# GROWTH ANALYSIS OF WRONSKIANS BASED ON RELATIVE $L^*$-ORDER AND RELATIVE $L^*$-TYPE

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**Abstract.** In this paper, we establish the relationship between the relative $L$-order (relative $L^*$-order), relative $L$-type (relative $L^*$-type) and relative $L$-weak type (relative $L^*$-weak type) of a transcendental meromorphic function $f$ with respect to an entire function $g$ and that of Wronskian generated by the meromorphic $f$ and entire $g$.

**Keywords and phrases:** transcendental entire function, transcendental meromorphic function, relative order (relative lower order), relative type (relative lower type), relative weak type, wronskian, slowly changing function.

**AMS Subject Classification (2010):** 30D20,30D30,30D35.

## 1. Introduction, definitions and notations

Let $\mathbb{C}$ be the set of all finite complex numbers. For a meromorphic function $f$ defined on $\mathbb{C}$, the Wronskian determinant $W(f) = W(a_1, a_2, ....a_k, f)$ is defined as

$$W(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1^{'} & a_2^{'} & . & . & . & a_k^{'} & f^{'} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix}$$

where $a_1, a_2, ....a_k$ are linearly independent meromorphic functions and small with respect to $f$ (i.e., $T_{a_i}(r) = S(r, f)$ or, in other words, $\frac{T_{a_i}(r)}{S(r,f)} \to 0$ as $r \to \infty$ for $i = 1, 2, 3...k$, $T_f(r)$ being the Nevanlinna Characteristic function of $f$).

We do not explain the standard notations and definitions in the theory of entire and meromorphic functions as those are available in [3] and [6]. From the Nevanlinna's second fundamental theorem, it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum\limits_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ (cf. [3], p. 43), where $\delta(a; f) = 1 - \limsup\limits_{r \to \infty} \frac{N(r,a;f)}{T_f(r)} = \liminf\limits_{r \to \infty} \frac{m(r,a;f)}{T_f(r)}$. If, in particular, $\sum\limits_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that $f$ has the maximum deficiency sum.

Somasundaram and Thamizharasi [5] introduced the notions of $L$-order and $L$-lower order for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant "$a$". Their definitions are as follows:

**Definition 1** [5] The $L$-order $\rho_f^L$ and the $L$-lower order $\lambda_f^L$ of a meromorphic function $f$ are defined as follows:

$$\rho_f^L = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [rL(r)]} \quad \text{and} \quad \lambda_f^L = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log [rL(r)]}.$$

For $L(r) \equiv 1$, the definitions of $L$-order and the $L$-lower order of a meromorphic function $f$ respectively reduce to the classical definitions of order and lower order of the same.

The more generalized concept of $L$-order and $L$-lower order of meromorphic functions are $L^*$-order and $L^*$-lower order respectively which are as follows:

**Definition 2** The $L^*$-order $\rho_f^{L^*}$ and the $L^*$-lower order $\lambda_f^{L^*}$ of a meromorphic function $f$ are defined by

$$\rho_f^{L^*} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log [re^{L(r)}]} \quad \text{and} \quad \lambda_f^{L^*} = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log [re^{L(r)}]}.$$

It is well known that the Nevanlinna's characteristic function $T_g(r)$ of an entire function $g$ is defined as

$$T_g(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|g(re^{i\theta})\right| d\theta,$$

where $\log^+ x = \max(0, \log x)$ for $x > 0$. If $g$ is non-constant then $T_g(r)$ is strictly increasing and continuous and its inverse $T_g^{-1} : (T_g(0), \infty) \to (0, \infty)$ exists and is such that $\lim_{s\to\infty} T_g^{-1}(s) = \infty$.

Lahiri and Banerjee [4] introduced the definition of relative order of a meromorphic function with respect to an entire function in the following way:

**Definition 3** [4]Let $f$ be meromorphic and $g$ be entire. The relative order of $f$ with respect to $g$ denoted by $\rho_g(f)$ is defined as

$$\rho_g(f) = \inf\left\{ \begin{array}{c} \mu > 0 : T_f(r) < T_g(r^\mu) \\ \text{for all sufficiently large } r \end{array} \right\} = \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

For $g(z) = \exp z$, the above definition coincides with the classical one [4].

Analogously, the relative lower order $\lambda_g(f)$ of a meromorphic function $f$ with respect to an entire function $g$ is defined.

Datta and Biswas [1] gave the definition of relative type and relative weak type of a meromorphic function with respect to an entire function $g$ which are as follows:

**Definition 4** [1] The relative type $\sigma_g(f)$ and lower relative type $\bar{\sigma}_g(f)$ of a meromorphic function $f$ with respect to an entire function $g$ are defined as

$$\sigma_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}} \quad \text{and} \quad \bar{\sigma}_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}},$$

where $0 < \rho_g(f) < \infty$.

**Definition 5** [1] The relative weak type $\tau_g(f)$ of a meromorphic function $f$ with respect to an entire function $g$ with finite positive relative lower order $\lambda_g(f)$ is defined by

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\lambda_g(f)}}.$$

Similarly, one can define the growth indicator $\bar{\tau}_g(f)$ by replacing "$\liminf$" with "$\limsup$" in Definition 5.

In order to prove our results, we require the following definitions:

**Definition 6** The relative $L$-order $\rho_g^L(f)$ (the relative $L$-lower order $\lambda_g^L(f)$), relative $L$-type $\sigma_g^L(f)$ (relative $L$-lower type $\overline{\sigma}_g^L(f)$) and relative $L$-weak type $\tau_g^L(f)$ (growth indicator $\overline{\tau}_g^L(f)$) of a meromorphic function $f$ with respect to an entire function $g$ are defined as follows:

$$\rho_g^L(f) = \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log[rL(r)]} \quad \left(\lambda_g^L(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log[rL(r)]}\right),$$

$$\sigma_g^L(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{[rL(r)]^{\rho_g^L(f)}} \quad \left(\overline{\sigma}_g^L(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{[rL(r)]^{\rho_g^L(f)}}\right),$$

where $0 < \rho_g^L(f) < \infty$, and

$$\tau_g^L(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{[rL(r)]^{\lambda_g^L(f)}} \quad \left(\overline{\tau}_g^L(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{[rL(r)]^{\lambda_g^L(f)}}\right),$$

where $0 < \lambda_g^L(f) < \infty$.

The more generalized concept of relative $L$-order (relative $L$-lower order), relative $L$-type (relative $L$-lower type) and relative $L$-weak type of a meromorphic function with respect to an entire function are relative $L^*$-order (relative $L^*$-lower order), relative $L^*$-type ( relative $L^*$-lower type) and relative $L^*$-weak type respectively which may be defined as follows:

**Definition 7** The relative $L^*$-order $\rho_f^{L^*}$( relative $L^*$-lower order $\lambda_f^{L^*}$), relative $L^*$-type $\sigma_g^{L^*}(f)$ (relative $L^*$-lower type $\overline{\sigma}_g^{L^*}(f)$) and relative $L^*$-weak type $\tau_g^{L^*}(f)$ (the growth indicator $\overline{\tau}_g^{L^*}(f)$) of a meromorphic function $f$ with respect to entire $g$ are respectively defined as follows:

$$\rho_g^{L^*}(f) = \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log[re^{L(r)}]} \quad \left(\lambda_g^{L^*}(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log[re^{L(r)}]}\right),$$

$$\sigma_g^{L^*}(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{[re^{L(r)}]^{\rho_g^{L^*}(f)}} \quad \left(\overline{\sigma}_g^{L^*}(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{[re^{L(r)}]^{\rho_g^{L^*}(f)}}\right)$$

where $0 < \rho_g^{L^*}(f) < \infty$, and

$$\tau_g^{L^*}(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{[re^{L(r)}]^{\lambda_g^{L^*}(f)}} \quad \left(\overline{\tau}_g^{L^*}(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{[re^{L(r)}]^{\lambda_g^{L^*}(f)}}\right),$$

where $0 < \lambda_g^{L^*}(f) < \infty$.

Since the natural extension of a derivative is a differential polynomial, in this paper we prove our results for a special type of linear differential polynomials viz. the Wronskians. In the paper, we establish the relationship between the relative $L$-order (relative $L^*$-order), relative $L$-type (relative $L^*$-type) and relative $L$-weak type (relative $L^*$-weak type) of a transcendental meromorphic function $f$ with respect to an entire function $g$ and that of Wronskian generated by the meromorphic $f$ and entire $g$.

## 2. Lemmas

In this section, we present a lemma which will be needed in the sequel.

**Lemma 1** [2] *Let $f$ be a transcendental meromorphic function having the maximum deficiency sum and $g$ be a transcendental entire function with regular growth and non zero finite order. Also, let $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then*

$$\lim_{r\to\infty}\frac{\log T_{W(g)}^{-1}T_{W(f)}(r)}{\log T_g^{-1}T_f(r)}=1.$$

**Lemma 2** [2] *Let $f$ be a transcendental meromorphic function having the maximum deficiency sum and $g$ be a transcendental entire function of regular growth and non zero finite type. Also let $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then*

$$\lim_{r\to\infty}\frac{T_{W(g)}^{-1}T_{W(f)}(r)}{T_g^{-1}T_f(r)}=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}},$$

*where $W(f)=W(a_1,a_2,....a_{k_1},f)$ and $W(g)=W(a_1,a_2,....a_{k_2},g)$.*

## 3. Theorems

In this section, we present the main results of the paper.

**Theorem 1** *Let $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be a transcendental entire function of regular growth having non zero finite order and $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then the relative $L$-order and relative $L$-lower order of $W(f)$ with respect to $W(g)$ are same as those of $f$ with respect to $g$.*

**Proof.** By Lemma 1 we obtain that

$$\begin{aligned}
\rho^L_{W(g)}(W(f)) &= \limsup_{r\to\infty} \frac{\log T^{-1}_{W(g)} T_{W(f)}(r)}{\log[rL(r)]} \\
&= \limsup_{r\to\infty} \left\{ \frac{\log T_g^{-1} T_f(r)}{\log[rL(r)]} \cdot \frac{\log T^{-1}_{W(g)} T_{W(f)}(r)}{\log T_g^{-1} T_f(r)} \right\} \\
&= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log[rL(r)]} \cdot \lim_{r\to\infty} \frac{\log T^{-1}_{W(g)} T_{W(f)}(r)}{\log T_g^{-1} T_f(r)} \\
&= \rho^L_g(f) \cdot 1 = \rho^L_g(f) \; .
\end{aligned}$$

In a similar manner,

$$\lambda^L_{W(g)}(W(f)) = \lambda^L_g(f) \, .$$

This proves the theorem. ■

**Theorem 2** *Let $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be a transcendental entire function of regular growth having non zero finite order and $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$. Then the relative $L^*$-order and relative $L^*$-lower order of $W(f)$ with respect to $W(g)$ are same as those of $f$ with respect to $g$.*

We omit the proof of Theorem 2 because it can be carried out in the line of Theorem 1.

**Theorem 3** *Let $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty} \delta(a;g) + \delta(\infty;g) = 2$. Then the relative $L$-type and relative $L$-lower type of $W(f)$ with respect to $W(g)$ are $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of $f$ with respect to $g$ if $\rho^L_g(f)$ is positive finite.*

**Proof.** From Lemma 2 and Theorem 1, we get that

$$\begin{aligned}
\sigma^L_{W(g)}(W(f)) &= \limsup_{r\to\infty} \frac{T^{-1}_{W(g)} T_{W(f)}(r)}{[rL(r)]^{\rho_{w(g)}(w(f))}} \\
&= \lim_{r\to\infty} \frac{T^{-1}_{W(g)} T_{W(f)}(r)}{T_g^{-1} T_f(r)} \cdot \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{[rL(r)]^{\rho^L_g(f)}} \\
&= \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}} \sigma^L_g(f) \; .
\end{aligned}$$

Similarly,

$$\overline{\sigma}^L_{W(g)}(W(f)) = \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}} \cdot \overline{\sigma}^L_g(f) \, .$$

Thus, the theorem is established. ■

**Theorem 4** *If $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$ then the relative $L^*$-type and relative $L^*$-lower type of $W(f)$ with respect to $W(g)$ are $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of $f$ with respect to $g$ if $\rho_g^{L^*}(f)$ is positive finite.*

We omit the proof of Theorem 4 because it can be carried out in the line of Theorem 3.

Now, we state the following two theorems without their proofs because those can be carried out in the line of Theorem 3 and Theorem 4 respectively.

**Theorem 5** *Let $f$ be a transcendental meromorphic function with the maximum deficiency sum $g$ be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then $\tau_{W(g)}^{L}(W(f))$ and $\overline{\tau}_{W(g)}^{L}(W(f))$ are $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of $f$ with respect to $g$ i.e.,*

$$\tau_{W(g)}^{L}(W(f))=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\tau_g^{L}(f)$$

*and*

$$\overline{\tau}_{W(g)}^{L}(W(f))=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\overline{\tau}_g^{L}(f)$$

*when $\lambda_g^{L}(f)$ is positive finite.*

**Theorem 6** *If $f$ be a transcendental meromorphic function with the maximum deficiency sum and $g$ be a transcendental entire function of regular growth having non zero finite type and $\sum_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$, then $\tau_{W(g)}^{L^*}(W(f))$ and $\overline{\tau}_{W(g)}^{L^*}(W(f))$ are $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of $f$ with respect to $g$, i.e.,*

$$\tau_{W(g)}^{L^*}(W(f))=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\tau_g^{L^*}(f)$$

*and*

$$\overline{\tau}_{W(g)}^{L^*}(W(f))=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\overline{\tau}_g^{L^*}(f)$$

*when $\lambda_g^{L^*}(f)$ is positive finite.*

## 4. Conclusion

The notions of order (type), $L$-order ($L$-type) and $L^*$-order ($L^*$-type) which are the main tools to study the composite growth properties of entire and meromorphic functions are very much classical in complex analysis. On the basis of the order (type), $L$-order ($L$-type) and $L^*$-order ($L^*$-type) of entire or meromorphic functions, several researchers have already explored their works in the area of comparative growth rates of composite entire and meromorphic functions in different directions. In fact, the main aim of this paper is actually to extend these notions to the relativeness of growth indicators in case of wronskians. Actually, the relative order, relative type etc. are the gradation of the growth indicators of entire and meromorphic functions. So keeping all these in mind, it is quite expected to explore and establish similar strong results using the existing literature and theorems of this paper in the field of growth analysis of complex valued, bi-complex valued and fuzzy complex valued functions.

## References

[1] Datta, S.K., Biswas, A., *On relative type of entire and meromorphic functions, Advances in Applied Mathematical Analysis*, 8 (2) (2013), 63-75.

[2] Datta, S.K., Biswas, T., Ali, S., *Some growth properties of wronskians using their relative order*, Journal of Classical Analysis, 3 (1) (2013), 91-99.

[3] Hayman, W.K., *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[4] Lahiri, B.K., Banerjee, D., *Relative order of entire and meromorphic functions*, Proc. Nat. Acad. Sci. India, 69 (A) III (1999), 339-354.

[5] Somasundaram, D., Thamizharasi, R., *A note on the entire functions of L-bounded index and L-type*, Indian J. Pure Appl. Math., 19 (3) (March 1988), 284-293.

[6] Valiron, G., *Lectures on the general theory of integral functions*, Chelsea Publishing Company, 1949.

Accepted: 10.07.2014