FACTOR BISEMIRINGS

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Abstract. In this paper we define congruence relations on bisemirings and bisemiring homomorphisms. We show that each bisemiring homomorphism defines a congruence relation on a bisemiring and then we introduce factor bisemirings. In the last section, we prove analogue of the isomorphism theorems.

Keywords: congruences, homomorphisms, analogue of the isomorphism theorems.

1. Introduction

In 1934, Vandever introduced the notion of a semiring which was a common generalization of rings and distributive lattices. The following definition has been taken from [4].

A semiring $(R, +, \cdot)$ is a non-empty set in which $(R, +)$ and $(R, \cdot)$ are semigroups such that "·" is distributive over " + ".

Corresponding to semiring in 2001, M.K. Sen, Shamik Ghosh and Suma Ghosh introduced the concept of a bisemiring in [3]. A bisemiring $(R, +, \cdot, \times)$ is a non-empty set in which $(R, +, \cdot)$ and $(R, +, \times)$ are semirings. In other words $(R, +)$, $(R, \cdot)$ and $(R, \times)$ are semigroups such that for all $a, b, c \in R$,

$$a \cdot (b + c) = a \cdot b + a \cdot c$$
$$(b + c) \cdot a = b \cdot a + c \cdot a$$

and

$$a \times (b \cdot c) = (a \times b) \cdot (a \times c)$$
$$(b \cdot c) \times a = (b \times a) \cdot (c \times a)$$

where " + " is called addition " · " is called multiplication and " × " is called product. To understand the above concept we give some examples. The following examples have been taken from [3].

Example 1.1.

(i) Let $P(X)$ be the power set on a non-empty set X. Then $(P(X), \triangle, \cap, \cup)$ is a bisemiring.

(ii) Let N be the set of natural numbers. Then $(N, \min, \max, +)$, $(N, \min, \max, \cdot)$ are bisemirings.

(iii) Consider again the set N of natural numbers. Define $a + b = gcd(a, b)$ and $a \cdot b = lcm(a, b)$ and $a \times b = ab$. Then $(N, +, \cdot, \times)$ is a bisemiring.

2. Congruences on Bisemirings

In this section we define left compatible (left congruence), right compatible (right congruence) and compatible (congruence) relations on bisemirings. The idea comes from the book [2] in which the author has defined these terms for semigroups. At the end of this section we prove a result which gives equivalent conditions for congruence relations on bisemirings by using the same idea of [2].

Definition 2.1. Let $(R, +, \cdot)$ be a bisemiring. A relation ρ on R is said to be left compatible if for all s, t and $a \in R$ such that $(s, t) \in \rho$ implies that $(a + s, a + t)$, $(a \cdot s, a \cdot t)$ and $(a \times s, a \times t) \in \rho$.

The relation ρ is said to be right compatible if for all s, t and $b \in R$ such that $(s, t) \in \rho$ implies $(s + b, t + b)$, $(s \cdot b, s \cdot b)$ and $(s \times b, t \times b) \in \rho$.

It is called compatible if for all s, t, u and $v \in R$, (s, t) and $(u, v) \in \rho$ implies that $(s + u, t + v)$, $(s \cdot u, t \cdot v)$ and $(s \times u, t \times v) \in \rho$.

A left (right) compatible equivalence relation is called a left (right) congruence relation. A compatible equivalence relation is called a congruence relation.

To understand the above notion we give an example.

Example 2.2. Let $P(X)$ be the power set on a non-empty set X. Then $(P(X), \triangle, \cap, \cup)$ is a bisemiring as discussed in Section 1. Let $\rho = \{(A, B) : A = B\}$ be a relation on $P(X)$. Then one can easily verify that ρ is a congruence relation on $P(X)$.

We are now going to state and prove a result which gives equivalent conditions between left (right) congruence relations and congruence relations.

Proposition 2.3. *A relation ρ on a bisemiring R is a congruence relation if and only if it is both a left and a right congruence relation.*

Proof. Suppose that ρ is a congruence relation on R. Let s, t and $a \in R$ such that $(s, t) \in \rho$, then $(a + s, a + t)$, $(a \cdot s, a \cdot t)$ and $(a \times s, a \times t) \in \rho$, since $(a, a) \in \rho$. This shows that ρ is a left congruence relation. In the same way we can show that ρ is a right congruence relation. Conversely, assume that ρ is both right and left congruence relation. Let s, t, u and $v \in R$ such that (s, t) and $(u, v) \in \rho$. This implies that $(s + u, t + u)$, $(s \cdot u, t \cdot u)$ and $(s \times u, t \times u) \in \rho$, as ρ is a right compatible relation and $(t+u, t+v)$, $(t \cdot u, t \cdot v)$ and $(t \times u, t \times v) \in \rho$, as ρ is a left compatible realtion. This implies that $(s+u, t+v)$, $(s \cdot u, t \cdot v)$ and $(s \times u, t \times v) \in \rho$, as ρ is transitive. Thus ρ is a congruence relation. ∎

Like homomorphisms of other algebraic structures such as groups and rings, homomorphisms of bisemirings are maps which preserve binary operations. In the following section we give a proper definition of a bisemiring homomorphism.

3. Homomorphisms of bisemirings

Definition 3.1. Let $(R, +, \cdot, \times)$ and $(S, \oplus, \circ, \otimes)$ be two bisemirings. A function $f : R \to S$ is said to be a bisemiring homomorphism or simply a homomorphism if it satisfies the following conditions:

(i) $f(r+s) = f(r) \oplus f(s)$ for all $r, s \in R$;

(ii) $f(r \cdot s) = f(r) \circ f(s)$ for *all* $r, s \in R$;

(iii) $f(r \times s) = f(r) \otimes f(s)$ for all $r, s \in R$.

The terms monomorphism, epimorphism, isomorphism, endomorphism and automorphism can be defined in the same way. If there is an isomorphism from a bisemiring R to a bisemiring S, then we say that R is isomorphic to S and write $R \cong S$.

To understand the above concept we give an example.

Example 3.2. Let N be the set of natural numbers. Then $(N, \min, \max, +)$ and $(2N, \min, \max, +)$ are bisemirings as discussed in Section 1. Now define a map $\theta : N \to 2N$ by $\theta(n) = 2n$. Then it can be easily verified that θ is a bisemiring homomorphism.

We are now going to state a result in which we prove that corresponding to every homomorphism there is a congruence relation. The result is important because once we get this congruence relation we can get factor bisemiring.

Theorem 3.3. *If f is a homomorphism from a bisemiring R to a bisemiring S, then f defines a congruence relation ρ on R given by $r \rho s$ if and only if $f(r) = f(s)$.*

Proof. First, we show that this is an equivalence relation. Since $f(r) = f(r)$ for all $r \in R$, therefore $r \rho r$ and the relation is reflexive. If $r \rho s$, then $f(r) = f(s)$ and this implies that $f(s) = f(r)$. Thus $s \rho r$ and so the relation is symmetric. Now, if $r \rho s$ and $s \rho t$, then $f(r) = f(s)$ and $f(s) = f(t)$ and this gives us $f(r) = f(t)$. This shows that $r \rho t$ and so the relation is transitive. Now, let $r \rho s$ and $t \rho u$, then $f(r) = f(s)$ and $f(t) = f(u)$. As $f(r+t) = f(r) + f(t) = f(s) + f(u) = f(s+u)$. So we get $r + t \ \rho \ s + u$. Similarly $r \cdot t \ \rho \ s \cdot u$ and $r \times t \ \rho \ s \times u$. Thus the relation is compatible. This completes the proof. ∎

Let ρ be an equivalence relation on a set A. Then the equivalence class corresponding to an element a of A is denoted by the symbol $a\rho$ and is defined as:

$$a\rho = \{x \in A = (a, x) \in \rho\}.$$

If ρ is a congruence relation on a bisemiring R, then we say that $a\rho$ is a congruence class corresponding to the element a of R. Let R/ρ denote the set of all congruence classes, i.e., $R/\rho = \{a\rho : a \in R\}$. We are now going to state a result which has been taken from [1] and will be used later. The result is true for classes but since we know that every class is a set, so in particular it is true for sets as well.

Lemma 3.4. *Let ρ be an equivalence relation on a set A, then $a\rho = b\rho$ if and only if $(a, b) \in \rho$.*

Let $a, b \in R$ and $a\rho$, $b\rho$ represent the congruence classes corresponding to a and b, then we can define binary operations on the quotient set R/ρ as follows:

$$a\rho + b\rho = (a + b)\rho,$$
$$a\rho \cdot b\rho = (a \cdot b)\rho$$

and

$$a\rho \times b\rho = (a \times b)\rho.$$

These operations are well defined, since for all a, b, c and $d \in R$ if $a\rho = c\rho$ and $b\rho = d\rho$, then by the above lemma, $(a, c) \in \rho$ and $(b, d) \in \rho$. Thus $(a + b, c + d), (a \cdot b, c \cdot d), (a \times b, c \times d) \in \rho$, as ρ is a congruence relation. Thus again by the above lemma this implies that $(a + b)\rho = (c + d)\rho$, $(a \cdot b)\rho = (c \cdot d)\rho$ and $(a \times b)\rho = (c \times d)\rho$. Further,

Associative laws:

(i) With respect to addition:

Let a, b and $c \in R$ such that $a\rho, b\rho$ and $c\rho \in R/\rho$, then

$$\begin{aligned}(a\rho + b\rho) + c\rho &= (a + b)\rho + c\rho \\ &= ((a + b) + c))\rho \\ &= ((a + (b + c))\rho \\ &= a\rho + (b + c)\rho \\ &= a\rho + (b\rho + c\rho).\end{aligned}$$

(ii) With respect to multiplication:

$$\begin{aligned}(a\rho \cdot b\rho) \cdot c\rho &= (a \cdot b)\rho \cdot c\rho \\ &= ((a \cdot b) \cdot c))\rho \\ &= ((a \cdot (b \cdot c))\rho \\ &= a\rho \cdot (b \cdot c)\rho \\ &= a\rho \cdot (b\rho \cdot c\rho).\end{aligned}$$

(iii) With respect to product:

$$\begin{aligned}(a\rho \times b\rho)c\rho &= (a \times b)\rho \times c\rho \\ &= ((a \times b) \times c))\rho \\ &= ((a \times (b \times c))\rho \\ &= a\rho \times (b \times c)\rho \\ &= a\rho \times (b\rho \times c\rho).\end{aligned}$$

Thus $(R/\rho, +)$, $(R/\rho, \cdot)$ and $(R/\rho, \times)$ are semigroups.

Distributive laws:

(i) Multiplication is distributive over addition:

$$\begin{aligned}a\rho \cdot (b\rho + c\rho) &= a\rho \cdot \ (b + c)\rho \\ &= (a \cdot (b + c))\rho \\ &= (a \cdot b + a \cdot c)\rho \\ &= (a \cdot b)\rho + (a \cdot c)\rho \\ &= a\rho \cdot b\rho + a\rho \cdot c\rho\end{aligned}$$

Similarly,

$$(a\rho + b\rho) \cdot c\rho = a\rho \cdot c\rho + b\rho \cdot c\rho$$

(ii) Product is distributive over multiplication:

$$\begin{aligned}a\rho \times (b\rho \cdot c\rho) &= a\rho \times (b \cdot c)\rho \\ &= (a \times (b \cdot c))\rho \\ &= (a \times b \cdot a \times c)\rho \\ &= (a \times b)\rho \cdot (a \times c)\rho \\ &= a\rho \times b\rho \cdot a\rho \times c\rho.\end{aligned}$$

Similarly,

$$(a\rho \cdot b\rho) \times c\rho = a\rho \times c\rho \cdot b\rho \times c\rho.$$

Thus $(R/\rho, +, \cdot, \times)$ is a bisemiring which is called a factor bisemiring.

We are now going to state and prove the bisemiring analogues of the first, second and third isomorphism theorems. The semigroup equivalents can be found in [2].

Theorem 3.5. (First Isomorphism Theorem) *If ρ is a congruence on a bisemiring R. Then R/ρ is a bisemiring with respect to the operations*

$$\begin{aligned}a\rho + b\rho &= (a + b)\rho, \\ a\rho \cdot b\rho &= (a \cdot b)\rho \\ a\rho \times b\rho &= (a \times b)\rho.\end{aligned}$$

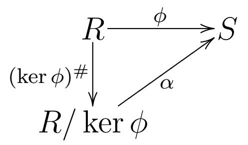
The mapping $\rho : R \to R/\rho$ *defined by* $\rho^{\#}(a) = a\rho$ *for all* $a \in R$ *is an epimorphism. If*

$$\phi : R \to S$$

is a homomorphism where R *and* S *are bisemirings, then the relation*

$$ker\phi = \{(a, b) \in R \times R : \phi(a) = \phi(b)\}$$

is a congruence relation on R *and there is a monomorphism* $\alpha : R/ker \to S$ *such that* $ran\alpha = ran\phi$ *and the diagram*



commutes.

Proof. We have already proved that R/ρ is a bisemiring. Now, let a and $b \in R$, then

$$\begin{aligned} \rho^{\#}(a+b) &= (a+b)\rho = a\rho + b\rho = \rho^{\#}(a) + \rho^{\#}(b), \\ \rho^{\#}(a \cdot b) &= (a \cdot b)\rho = a\rho \cdot b\rho = \rho^{\#}(a) \cdot \rho^{\#}(b) \\ \rho^{\#}(a \times b) &= (a \times b)\rho = a\rho \times b\rho = \rho^{\#}(a) \times \rho^{\#}(b) \end{aligned}$$

Thus ρ is a homomorphism. $ker\phi$ is a congruence relation on R by Theorem 3.3. Next we define $\alpha : R/ker\phi \to S$ by $\alpha(aker\phi) = \phi(a)$. Then α is both well defined and one-one, since for all $a, b \in R$

$$aker\phi = bker\phi \Leftrightarrow (a, b) \in ker\phi \Leftrightarrow \phi(a) = \phi(b) \Leftrightarrow \alpha(aker\phi) = \alpha(bker\phi).$$

It is a homomorphism, since for all $a, b \in R$

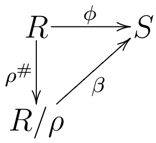
$$\begin{aligned} \alpha[(aker\phi) + (bker\phi)] &= \alpha[(a+b)ker\phi] \\ &= \phi(a+b) \\ &= \phi(a) + \phi(b) \\ &= \alpha(aker\phi) + \alpha(bker\phi), \\ \alpha[(aker\phi) \cdot (bker\phi)] &= \alpha[(a \cdot b)ker\phi] \\ &= \phi(a \cdot b) \\ &= \phi(a) \cdot \phi(b) \\ &= \alpha(aker\phi) \cdot \alpha(bker\phi) \end{aligned}$$

and

$$\begin{aligned} \alpha[(aker\phi) \times (bker\phi)] &= \alpha[(a \times b)ker\phi] \\ &= \phi(a \times b) \\ &= \phi(a) \times \phi(b) \\ &= \alpha(aker\phi) \times \alpha(bker\phi). \end{aligned}$$

Clearly, $ran\alpha = ran\phi$ and from the definition it is obvious that for all $a \in R$, $\alpha[(ker\phi)^{\#}(a)] = \alpha(aker\phi) = \phi(a)$. That is, the diagram commutes. ∎

Theorem 3.6. (Second Isomorphism Theorem) *Let ρ be congruence relation on a bisemiring R. If $\phi : R \longrightarrow S$ is a homomorphism where R and S are bisemirings such that $\rho \subseteq ker\phi$, then there is a unique homomorphism $\beta : R/\rho \longrightarrow S$ such that $ran\beta = ran\phi$ and the diagram*



commutes.

Proof. Define $\beta : R/\rho \longrightarrow S$ by $\beta(a\rho) = \phi(a)$ where $a\rho \in R/\rho$. Then β is well defined, since for all $a, b \in R$,

$$a\rho = b\rho \Rightarrow (a, b) \in \rho \subseteq ker\phi \Rightarrow \phi(a) = \phi(b) \Rightarrow \beta(a\rho) = \beta(b\rho)\beta.$$

β is a homomorphism, because if $a\rho, b\rho \in R/\rho$, then

$$\begin{aligned} \beta(a\rho + b\rho) &= \beta[(a + b)\rho] = \phi(a + b) = \phi(a) + \phi(b) = \beta(a\rho) + \beta(a\rho), \\ \beta(a\rho \cdot b\rho) &= \beta[(a \cdot b)\rho] = \phi(a \cdot b) = \phi(a) \cdot \phi(b) = \beta(a\rho) \cdot \beta(a\rho) \\ \beta(a\rho \times b\rho) &= \beta[(a \times b)\rho] = \phi(a \times b) = \phi(a) \times \phi(b) = \beta(a\rho) \times \beta(a\rho). \end{aligned}$$

Now, $\beta[\rho^{\#}(a)] = \beta(a\rho) = \phi(a)$. That is the diagram commutes and it is obvious that $ran\beta = ran\phi$. Finally, let $\beta_1 : R/\rho \longrightarrow S$ be another homomorphism such that $\beta_1\rho^{\#} = \phi$. Let $a \in R$, then

$$\beta_1[\rho^{\#}(a)] = \phi(a) = \beta[\rho^{\#}(a)].$$

So

$$\beta_1(a\rho) = \beta(a\rho),$$

i.e.

$$\beta_1 = \beta.$$

■

Theorem 3.7. (Third Isomorphism Theorem) *Let ρ and σ be congruence relations on a bisemiring R such that $\rho \subseteq \sigma$. Then*

$$\sigma/\rho = \{(x\rho, y\sigma) \in R/\rho \times R/\rho : (x, y) \in \sigma\}$$

is a congruence relation on R/ρ and

$$R/\rho|\sigma/\rho \cong R/\sigma.$$

Proof. First, we show that σ/ρ is a congruence relation. Let $x \in R$, then $(x, x) \in \sigma$, as σ is reflexive. Then $(x\rho, x\rho) \in \sigma/\rho$, so σ/ρ is reflexive. Now, let $x, y \in R$ such that $(x\rho, y\rho) \in \sigma/\rho$, then $(x, y) \in \sigma$ and this implies $(y, x) \in \sigma$, as σ is symmetric. Then $(y\rho, x\rho) \in \sigma/\rho$, so σ/ρ is symmetric. Now, let x, y and $z \in R$ such that $(x\rho, y\rho)$ and $(y\rho, z\rho) \in \sigma/\rho$, then (x, y) and $(y, z) \in \sigma$ and this implies $(x,z) \in \sigma$, as σ is transitive. Then $(x\rho, z\rho) \in \sigma/\rho$, so σ/ρ is transitive. This shows that σ/ρ is an equivalence relation.

Now, let w, x, y and $z \in R$ such that $(w\rho, x\rho)$ and $(y\rho, z\rho) \in \sigma/\rho$, then (w,x) and $(y,z) \in \sigma$.This implies that $(w+y, x+z), (w \cdot y, x \cdot z)$ and $(w \times y, x \times z) \in \sigma$, as σ is compatible, and this implies

$$((w+y)\rho, (x+z)\rho), ((w \cdot y)\rho, (x \cdot z)\rho) \text{ and } ((w \times y)\rho, (x \times z)\rho) \in \sigma/\rho.$$

This shows that σ/ρ is compatible.

Now, define $\beta : R/\rho \longrightarrow R/\sigma$ by $\beta(a\rho) = a\sigma$. Let $a\rho$ and $b\rho \in R/\rho$, then

$$\begin{aligned} \beta(a\rho + b\rho) &= \beta((a+b)\rho) = (a+b)\sigma = a\sigma + b\sigma = \beta(a\rho) + \beta(b\rho) \\ \beta(a\rho \cdot b\rho) &= \beta((a \cdot b)\rho) = (a \cdot b)\sigma = a\sigma \cdot b\sigma = \beta(a\rho) \cdot \beta(b\rho). \end{aligned}$$

Similarly,

$$\beta(a\rho \times b\rho) = \beta((a \times b)\rho) = (a \times b)\sigma = a\sigma \times b\sigma = \beta(a\rho) \times \beta(b\rho)$$

Thus, β is homomorphism. So, by Theorem 3.5, there is a monomorphism

$$\alpha : R/\rho | ke\beta \longrightarrow R/\sigma$$

defined by $\alpha((a\rho)\ker\beta) = a\sigma$. Clearly, it is onto. Thus, $R/\rho|\ker\beta \cong R/\sigma$. Now,

$$\begin{aligned} \ker\beta &= \{(x\rho, y\rho) \in R/\rho \times R/\rho : \beta(x\rho) = \beta(y\rho)\} \\ &= \{(x\rho, y\rho) \in R/\rho \times R/\rho : x\sigma = y\sigma\} \\ &= \{(x\rho, y\rho) \in R/\rho \times R/\rho : (x,y) \in \sigma\} \\ &= \sigma/\rho. \end{aligned}$$

Thus, $R/\rho|\sigma/\rho \cong R/\sigma$, as required. ∎

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