SOME RESULTS ON LAGUERRE TRANSFORM IN TWO VARIABLES

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Abstract. An attempt is made to investigate some results on Laguerre transform in two variables [8]. In this paper, Laguerre transform of some particular functions and integral formulas have been obtained.

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1. Introduction

Basic concepts and applications of the Laguerre transform and the generalized Laguerre transform can be found in Debnath et al. [2]. Recently Shukla et al. [8] introduced the Laguerre transform of $f(x,y)$ as

$$L\{f(x,y)\} = F_n(\alpha,\beta) = \int_0^\infty \int_0^\infty e^{-(x+y)} x^\alpha y^\beta K_n^{(\alpha,\beta)}(x,y)\, f(x,y)\, dxdy, \tag{1.1}$$

where $f(x,y)$ be a Riemann integrable function (see [7]) defined on the set $S = R^+ \times R^+$,$\alpha > -1$, $\beta > -1$, n is non-negative integer and $K_n^{(\alpha,\beta)}(x,y) = L_n^\alpha(x)\, L_n^\beta(y)$.

Shukla et al. [8] proved the following theorem:

Theorem 1.1. *If* $K_n^{(\alpha,\beta)}(x,y) = L_n^{\alpha}(x)\, L_n^{\beta}(y)$, *then*

$$\int_0^{\infty}\int_0^{\infty} e^{-(x+y)} x^{\alpha} y^{\beta} K_n^{(\alpha,\beta)}(x,y) K_m^{(\alpha,\beta)}(x,y)\, dxdy = \delta_n \delta_{m\,n}, \tag{1.2}$$

where $\delta_{m\,n}$ *(Kronecker delta symbol) is defined as*

$$\delta_{m\,n} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases},$$

$$\delta_n = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(n!)^2},$$

$$\alpha > -1 \text{ and } \beta > -1.$$

Howell [4] proved the following results:

$$\int_0^{\infty} x^{\alpha+a-1} e^{-x} L_n^{\alpha}(x) dx = \frac{\Gamma(a+\alpha)\Gamma(n-a+1)}{n!\Gamma(1-a)} \tag{1.3}$$

$$\int_0^{\infty} e^{-(1+a)x} y^{\beta} L_n^{\beta}(y) L_m^{\beta}(y) dy = \frac{\Gamma(n+\alpha+1)\Gamma(m+\alpha+1)(a-1)^{n-m+\alpha+1}}{n!m!\Gamma(1+\alpha)a^{n+m+2\alpha+2}}$$

$$\times {}_2F_1\left(n+\alpha+1, \frac{m+a+1}{a+1}, \frac{1}{a^2}\right) \tag{1.4}$$

$$\int_0^{\infty} e^{-st} t^{\alpha} L_n^{\alpha}(t)\, dt = \frac{\Gamma(n+\alpha+1)(s-1)^n}{n! s^{n+\alpha+1}}, \tag{1.5}$$

where $\mathrm{Re}(\alpha) > -1$,$\mathrm{Re}(s) > 0$, and

$$e^{x+y}(xy)^{-\alpha}\Gamma(\alpha, \max(x,y))\,\gamma(\alpha, \min(x,y)) = \sum_{m=0}^{\infty} \frac{m!}{(m+1)(\alpha)_{m+1}} L_m^{\alpha}(x)\, L_m^{\alpha}(y), \tag{1.6}$$

where $\gamma(\alpha, x)$ is incomplete gamma functions and $\Gamma(\alpha,x) = \Gamma(\alpha) - \gamma(\alpha,x)$.

$$e^{x+y}(xy)^{-\alpha}\left\{\Gamma(\alpha, \max(x,y)) - \Gamma(\alpha,x)\Gamma(\alpha,y)/\Gamma(\alpha)\right\} = \sum_{m=0}^{\infty} \frac{m!}{(m+1)\Gamma(m+\alpha+1)} L_m^{\alpha}(x)\, L_m^{\alpha}(y) \tag{1.7}$$

$$e^{-\frac{1}{2}(x+y)}(xy)^{-\frac{1}{2}\alpha} e^{-\alpha\pi i}\gamma\left(\alpha, e^{i\pi}\min(x,y)\right) = \sum_{m=0}^{\infty} \frac{m!}{(m+\alpha)\Gamma(m+\alpha+1)} L_m^{\alpha}(x)\, L_m^{\alpha}(y). \tag{1.8}$$

2. Laguerre Transforms of some particular functions

In this section, some properties of Laguerre transform in two variables [8] have been obtained, this work can be considered as an extension of [8] and [9].

By using the orthogonal property (1.2), we can prove:

$$L\left\{K_n^{(\alpha,\beta)}(x,y)\right\} = \begin{cases} 0, & m \neq n \\ \dfrac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(n!)^2}, & m = n \end{cases} \tag{2.1}$$

From definition (1.1) and result (1.3) of Howell [4], we get the following result as:

If $f(x,y) = x^{a-1}y^{b-1}$, where a and b are positive numbers, then

$$L\{f(x,y)\} = \frac{\Gamma(a+\alpha)\,\Gamma(b+\beta)\,\Gamma(n-a+1)\,\Gamma(n-b+1)}{(n!)^2\Gamma(1-a)\,\Gamma(1-b)} \tag{2.2}$$

From (1.5), we can prove the following result as:

If $f(x,y) = e^{-(a\,x\,+\,b\,y)}$, where $a > -1$ and $b > -1$ then,

$$L\{f(x,y)\} = \frac{(ab)^n\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(n!)^2(a+1)^{n+\alpha+1}(b+1)^{n+\beta+1}} \tag{2.3}$$

Applying (1.1) to equation (1.4) and further simplification gives the following result.

If $f(x,y) = e^{-(a\,x\,+\,b\,y)}K_m^{(\alpha,\beta)}(x,y)$, where $a > -1$ and $b > -1$ then,

$$L\{f(x,y)\} = \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{(n!)^2(m!)^2\Gamma(1+\alpha)\,\Gamma(1+\beta)}$$

$$\times\frac{\Gamma(m+\alpha+1)\,\Gamma(m+\beta+1)\,(a-1)^{n-m+\alpha+1}(b-1)^{n-m+\beta+1}}{a^{n+m+2\alpha+2}b^{n+m+2\beta+2}}$$

$$\times{}_2F_1\left(n+\alpha+1, \frac{m+a+1}{a+1}, \frac{1}{a^2}\right){}_2F_1\left(n+\beta+1, \frac{m+b+1}{b+1}, \frac{1}{b^2}\right). \tag{2.4}$$

From (1.2) and (1.6), we arrive at following result:

If $f(x,y) = e^{x+y}(xy)^{-\alpha}\Gamma(\alpha, \max(x,y))\,\gamma(\alpha, \min(x,y))$ and $F_n(\alpha,\beta) = L\{f(x,y), \alpha, \beta, n\}$ then,

$$F_n(\alpha,\alpha) = \frac{\Gamma(\alpha)\,\Gamma(n+\alpha+1)}{(n+1)!} \tag{2.5}$$

Also from (1.7) and (1.2) further proceeding as above, this yields:

If $f(x,y) = e^{x+y}(xy)^{-\alpha}\{\Gamma(\alpha, \max(x,y)) - \Gamma(\alpha,x)\Gamma(\alpha,y)/\Gamma(\alpha)\}$ and $F_n(\alpha,\beta) = L\{f(x,y), \alpha, \beta, n\}$ then,

$$F_n(\alpha,\alpha) = \frac{\Gamma(n+\alpha+1)}{(n+1)!}. \tag{2.6}$$

By using (1.8) and (1.2), we can say:

If $f(x,y) = e^{-\frac{1}{2}(x+y)}(xy)^{-\frac{1}{2}\alpha}e^{-\alpha\pi i}\gamma(\alpha, e^{i\pi}\min(x,y))$ then,

$$L\{f(x,y), \alpha, \alpha, n\} = \frac{\Gamma(n+\alpha+1)}{(n+1)!} \tag{2.7}$$

3. Some integral formula

Laguerre polynomials occur in many fields of research in science, engineering and numerical mathematics such as, in quantum mechanics [5], communication theory [1] and numerical inverse Laplace transform [6]. Explicit evaluation of integrals involving Laguerre polynomials is very often required in these and other applied areas of research. In this section, we derive some integral formula.

This is interesting to write (2.5) in the following form as,

$$\int_0^\infty\int_0^\infty L_n^\alpha(x)L_n^\alpha(y)\Gamma(\alpha, \max\,(x,y))\gamma(\alpha,\min(x,y))dxdy = \frac{\Gamma(\alpha)\Gamma(n+\alpha+1)}{(n+1)!} \tag{3.1}$$

Proof of (3.1). To prove (3.1), first replace β by α in the definition (1.1), and we have

$$F_n(\alpha,\alpha) = \int_0^\infty\int_0^\infty e^{-(x+y)}(xy)^\alpha L_n^\alpha(x)\,L_n^\alpha(y)\,f(x,y)\,dxdy$$

By substituting the value of $f(x,y)$, we get

$$F_n(\alpha,\alpha) = \int_0^\infty\int_0^\infty e^{-(x+y)}(xy)^\alpha L_n^\alpha(x)\,L_n^\alpha(y)\sum_{m=0}^\infty \frac{m!}{(m+1)\,(\alpha)_{m+1}} L_m^\alpha(x)\,L_m^\alpha(y)dxdy$$

By applying the orthogonal property (1.2),

$$F_n(\alpha,\alpha) = \sum_{m=0}^\infty \frac{m!}{(m+1)\,(\alpha)_{m+1}}\delta_n\delta_{m\,n}$$

and using the definition of Kronecker delta $\delta_{m\,n}$ and δ_n , we have

$$F_n(\alpha,\alpha) = \frac{n!}{(n+1)\,(\alpha)_{n+1}}\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\alpha+1)}{(n!)^2}$$

Further simplification yields (3.1).

By using (1.2),(1.7) and (2.6),we get

$$(3.2)\quad \int_0^\infty\int_0^\infty L_n^\alpha(x)L_n^\alpha(y)\{\Gamma(\alpha,\max(x,y))-\Gamma(\alpha,x)\Gamma(\alpha,y)/\Gamma(\alpha)\}dxdy=\frac{\Gamma(n+\alpha+1)}{(n+1)!}$$

$$(3.3)\quad \int_0^\infty\int_0^\infty e^{\frac{1}{2}(x+y)}(xy)^{\frac{1}{2}\alpha}e^{-\alpha\pi i}\gamma(\alpha,e^{i\pi}\min(x,y))L_n^\alpha(x)L_n^\alpha(y)dxdy=\frac{\Gamma(n+\alpha+1)}{(n+1)!}$$

Proof (3.3). By setting $\beta=\alpha$ in (1.1), this reduces to

$$F_n(\alpha,\alpha)=\int_0^\infty\int_0^\infty e^{-(x+y)}(xy)^\alpha L_n^\alpha(x)\,L_n^\alpha(y)\,f(x,y)\,dxdy$$

and substituting the value of $f(x,y)$, we arrive at

$$F_n(\alpha,\alpha)$$
$$=\int_0^\infty\int_0^\infty e^{-(x+y)}(xy)^\alpha L_n^\alpha(x)\,L_n^\alpha(y)\,e^{-\frac{1}{2}(x+y)}(xy)^{-\frac{1}{2}\alpha}e^{-\alpha\pi i}\gamma\left(\alpha,e^{i\pi}\min(x,y)\right)dxdy$$

Now, applying the same argument as in proof (3.1), gives

$$F_n(\alpha,\alpha)=\frac{n!}{(n+1)\,\Gamma(n+\alpha+1)}\frac{\Gamma(n+\alpha+1)\,\Gamma(n+\alpha+1)}{(n!)^2}$$

Further simplification yields (3.3).

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