NORMAL INDUCED FUZZY TOPOLOGICAL SPACES

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Abstract. The motto of the present treatise is to introduce and characterize the concept of n-infy induced fuzzy topological spaces generated by normal lower semi-continuous functions. Examples of n-infy induced fuzzy topological spaces are given and its properties are studied. Interrelationship between the newly defined induced spaces and their corresponding topological spaces are examined.

Keywords: Regular open and regular closed subsets, normal lower semi-continuous function, topology, induced fuzzy topological space, n-continuous mapping, r-continuous mapping.

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1. Introduction

In 1965, L.A. Zadeh introduced the fuzzy set in his classical paper [16]. Since then many researchers used this tool to generalize different concepts of Mathematics. One such successful generalization is fuzzy topology from general topology. With the help of fuzzy set, C.L. Chang [6] defined fuzzy topological space as a generalization of topological space as follows:

Let X be a non-empty set. A family $\mathscr{F}$ of fuzzy subsets of X is called fuzzy topology on X if

(i) $0(\equiv \mu_\phi)$, $1(\equiv \mu_X) \in \mathscr{F}$

(ii) arbitrary supremum of members of $\mathscr{F}$ is in $\mathscr{F}$ (iii) finite infimum of members of $\mathscr{F}$ is in $\mathscr{F}$.

Later, R. Lowen [11] suggested an alternative and more natural definition of fuzzy topology for achieving more results which are compatible to the general case in topology by incorporating all constant functions instead of only 0 and 1 in (i) of Chang's definition. It has been seen that fuzzy topology has numerous applications in science and technology including medical sciences.

After this, several researchers successfully generalized the concepts of fuzzy topology further. Fuzzy supra topology and fuzzy minimal structure are two examples of these kinds. Monsef and Ramadan [12] introduced the concept of fuzzy supra topology and Alimohammady and Roohi [1] introduced fuzzy minimal structure. Again it has been seen that these generalized fuzzy structures also have applications in various branches of science and technology.

Again Weiss [15] showed that there is a natural way to associate a fuzzy topology $\mathscr{F}$ on a set X with a given topology T on X by means of collections of lower semi continuous (LSC) functions from a topological space (X, T) to unit closed interval I and he called this fuzzy topology as induced fuzzy topology on X. In 1976, R. Lowen [11] noticed the natural association between a topological space and a fuzzy topological space on a set X and introduced the notion of topologically generated space which is same as the induced fuzzy topological space of Weiss and further studied category of the fuzzy topological space. The notion of induced fuzzy topological spaces was further studied in [4], [5], [13].

Defining all the concepts stated above, the LSC function and their stronger forms (viz. Completely LSC, δ-LSC) played the key role. But it is interesting to notice that all the stronger forms of LSC functions fails to satisfy the conditions to become a fuzzy topology. So, it would be interesting to study such structures formed by other forms of LSC functions which either

(i) fail to preserve point wise arbitrary suprema but is closed under countable point wise suprema and finite infima, or

(ii) is closed under finite infima only but not closed even under finite suprema.

In this paper, we study the structure formed by the functions that are of category (ii) only. The structure formed by the functions of the category (i) has been defined as countable fuzzy topology in [2] and with the help of regular lower semi continuous (RLSC) functions [10] a countably induced fuzzy topological space viz. r-countably induced fuzzy topological space has been introduced and studied in [2], [3].

The motivation of the present paper is to search for a function which is not closed even under finite suprema but is closed under finite infima and to form a generalized induced fuzzy topological structure viz. n-infy induced fuzzy topological space. This has further tempted us to define infy fuzzy topology in this paper and the properties of this generalized fuzzy structure will be studied elsewhere.

Let us first introduce the concept of fuzzy infy topological space in the following way.

Definition 1.1: A family $\mathscr{F}$ of fuzzy sets in X is said to form a fuzzy infy topology in X if i) $r1_X \in \mathscr{F}$, for r $\in$ I. ii) For any two fuzzy subset λ and μ of $\mathscr{F}$, $\lambda \wedge \mu \in \mathscr{F}$.

The space (X, $\mathscr{F}$) is called fuzzy infy topological space. Every member of $\mathscr{F}$ is called fuzzy i-open set and complement of an i-open set is called i-closed set.
In section 2, we recall the definition of NLSC function and then prove that the collection of all NLSC functions from a topological space X to I is closed under finite point wise infima but fails to be closed even under finite suprema. This led us to define a new structure viz. infy induced fuzzy topological space.
The infy induced fuzzy topological spaces generalize the usual notion of induced fuzzy topological spaces; for, every induced fuzzy topological space is infy induced fuzzy topological space but not conversely (Example 2.14). So, the properties possessed by infy induced fuzzy topological spaces will also valid for induced fuzzy topological spaces, which makes the study of such spaces worthy and meaningful.

2. n-infy induced fuzzy space

In this section, we introduce the concept of n-infy induced fuzzy topological space with the help of NLSC functions. Before proceeding further, we define the concepts which are relevant for our proposed study and the results will be used in the sequel without any specific reference.

Definition 2.1 [14] A subset A of X is said to be regular open if $A = \text{int}(\text{cl } A)$. Alternatively, A is regular open if A is the interior of some closed set.

Clearly, every regular open set is open but the converse is not true, e.g. let $X = [0, 1]$, the set $A = (1/2, 1)$ is open in X. But cl $A = [1/2, 1]$ and int (cl A) $= (1/2, 1] \neq A$. Thus A is open but not regular open in X.

Definition 2.2. [14] A subset A of X is said to be regular closed if $A = \text{cl}(\text{int } A)$. Alternatively, A is regular closed if A is the closure of some open set.

Clearly, every regular closed set is closed but the converse is not true, e.g., let $X = [0, 1]$, $A = [0, 1/2] \cup \{1\}$ which is closed in X. Again, int $A = (0, 1/2)$ and cl (int A) $= [0, 1/2] \neq A$. Thus, A is closed but not regular closed in X.

It may be remarked here that the intersection of two regular open sets is regular open and union of two regular closed sets is regular closed. The complement of a regular open set is regular closed and conversely.

Definition 2.3. [9] A function $f : X \to R$ is called upper semi-continuous (respectively, lower semi-continuous) if for each $r \in R$, the set $\{x : f(x) < r\}$ (respectively $\{x : f(x) > r\}$) is open in X, i.e. the set $\{x : f(x) \geq r\}$ (respectively, $\{x : f(x) \leq r\}$) is closed in X.

Definition 2.4. [7] An upper semi-continuous (USC) function ϕ on X is normal iff for each $x \in X$, $\Phi(x) > \lambda$ and an open set U containing x, there exists a non-void open set V, $\overline{V} \subset U$ with $\phi(v) < \lambda$ for each $v \in V$, here $\overline{V} = cl(V)$.

Definition 2.5. [7] An USC function Φ on X is normal iff for each $\lambda \in R$, the set $\{x : \Phi(x) > \lambda\}$ is a union of regular closed sets.

Dually, one can define and characterize normal lower semi-continuous function as follows:

Definition 2.6. [7] A lower semi-continuous (LSC) function ϕ on X is normal iff for each $x \in X$, $\phi(x) < \lambda$ and an open set U containing x, there exists a non-void open set V, $\overline{V} \subset U$ with $\phi(v) < \lambda$ for each $v \in V$.

Theorem 2.7. [7] *An LSC function ϕ on X is normal iff for each $\lambda \in R$, the set $\{x : \phi(x) < \lambda\}$ is a union of regular closed sets.*

Corollary 2.8. *The characteristic function of a regular open* (*resp. regular closed*) *set is NLSC* (*resp. NUSC*) [8].

It can be easily seen that every continuous function is NLSC but the converse is not true, which is shown in the following example.

Example 2.9. Let $X = [0, 1]$ and $A = (1/2, 1]$. Then $\text{cl}(A) = [1/2, 1]$ and int $\text{cl}(A) = (1/2, 1] = A$. Thus, A is regular open. Let us now define a function $f : X \to I$ such that

$$f(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A. \end{cases}$$

Clearly, f is NLSC. Let us now consider an open set B of I, given by $B = [0, 2/3)$. So $f^{-1}(B) = X - A$, which is not open in X. Hence, f is not continuous.

Next, we study some properties of NLSC functions.

Theorem 2.10. *If f is NLSC, then for a ¿ 0, af is also NLSC.*

Proof. Let f is NLSC, so for each $r \in R$, the set $\{x : f(x) < r\}$ is a union of regular closed sets. Again, let us consider the set $\{x : af(x) < t\}$, $t \in R$. Now, $\{x : af(x) < t\} = \{x : f(x) < t/a\} = \{x : f(x) < p\}, p = t/a$, which is a union of regular closed sets, since f is NLSC. Similarly, we can check that, since f is LSC, af $(a > 0)$ is also LSC. Hence, af is NLSC. ∎

That the finite suprema of NLSC functions may not be NLSC is shown in the following example.

Example 2.11. Let $X = [0, 1]$, then $A_1 = [0, 1/3)$ and $A_2 = (1/3, 1]$ be two regular open subset of X. Now, we define

$$\phi_1(x) = \begin{cases} 1/2, & x \in A_1 \\ 0, & x \notin X - A_1 \end{cases}$$

and

$$\phi_2(x) = \begin{cases} 1/3, & x \in A_2 \\ 0, & x \notin X - A_2 \end{cases}$$

Then, ϕ_1, ϕ_2 are two NLSC functions on X. Now, $\Phi = \text{Sup}\{\phi_1, \phi_2\}$ is given by

$$\Phi(x) = \begin{cases} 1/2, & x \in [0, 1/3) \\ 0, & x \in 1/3 \\ 1/3 & x \in (1/3, 1] \end{cases}.$$

Now, we see that Φ is not NLSC at $x = 1/3$. For this, we note that $\Phi(1/3) = 0 < 1/6$ ($= \lambda$, say) and $U = (1/6, 1/2)$ is an open set containing 1/3. It can be seen that there does not exist any non-void open subset V of U with $\overline{V} \subset U$ such that $\Phi(v) < \lambda$, for all $v \in V$. Thus suprema of two NLSC functions may not be NLSC.

Remark. The above example further shows that the sum of two NLSC functions may not be an NLSC function, because the sum of the two functions $\phi_1(x)$ and $\phi_2(x)$ is same as their supremum $\Phi(x)$, which is not NLSC.

That the finite pointwise infima of NLSC functions is NLSC is shown in the following theorem.

Theorem 2.12. *The finite infima of NLSC functions is NLSC.*

Proof. Let $\psi(x) = \inf\{\phi_i(x)\}$, $i = 1, 2, ..., n$ and $x \in X$, where each $\phi_i(x)$ is NLSC. We prove that $\psi(x)$ is NLSC. Let for $x \in X, \psi(x) < \lambda$ and U be an open set containing x. Now, $\psi(x) < \lambda$ i.e. $\inf(\phi_i(x)) < \lambda$ implies that, there exists at least one i, say j, for which $\phi_j(x) < \lambda$. Again, ϕ_j being NLSC, $\phi_j(x) < \lambda$ and U is an open set containing x, so there exists a non-void open set V_j such that $\overline{V_j} \subset U$ and $\phi_j(v_{\alpha,j}) < \lambda$ for all $v_{\alpha,j} \in V_j$. Now, $\phi_j(v_{\alpha,j)} < \lambda$ for all $v_{\alpha,j} \in V_j$ implies $inf\phi_i(v) < \lambda$ for all $v \in \cup_j V_j$, i.e. $\psi(v) < \lambda$ for all $v \in \cup_j V_j = V$ (say), where $\overline{V} = \overline{\cup_j V_j} \subseteq \cup_j \overline{V_j} \subset U$. Again, since the finite infimum of LSC functions is also LSC, so ψ is also LSC. Hence, $\psi = inf\phi_i$, $i = 1, 2, ..., n$ is NLSC.

Now, we cite examples of infy induced fuzzy topological space.

Example 2.13. Let A be a regular open set of a topological space X. We define a family $\mathscr{F} = \{f_a : a \in I\}$ of functions as follows:

$$f_a(x) = \begin{cases} a & \text{if } x \in A \\ 0 & \text{if } x \in X - A \end{cases}$$

Clearly, all the members of $\mathscr{F}$ are NLSC. Now, we construct a family $\mathscr{F}^*$ of functions as follows: $\mathscr{F}^* = \{rf_a : f_a \in \mathscr{F}, 0 \le r \le 1\}$

So, $\mathscr{F}^*$ is a collection of NLSC functions on X such that

(i) $t1_X \in \mathscr{F}^*$, for all $t \in I$.

(ii) Let, $r_1 f_{a^1}, r_2 f_{a^2}, ..., r_n f_{a^n} \in \mathscr{F}^*$.

Then, $\wedge_i r_i f_{a^i} = g$ (say) is given by

$$g(x) = \begin{cases} \wedge_i a^i & \text{if } x \in A \\ 0 & \text{if } x \in X - A \text{ where } 0 \leq a^i \leq 1, \end{cases}$$

which shows that $g(x)$ belongs to $\mathscr{F}^*$.

Thus, we see that $(X, \mathscr{F}^*)$ forms an infy induced fuzzy topological space.

Example 2.14. Let us consider the family of functions $\Sigma = \{0, 1, \phi_1, \phi_2\}$ defined on the topological space $X = [0, 1]$ with usual topology, where ϕ_1 and ϕ_2 are defined as in Example 2.11. Then Σ forms an infy induced fuzzy topological space on X but (X, Σ) is not a induced fuzzy topological space.

Now, let us consider the family $\mathscr{F} = \{f_\alpha : \alpha \in \wedge\}$ of all NLSC functions from a topological space X to the closed unit interval I. Thus, by Theorems 2.10 and 2.12, we see that $\mathscr{F}$ satisfies the following properties:

(i) $t1_X \in \mathscr{F}$, for all $t \in I$.

(ii) $f_i \in \mathscr{F} \Rightarrow \inf_i f_i \in \mathscr{F}, \qquad i = 1, 2, ..., n.$

Thus, by Definition 1.1, the family $\mathscr{F}$ of all NLSC functions from X to I (henceforth denoted by $nL(X)$) forms a fuzzy infy topological space.

Definition 2.15. The family $\mathscr{F} = \{f_\alpha : \alpha \in \wedge\}$ of all NLSC functions from a topological space X to the closed unit interval I forms a fuzzy infy topology on X, this induced fuzzy infy topology is called normal infy induced fuzzy topology on X and the corresponding space denoted by $(X, n(T))$ is called normal infy induced fuzzy topological space or in short n-infy induced fts. The members of $n(T)$ are called n-open sets.

Definition 2.16. A subset H of X is said to be G^* if it can be expressed as intersection of regular open sets, i.e. H is G^* if $H = \cap_a \operatorname{int} F_a$, where $a \in \wedge$ and each Fa is closed. Here, we note that every regular open subset is clearly open G^*. With the help of G^* subset we have the following characterization of NLSC functions.

Definition 2.17. An LSC function $f : X \to R$ is NLSC iff the set $\{x : f(x) \geq r\}$ is a G^* set.

Proof. Let f be NLSC. So the set $A = \{x \in X : f(x) < r\}$ is a union of regular closed sets. i.e., $A = \cup_a clG_a$, where each G_a is open.

Now, $B = \{x \in X : f(x) \geq r\} = X - \{x \in X : f(x) < r\} = X - \cup_a clG_a = \cap_a(X - clG_a) = \cap_a \operatorname{int}(X - G_a) = \cap_a \operatorname{int} F_a$, where each $F_a = X - G_a$ is closed. Thus, B is G^* set.

Conversely, let f be LSC and the set $B = \{x \in X : f(x) \geq r\}$ be a G^* set. So $B = \cap_a \operatorname{int} F_a$, where each F_a is closed. Let us consider the set $A = \{x \in X : f(x) < r\}$. Then, $A = X - \{x \in X : f(x) \geq r\} = X - \cap_a \operatorname{int} F_a = \cup_a(X - \operatorname{int} F_a) = \cup_a cl(X - F_a) = \cup_a clG_a$, where each $G_a = X - F_a$ is open. Thus A is a union of regular open subsets of X and hence NLSC. ∎

Theorem 2.18. *Characteristic function of an open G^* set is NLSC.*

Proof. It is straightforward. ■

Again, since every regular open set is open G^*, as a corollary of this result it follows that the characteristic function of a regular open set is NLSC.

For further discussion of the newly introduced n-infy induced fuzzy space, we assume the following property to hold in its underlying topological space (X, T).

Definition 2.19. A topological space (X, T) is said to have the property $*$ if for each NLSC function f on X, $\{x \in X : f(x) \leq r\} = cl_X\{x \in X : f(x) < r\}$.

Since f is LSC, so the set $A = \{x \in X : f(x) > r\}$ is open and so no point of A is the limit point of $\{x \in X : f(x) < r\}$. i.e., $\text{cl}_X\{x \in X : f(x) < r\} \subseteq \{x \in X : f(x) \leq r\}$.

Now, let $x_0 \in \{x \in X : f(x) \leq r\} - \text{cl}_X\{x \in X : f(x) < r\}$ which implies that $f(x_0) = r$ and there exists a neighbourhood of x_0, say $N(x_0)$ such that $N(x_0) \cap \{f(x) < rl\} = \phi$.

Thus, we infer that if there exists a point $x_0 \in X, f(x_0) = r$ such that $N(x_0)$ is a subset of $\{x : f(x) > r\} \cup \{x_0\}$, then the reverse inclusion is not true. Thus the property $*$ follows from the condition that the set of the form $\{x : f(x) < r\}$, which is a union of regular closed sets, is not closed in X, i.e. the set$\{x : f(x) \geq r\}$ is not open.

In the sequel all the topological spaces considered are assumed to have the property $*$.

Theorem 2.20. *A fuzzy subset $\lambda \in I^X$ in an n-infy induced fts $(X, n(T))$ is n-open iff for each $r \in I$, the strong r-cut $\sigma_r(\lambda)$ is regular open in the topological space (X, T).*

Proof. Let $\lambda \in n(T)$, i.e., λ be an NLSC function. Now, as λ is NLSC, so λ is LSC and for each $r \in R$, the set $\{x \in X : \lambda(x) < r\}$ is a union of regular closed sets. i.e., $\{x \in X : \lambda(x) < r\} = \cup_\alpha(\text{cl}_X G_\alpha)$, where each G_α is open. Again, the set $\{x \in X : \lambda(x) > r\}$ is open, i.e., the set $\{x \in X : \lambda(x) \leq r\}$ is closed. But, by property $*$, we have

$$\{x \in X : \lambda(x) \leq r\} = \text{cl}_X\{x \in X : \lambda(x) < r\} = \text{cl}_X(\cup_\alpha(\text{cl}_X G_\alpha)),$$

which is regular closed [8]. Hence, $\sigma_r(\lambda) = \{x \in X : \lambda(x) > r\}$ is regular open.

Conversely, let us assume that the strong r-cut $\sigma_r(\lambda) = \{x \in X : \lambda(x) > r\}$ is regular open, i.e., the sets of the form $\{x \in X : \lambda(x) \leq r\}$ is regular closed. So λ is LSC (since the regular open sets are open). Now, let us consider the set $B = \{x \in X : \lambda(x) < r\}$. We see that

$$B = \{x \in X : \lambda(x) < r\} = \cup_{n \in N}\{x \in X : \lambda(x) \leq r - 1/n\}.$$

But $\{x \in X : \lambda(x) \leq r - 1/n\}$ is a regular closed set for each n. Thus B is a union of regular closed sets and hence λ is NLSC and thus belongs to $n(T)$.

Lemma 2.21. *A function $f : (X, T) \to (R, \sigma_1)$, where $\sigma_1 = \{(r, \infty) : r \in R\}$ is NLSC function iff the inverse image of an open subset of (R, σ_1) is regular open in (X, T).*

Proof. Let $f : (X, T) \to (R, \sigma_1)$ be NLSC. We consider the open subset (r, ∞) of (R, σ_1), where $r \in R$. We are to show that $f^{-1}(r, \infty) = \{x \in X : f(x) > r\}$ is regular open. Since f is NLSC, we have $\{x \in X : f(x) < r\} = \cup_\alpha (cl_X G_\alpha)$, where each G_α is open in X. Now, let us consider the set $A = \{x \in X : f(x) \leq r\}$. Then, $A = \{x \in X : f(x) \leq r\} = cl_X\{x \in X : f(x) < r\} = cl_X(\cup_\alpha(cl_X G_\alpha))$ [8], which is regular closed.

Again, $f^{-1}(r, \infty) = \{x \in X : f(x) > r\} = X - \{x \in X : f(x) \leq r\}$, which is regular open.

Conversely, let $f^{-1}(r, \infty) = \{x \in X : f(x) > r\}$ be regular open, i.e., $\{x \in X : f(x) \leq r\}$ is regular closed.

But, $\{x \in X : \lambda(x) < r\} = \cup_{n \in N}\{x \in X : \lambda(x) \leq r - 1/n\}$, which is a union of regular closed sets. Also, since $\{x \in X : f(x) > r\}$ is regular open and hence open; so f is LSC. Therefore, $f : (X, T) \to (R, \sigma_1)$ is NLSC.

Definition 2.22. Let $f : (X, n(T_1)) \to (Y, n(T_2))$ be a mapping between two n-infy induced fuzzy topological spaces. Then f is called fuzzy n-continuous if the inverse image of an n-open fuzzy subset of $n(T_2)$ is an n-open fuzzy subset of $n(T_1)$.

Definition 2.23. A mapping $f : X \to Y$ is said to be an r-continuous if the inverse image under f of any regular open subset of Y is regular open in X.

Theorem 2.24. *A mapping $f : (X, n(T_1)) \to (Y, n(T_2))$ is fuzzy n-continuous iff the mapping $f : (X, T_1) \to (Y, T_2)$ is r-continuous.*

Proof. Let $f : (X, n(T_1)) \to (Y, n(T_2))$ be a fuzzy n-continuous mapping and let B be a regular open subset of (Y, T_2). Now,

$$\begin{aligned} f^{-1}(B) &= \{x \in X : f(x) \in B\} \\ &= \{x \in X : \mu_B f(x) = 1\}, \\ &\quad \text{where } \mu_B \text{ is the characteristic function of the crisp set B.} \\ &= \{x \in X : \mu_B f(x) > r, \ \ 0 \leq r < 1\} \\ &= \{x \in X : (f^{-1}(\mu_B))(x) > r, \ \ 0 \leq r < 1\} \\ &= \sigma_r(f^{-1}(\mu_B)) \end{aligned}$$

Now μ_B, being the characteristic function of a regular open set of Y, is an NLSC function. So, $\mu_B \in n(T_2)$. Again, $f : (X, n(T_1)) \to (Y, n(T_2))$ being a fuzzy n-continuous mapping, so $f^{-1}(\mu_B) \in n(T_1)$.

Hence, by Theorem 2.20, $f^{-1}(B) = \sigma_r(f_{-1}(\mu_B))$ is a regular open subset of X, i.e. $f : (X, T_1) \to (Y, T_2)$ is r-continuous.

Conversely, let $f : (X, T_1) \to (Y, T_2)$ be an r-continuous mapping and β be an n-open fuzzy subset of $(Y, n(T_2))$. We are to show that $f^{-1}(\beta)$ is a member of $n(T_1)$. Now, for $0 < r < 1$,

$$\begin{aligned}\sigma_r(f^{-1}(\beta)) &= \{x \in X : (f^{-1}(\beta))(x) > r\} \\ &= \{x \in X : \beta(f(x)) > r\} \\ &= (\beta f)^{-1}(r, 1] \\ &= f^{-1}\beta^{-1}(r, 1] \\ &= f^{-1}(\beta^{-1}(r, 1]) \\ &= f^{-1}(\sigma_r(\beta))\end{aligned}$$

Now, since $\beta \in n(T_2)$ i.e. β is NLSC in Y. So $\sigma_r(\beta) = \{y \in Y : \beta(y) > r\}$ is a regular open subset of Y. Finally, $f : (X, T_1) \to (Y, T_2)$, being an r-continuous mapping, $\sigma_r(f^{-1}(\beta)) = f^{-1}(\beta^{-1}(r, 1])$ is a regular open subset of X.

Hence, $f^{-1}(\beta) \in n(T_1)$ and thus f is fuzzy n-continuous.

To conclude, we investigate the conditions under which a fuzzy topological space $(X, \mathscr{F})$ on a given set X becomes an infy-induced fuzzy topological space of the topology formed by the crisp members of $\mathscr{F}$. We first recall a theorem from [5].

Theorem 2.25. *Let $(X, \mathscr{F})$ be a fuzzy topological space and $T = \mathscr{F} \cap 2^X$ be the crisp members of $\mathscr{F}$. Then, the following two statements are equivalent:*

(a) *For any fuzzy subset λ and $r \in I$,*
$\omega_r(cl_{\mathscr{F}}(\lambda)) = \cap\{cl_T(\omega_s(\lambda)) : s < r\}$ is regular closed in X.

(b) *For any fuzzy subset μ and $p \in I$,*
$\sigma_p(int_{\mathscr{F}}(\mu)) = \cup\{int_T(\sigma_t(\mu)) : p < t\}$ is regular open in X.

Theorem 2.26. *Let $(X, \mathscr{F})$ be a fuzzy topological space and let $T = \mathscr{F} \cap 2^X$ the crisp part of $\mathscr{F}$. Then, the following statements are equivalent:*

(a) *For any fuzzy subset λ of X,*
$\omega_r(\lambda) = \sigma_r(cl_{\mathscr{F}}(\lambda)) = \cup\{cl_T(\omega_s(\lambda)) : s > r\}$is regular closed in (X, T).

(b) *For any fuzzy subset μ of X,*
$\sigma_p(\mu) = \omega_p(int_{\mathscr{F}}(\mu)) = \cap\{(\sigma_t(\mu)) : p > t\}$ is regular open in (X, T).

Proof. Let λ be a closed subset of $(X, \mathscr{F})$ and $r \in I$. Then, by the given condition,

$$\omega_r(\lambda) = \omega_r(cl_{\mathscr{F}}(\lambda)) = \cap\{cl_T(\omega_t(\lambda)) : t < r\}$$

is a regular closed subset of (X, T). Taking complement, we get

$$X - \omega_r(\lambda) \text{ is a regular open subset of (X,T),}$$

i.e., $\sigma_{1-r}(1_X - \lambda)$ is a regular open subset of (X, T), i.e., $(1_X - \lambda)$ is n-open in $(X, n(T))$, i.e., λ is closed in $(X, n(T))$.

Again, let μ be a closed fuzzy subset in $(X, n(T))$ and $r \in I$. Then, by given condition,

$$\omega_r(cl_{\mathscr{F}}(\mu)) = \cap\{cl_T(\omega_t(\mu)) : t < r\} \text{ is a regular closed subset of } (X, T).$$

Since μ is closed in $n(T)$, it can be seen that $\omega_t(\mu)$ is regular closed in (X, T) and hence closed in (X, T).

Thus, $\mathrm{cl}_T(\omega_t(\mu)) = \omega_t(\mu)$.

Hence, $\omega_r(cl_{\mathscr{F}}(\mu)) = \cap\{(\omega_t(\mu)) : t < r\} = \omega_r(\mu)$, i.e., $\mu = cl_{\mathscr{F}}(\mu)$.

Hence, μ is closed in $(X, \mathscr{F})$, which implies that the above two spaces $(X, \mathscr{F})$ and $(X, n(T))$ are equivalent, i.e., $\mathscr{F} = n(T)$.

Theorem 2.27. *Let $(X, \mathscr{F})$ be a infy induced fuzzy topological space and $T = \mathscr{F} \cap 2^X$, the crisp part of $\mathscr{F}$. Then the following statements are equivalent:*

(a) *For any fuzzy subset λ of X,*
$\omega_r(\lambda) = \sigma_r(cl_{\mathscr{F}}(\lambda)) = \cup\{cl_T(\omega_s(\lambda)) : s > r\}$ *is regular closed in (X, T).*

(b) *For any fuzzy subset μ of X,*
$\sigma_p(\mu) = \omega_p(int_{\mathscr{F}}(\mu)) = \cap\{int_T(\sigma_t(\mu)) : p > t\}$ *is regular open in (X,T).*

Proof. (**a**) $\Rightarrow$ (**b**). $\omega_r(\lambda) = \sigma_r(cl\mathscr{F}(\lambda)) = \cup\{cl_T(\omega_s(\lambda)) : s > r\}$ is regular closed in (X, T). i.e., $X - \omega_r(\lambda) = X - \sigma_r(cl_{\mathscr{F}}(\lambda)) = X - \cup\{cl_T(\omega_s(\lambda)) : s > r\}$ is regular open in (X, T).

Now, $X - \omega_r(\lambda) = \sigma_{1-r}(1_X - \lambda)$. Thus, $X - \omega_r(cl_{\mathscr{F}}(\lambda)) = \omega_{1-r}(1_X - cl_{\mathscr{F}}(\lambda)) = \omega_{1-r}(\mathrm{int}_{\mathscr{F}}(1_X - \lambda))$. Again,

$$\begin{aligned} X - \cup\{\mathrm{cl}_T(\omega_s(\lambda)) : s > r\} &= \cap(X\{\mathrm{cl}_T(\omega_s(\lambda)) : s > r\}) \\ &= \cap\{\mathrm{int}_T(X(\omega_s(\lambda))) : s > r\} \\ &= \cap\{\mathrm{int}_T(\sigma_{1-s}(1_X - \lambda)) : s > r\} \end{aligned}$$

i.e., $\sigma_{1-r}(1_X - \lambda) = \omega_{1-r}(\mathrm{int}_{\mathscr{F}}(1_X\lambda)) = \cap(\omega_{1-s}(1_X - \lambda)) : s > r, \}$ is regular open in (X, T). Putting $1 - r = p$, $1 - s = t$ and $1_X - \lambda = \mu$, we get, $\sigma_p(\mu) = \omega_p(\mathrm{int}_{\mathscr{F}}(\mu)) = \cap\mathrm{int}_(\sigma_t(\mu)) : p > t,\ p, t \in I$ is regular open in (X, T).

(**b**) $\Rightarrow$ (**a**). $\sigma_p(\mu) = \omega_p(\mathrm{int}_{\mathscr{F}}(\mu)) = \{\mathrm{int}_T(\sigma_t(\mu)) : p > t,\ p, t \in I\}$ is regular open in (X,T). i.e., $X - \sigma_p(\mu) = X - \omega_p(\mathrm{int}_{\mathscr{F}}\mu)) = X - \cap\{\mathrm{int}_T(\sigma_t(\mu)) : p > t,$ $p, t \in I\}$ is regular closed in (X,T). Now, $X - \sigma_p(\mu) = \omega_{1-p}(1_X - \mu) = \omega_r(\lambda)$. Again,

$$\begin{aligned} X - \omega_p(\mathrm{int}_{\mathscr{F}}(\mu)) &= \sigma_{1-p}(1_X - int_{\mathscr{F}}(\mu)) \\ &= \sigma_{1-p}(\mathrm{cl}_{\mathscr{F}}(1_X - \mu)) \\ &= \sigma_r(cl_{\mathscr{F}}(\lambda)) \end{aligned}$$

and

$$\begin{aligned} X - \cap\{\mathrm{int}_T(\sigma_t(\mu)) : p > t\} &= \cup\{X - int_T(\sigma_t(\mu)) : p > t\} \\ &= \cup\{cl_T(X - \sigma_t(\mu)) : p > t\} \\ &= \cup\{cl_T(\omega_{1-t}(1_X - \mu)) : p > t\} \\ &= \{cl_T(\omega_s(\lambda)) : s > r\} \end{aligned}$$

i.e., $\omega_r(\lambda) = \sigma_r(cl_{\mathscr{F}}(\lambda)) = \cup\{cl_T(\omega_s(\lambda)) : s > r\}$ is regular closed in (X,T).

This completes the proof. ■

Theorem 2.28. *Let $(X, \mathscr{F})$ be a fuzzy topological space and let $T = \mathscr{F} \cap 2^X$ be the crisp members of $\mathscr{F}$. Then the n-infy induced fuzzy topology $n(T)$ on X is equivalent to the fuzzy topology $\mathscr{F}$ if for any fuzzy subset λ and $r \in I$*

$$\omega_r(\lambda) = \sigma_r(\mathrm{cl}_{\mathscr{F}}(\lambda)) = \cup\{cl_T(\omega_s(\lambda)) : s > r \textit{ is regular closed in } (X, T).$$

Proof. We are to show that, under the given condition, $\mathscr{F} = n(T)$. Let the given condition holds and let λ be closed in $\mathscr{F}$. Then, $\sigma_r(\lambda) = \sigma_r(cl_{\mathscr{F}}(\lambda)) = \omega_r(\lambda)$ is regular closed in (X, T) and hence $\sigma_{1-r}(1_X - \lambda)$ is regular open in (X, T), i.e., $1_X - \lambda \in n(T)$, i.e., λ is closed in $n(T)$.

Again, let α be closed in $n(T)$ and $t \in I$. Then,

$$\sigma_t(\mathrm{cl}_{\mathscr{F}}(\alpha)) = \cup\{\mathrm{cl_T}(\omega_\mathrm{p}(\alpha)) : \mathrm{p} > \mathrm{t}\} = \cup\{\omega_\mathrm{p}(\alpha) : \mathrm{p} > \mathrm{t}\} = \sigma_\mathrm{t}(\alpha).$$

Thus, $\sigma_t(cl_{\mathscr{F}}(\alpha)) = \sigma_t(\alpha)$ for all $t \in I$, i.e., $\mathrm{cl}_{\mathscr{F}}(\alpha) = \alpha$, i.e., α is closed in $\mathscr{F}$. Hence, $\mathscr{F} = n(T)$. ■

In this paper, a new fuzzy topological structure namely infy induced fuzzy topological space has been introduced with the help of NLSC functions and under the influence of the space various properties of fuzzy subsets has been studied. Also, the conditions under which a fuzzy topological space becomes an infy induced fuzzy topological space has been investigated. The newly introduced space actually generalizes the concept of fuzzy topology and so there is ample scope to investigate various topological properties on this generalized space.

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