combinations of $\{t_1', \ldots, t_6'\}$. On the other hand, $\mathrm{rank}\left([N']_{U'} \right) = 2$ and $\mathrm{rank}\left([T']_{U'} \right) = 6$. Therefore, $8 + \dim(N' \cap T') = 8$ from which we can deduce

$$\mathrm{Ker}(\chi - 1) \cap \mathrm{Im}(\pi^*) = \emptyset,$$

and hence $\mathrm{Ker}(\chi_{\mathcal{A}^*} - 1) = \emptyset$.

We refer the reader to (http://www.skaji.org/code) for a computer-aided approach to obtain conjugation invariants in the dual Leibniz-Hopf algebra and the dual Steenrod algebra. None of the calculations in this present paper depends on the above computer-aided approach.

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