# ON INCLUSION BETWEEN $\Lambda BV^{(p)}$, CHANTURYIA AND $H_w^p$ CLASSES

*Dedicated to the memory of professor Parviz Azimi*

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**Abstract.** In this paper We prove inclusion relations between $\Lambda BV^{(p)}$ and V[v] and give a necessary condition for the inclusion of $\Lambda BV^{(p)}$ in classes $H_\omega^p$.

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## 1. Introduction

In 1972, Waterman [7] introduced the class of functions of $\Lambda BV$. In 1980, Shiba [5] generalized this class and introduced the class $\Lambda BV^{(p)}$ ($p \geq 1$). In 2006, Vyas [6] proved $\Lambda BV^{(p)}$ is a Banach space with suitable norm, the intersection of $\Lambda BV^{(p)}$, over all sequences $\Lambda$, is the class of functions $BV^{(p)}$ and the union of $\Lambda BV^{(p)}$, over all sequences $\Lambda$, is the class of functions having right- and left-hand limits at every point. In [1], Avdispahic gave inclusion relations between $\Lambda BV$ and V[v]. For $p \geq 1$, we give inclusion relation between $\Lambda BV^{(p)}$ and V[v]. Goginava in [3] gave a necessary condition for inclusion $\Lambda BV$ in $H_\omega^p$. Here, we give a necessary condition for inclusion $\Lambda BV^{(p)}$ in $H_\omega^p$. First, we define the classes $\Lambda BV^{(p)}$, Chanturia and $H_\omega^p$.

**Definition 1.1** Given an interval $I$, and a sequence of positive real numbers $\Lambda = \{\lambda_m\}$, (m=1,2,...) such that $\sum_{m=1}^{\infty}(1/\lambda_m)$ diverges and $1 \leq p < \infty$, we say that $f \in \Lambda BV^{(p)}$(I) (that is $f$ is a function of $p-\Lambda-$bounded variation over $I$) if

$$V(f) = V_\Lambda(f, p, I) = \sup_{\{I_m\}} V_\Lambda(\{I_m\}, f, p, I) < \infty,$$

where $V_\Lambda(\{I_m\}, f, p, I) = \left(\sum_m \frac{|f(a_m)-f(b_m)|^p}{\lambda_m}\right)^{1/p}$, and $\{I_m\}$ is a sequence of non-overlapping subintervals $I_m = [a_m, b_m] \subset I = [a, b]$. For any $x \in I = [a, b]$, we define

$$v(x) = v_\Lambda(f; x) = v_\Lambda(f, p, [a, x]).$$

For $f \in \Lambda BV^{(p)}(I)$, we define $||f|| = |f(a)| + V(f)$ where $I = [a, b]$.

**Definition 1.2** The modulus of variation of a $\Lambda$ function $f$ is the function $\nu_f(n)$ with domain the positive integers, defined by

$$\nu_f(n) = \sup_{\Pi_n} \sum_{k=1}^{n} |f(I_k)|$$

where $\Pi_n$ is an arbitrary system of $n$ disjoint intervals $I_k = [a_k, b_k] \subset (0, 1)$ and $f(I_k) = f(b_k) - f(a_k)$.

The modulus of variation of any function is nondecreasing and upwards convex. If the modulus of variation $\nu(n)$ is given, then $V[\nu]$ denotes the class of functions for which $\nu_f(n) = \mathrm{O}(\nu(n))$ when $n \to \infty$.

**Definition 1.3** If $\omega(\delta)$ is a modulus of continuity , then $H_p^\omega, p \geq 1$ , denotes the class of functions $f \in L^p([0, 1])$ for which $\omega(\delta, f)_p = O(\omega(\delta))$ as $\delta \to 0+$, where

$$\omega(\delta, f)_p = \sup_{0<h\leq\delta} \left(\int_0^1 |f(x + h) - f(x)|^p dx\right)^{1/p}.$$

## 2. On the classes $\Lambda BV^{(p)}$ and $V[\nu]$

In [1], Avdispahic gave inclusion relations between $\Lambda BV$ and $V[v]$. Here, we give inclusion relations between $\Lambda BV^{(p)}$ and $V[v]$. Theorem 1 of [4] shows that

**Theorem 2.1** $\Lambda BV^{(p)} \subset V[\left(n/\left(\sum\limits_{i=1}^{n}\right) 1/\lambda_i\right)]$.

**Theorem 2.2** *$\Lambda BV^{(p)}$ contains every class $V[\nu]$ such that the condition*

$$\sum_{k=1}^{\infty} \Delta(1/\lambda_k)\nu^p(k) < \infty$$

*is satisfied, where $\Delta a_k = a_k - a_{k+1}$.*

**Proof.** Let $\{I_k\}$, $k = 1, \ldots, n$, be an arbitrary collection of nonoverlapping intervals, $I_k \subset [0, 1]$. By partial summaion we obtain

$$\begin{aligned}\sum_{k=1}^{n} |f(I_k)|^p/\lambda_k &= \sum_{k=1}^{n-1} \Delta(1/\lambda_k) \sum_{i=1}^{k} |f(I_i)|^p + 1/\lambda_n \sum_{i=1}^{n} |f(I_i)|^p \\ &\leq \sum_{k=1}^{n-1} \Delta(1/\lambda_k)\nu^p(k) + \nu^p(n)/\lambda_n\end{aligned}$$

and $\nu^p(n)/\lambda_n \leq \sum_{k=n} \Delta(1/\lambda_k)\nu^p(k)$. ∎

**Theorem 2.3** *If* $p > 1$ *and* $k\Delta(\frac{1}{\lambda_k}) = O(1)$, *then*

$$V\left[\frac{n}{(\sum_{i=1}^n 1/\lambda_i)^p}\right] \subset \Lambda BV^{(p)}.$$

**Proof.** Let us denote $u_k = k\Delta(1/\lambda_k)$, $S_n = \sum_{k=1}^n u_k$. By [2, Theorem 2], $S_n \to \infty$ as $n \to \infty$. Hence by [1, p. 905, Corollary to theorem 1] we have

$$\sum_{k=1}^{\infty} \Delta\left(\frac{1}{\lambda_k}\right)\left(\frac{k^{\frac{1}{p}}}{\sum_{i=1}^k (1/\lambda_i)}\right)^p \le \sum_{k=1}^{\infty} \frac{u_k}{S_k^p} < \infty$$

The conclusion follows by Theorem 2.2. ∎

It can be observed that the sequence $\nu_n := \frac{n^{1/p}}{\sum_{i=1}^n \frac{1}{\lambda_i}}$ is equivalent to a modulus of variation. Thus, the class appearing in the statement of theorem 2.4 is a Chanturia class, in fact.

Immediately, from Theorem 2.3 and Theorem 2.1 we see that

**Theorem 2.4** $V[n^{\frac{1}{p}}/ln\ n] \subset HBV^{(p)} \subset V[n/\ln^{\frac{1}{p}} n]$.

## 3. On the imbedding of $\Lambda BV^{(p)}$ class in the class $H^p_\omega$

In [3], Goginava gave a necessary condition for inclusion $\Lambda BV$ in $H^p_\omega$. Here, we give a necessary condition for inclusion $\Lambda BV^{(p)}$ in $H^p_\omega$.

**Theorem 3.1** *Let* $\Lambda BV^{(p)} \subset H^\omega_p$ *for some* $p \in [1, \infty)$ *then*

$$\limsup_{n\to\infty} \frac{1}{\omega(1/n)n^{1/p}} \max_{1\le m\le n} \frac{m^{1/p}}{\sum_{i=1}^m 1/\lambda_i} < \infty. \tag{1}$$

**Proof.** Assume condition (1) is not satisfied. As an example, we construct a function from $\Lambda BV^{(p)}$ that is not in $H^\omega_p$. Since condition (1) is not satisfied, there exists a sequence of integers $\{\gamma_k, k \ge 1\}$ such that

$$\lim_{k\to\infty} \frac{1}{\omega(1/\gamma_k)\gamma_k^{1/p}} \max_{1\le m\le \gamma_k} \frac{m^{1/p}}{\sum_{i=1}^m 1/\lambda_i} = \infty\ .$$

Let $\{\gamma_k', k \ge 1\}$ be a sequence of integers for which $2^{\gamma_k'-1} \le \gamma_k < 2^{\gamma_k'}$. The fact that $\omega(\delta)$ is nondecreasing yields

$$\frac{2^{1/p}}{\omega(2^{-\gamma_k'})2^{\gamma_k'/p}} \max_{1\le m\le 2^{\gamma_k'}} \frac{m^{1/p}}{\sum_{i=1}^m 1/\lambda_i} \ge \frac{1}{\omega(1/\gamma_k)\gamma_k^{1/p}} \max_{1\le m\le \gamma_k} \frac{m^{1/p}}{\sum_{i=1}^m 1/\lambda_i}\ ,$$

where

$$\limsup_{k\to\infty} \frac{1}{\omega(2^{-\gamma_k'})2^{-\gamma_k'/p}} \max_{1\le m\le 2^{\gamma_k'}} \frac{m^{1/p}}{\sum_{i=1}^m 1/\lambda_i} < \infty.$$

Then, a sequence of integers $\{n_k^{'} : k \geq 1\} \subset \{\gamma_k^{'} : k \geq 1\}$ exist such that

$$\lim_{k\to\infty} \frac{1}{\omega(2^{-n_k^{'}})} \frac{1}{\sum_{i=1}^{m(n_k^{'})} 1/\lambda_i} \left(\frac{m(n_k^{'})}{2^{n_k^{'}}}\right)^{1/p} < \infty \ , \tag{2}$$

where

$$\max_{1\leq m\leq 2^{n_k^{'}}} \frac{m^{1/p}}{\sum_{i=1}^{m} 1/\lambda_i} = \frac{(m(n_k^{'}))^{1/p}}{\sum_{i=1}^{m(n_k^{'})} 1/\lambda_i}.$$

The following three cases are possible:

(a) (a) there exists a sequence of integers $\{s_k^{'} : k \geq 1\} \subset \{n_k^{'} : k \geq 1\}$ such that

$$m(s_k^{'}) < 2^{2s_{k-1}^{'}};$$

(b) there exists a sequence of integers $\{q_k^{'} : k \geq 1\} \subset \{n_k^{'} : k \geq 1\}$ such that

$$2^{2q_{k-1}^{'}} \leq m(q_k^{'}) < 2^{q_k^{'} - q_{k-1}^{'}};$$

(c) $2^{n_k^{'} - n_{k-1}^{'}} \leq m(n_k^{'}) < 2^{n_k^{'}}$ for all $k \geq k_0$.

First, consider case (a). We choose a sequence of integers $\{s_k : k \geq 1\}$ of $\{s_k^{'} : k \geq 1\}$ such that

$$\sum_{i=1}^{m(s_k)} \frac{1}{\lambda_i} \geq 2^{2s_{k-1}/p}.$$

Then, relation (2) yields

$$\lim_{k\to\infty} \omega\left(\frac{1}{2^{s_k}}\right) 2^{s_k/p} = 0.$$

Let $\{r_k : k \geq 1\} \subset \{s_k : k \geq 1\}$ be such that

$$\omega^p\left(\frac{1}{2^{r_k}}\right) 2^{r_k/p} \leq \omega\left(\frac{1}{2^{r_k}}\right) 2^{r_k/p} \leq 4^{-k}. \tag{3}$$

Consider the function $f$ defined as follows:

$$f(x) = \begin{cases} 2c_j(2^{r_j}x - 1), & x \in [2^{-r_j}, 3(2^{-r_j-1})), \\ -2c_j(2^{r_j}x - 2), & x \in [3(2^{-r_j-1}),\ 2(2^{-r_j})) \text{ for } j = 1, 2, ..., \\ 0, & \text{otherwise} \ , \end{cases}$$

and

$$f(x + l) = f(x), \quad l = \pm 1, \pm 2, ...,$$

where $c_j = \sqrt{\omega\left(\frac{1}{2^{r_j}}\right) 2^{r_j/p}}$. From the construction of the function $f$ and relation (3) it follows that $f \in \Lambda BV^{(p)}$.

Now, consider case (b). Let $\{q_k : k \geq 1\} \subset \{q_k^{'} : k \geq 1\}$ be such that

$$\frac{1}{\omega(2^{-q_k})} \frac{1}{\sum_{i=1}^{m(q_k)} 1/\lambda_i} \left(\frac{m(q_k)}{2^{q_k}}\right)^{1/p} \geq 4^k. \tag{4}$$

Consider the function $\mathrm{g}_k$ defined as follows:

$$\mathrm{g}_k(x) = \begin{cases} h_k(2^{q_k}x - 2j + 1), & x \in [(2j-1)/2^{q_k}, 2j/2^{q_k}), \\ -h_k(2^{q_k}x - 2j - 1), & x \in [2j/2^{q_k}, (2j+1)/2^{q_k}) \\ & \text{for } j = m(q_{k-1}), ..., m(q_k) - 1, \\ 0, & \text{otherwise }, \end{cases}$$

where $h_k = \frac{1}{2^k \sum_{j=1}^{m(q_k)} 1/\lambda_j}$.

Let

$$\mathrm{g}(x) = \sum_{k=2}^{\infty} \mathrm{g}_k(x), \qquad \mathrm{g}(x+l) = \mathrm{g}(x), \qquad l = \pm 1, \pm 2, ....$$

First, we prove that $\mathrm{g} \in \Lambda BV^{(p)}$. For each non overlapping intervals $\{I_n : n \geq 1\}$, we have

$$\sum_{j=1}^{\infty} \frac{|\mathrm{g}(I_j)|^p}{\lambda_j} \leq 2^p \sum_{i=1}^{\infty} h_i^p \sum_{j=1}^{m(q_i)} \frac{1}{\lambda_j} \leq 2^p \sum_{i=1}^{\infty} h_i \sum_{j=1}^{m(q_i)} \frac{1}{\lambda_j} = 2^p \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty.$$

Hence, $\mathrm{g} \in \Lambda BV^{(p)}$.

Finally, consider case (c). Let $\{n_k : k \geq 1\} \subset \{n_k^{'} : k \geq k_0\}$ be such that

$$n_k \geq 2n_{k-1} + 1, \tag{5}$$

$$\frac{1}{\omega(2^{-n_k})} \frac{1}{\sum_{i=1}^{m(n_k)} 1/\lambda_i} \left(\frac{m(n_k)}{2^{n_k}}\right)^{1/p} \geq 2^{2n_{k-1}/p+k}. \tag{6}$$

Consider the function $\phi_k$ defined as follows:

$$\phi_k(x) = \begin{cases} d_k(2^{n_k}x - 2j + 1), & x \in [(2j-1)/2^{n_k}, 2j/2^{n_k}), \\ -d_k(2^{n_k}x - 2j - 1), & x \in [2j/2^{n_k}, (2j+1)/2^{n_k}) \\ & \text{for } j = 2^{n_{k-1}-n_{k-2}}, ..., 2^{n_k - n_{k-1} - 1} - 1, \\ 0, & \text{otherwise }, \end{cases}$$

where $d_k = \frac{1}{2^k \sum_{j=1}^{m(n_k)} 1/\lambda_j}$.

Let

$$\phi(x) = \sum_{k=3}^{\infty} \phi_k(x), \qquad \phi(x+l) = \phi(x), \qquad l = \pm 1, \pm 2, ....$$

For every choice of nonoverlapping intervals $\{I_n, n \geq 1\}$, we have

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{|\phi(I_j)|^p}{\lambda_j} &\leq 2^p \sum_{i=2}^{\infty} d_i^p \sum_{j=1}^{2^{n_i-n_{i-1}-1}} \frac{1}{\lambda_j} \leq 2^p \sum_{i=2}^{\infty} d_i \sum_{j=1}^{2^{n_i-n_{i-1}-1}} \frac{1}{\lambda_j} \\ &\leq 2^p \sum_{i=2}^{\infty} d_i \sum_{j=1}^{m(n_i)} \frac{1}{\lambda_j} \leq 2^p \sum_{i=2}^{\infty} \frac{1}{2^i} < \infty.\end{aligned}$$

Hence $\phi \in \Lambda BV^{(p)}$, by [3, Theorem 1] we have $f, g$ and $\phi$ do not belong to $H_\omega^p$. Therefore the proof is completed. ∎

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