EXISTENCE OF THREE SOLUTIONS FOR NONLOCAL ELLIPTIC SYSTEM OF $(p_1, ..., p_n)$-KIRCHHOFF TYPE

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Abstract. In this paper, we establish the existence of at least three solutions to a Dirichlet boundary problem involving the $(p_1, ..., p_n)$-Kirchhoff type systems. Our technical approach is mainly based on the general three critical points theorem obtained by Ricceri.

Keywords: $(p_1, ..., p_n)$-Kirchhoff type system; multiple solutions; three critical points theory.

1. Introduction and main results

In the present paper, we deal with the existence of at least three solutions for nonlinear elliptic equations of $(p_1, ..., p_n)$-Kirchhoff type under Dirichlet boundary conditions:

$$
(1.1)\quad \begin{cases}
-\left[M_1\left(\int_\Omega |\nabla u_1|^{p_1}\right)\right]^{p_1-1}\Delta_{p_1}u_1=\lambda F_{u_1}(x,u_1,...,u_n)+\mu G_{u_1}(x,u_1,...,u_n), & \text{in } \Omega,\\
-\left[M_2\left(\int_\Omega |\nabla u_2|^{p_2}\right)\right]^{p_2-1}\Delta_{p_2}u_2=\lambda F_{u_2}(x,u_1,...,u_n)+\mu G_{u_2}(x,u_1,...,u_n), & \text{in } \Omega,\\
\cdots & \\
-\left[M_n\left(\int_\Omega |\nabla u_n|^{p_n}\right)\right]^{p_n-1}\Delta_{p_n}u_n=\lambda F_{u_n}(x,u_1,...,u_n)+\mu G_{u_n}(x,u_1,...,u_n), & \text{in } \Omega,\\
u_i=0 \quad \text{for } 1\le i\le n, & \text{on } \partial\Omega,
\end{cases}
$$

where $\Omega \subset R^N (N \geq 1)$ is a non-empty bounded open set with a sufficient smooth boundary $\partial\Omega$, $\lambda, \mu \in [0, +\infty)$, $p_i > N$, Δ_p is the p-Laplacian operator $\Delta_p u = div\left(|\nabla u|^{p-2}\nabla u\right)$. $F, G : \Omega\times R^n \mapsto \mathrm{R}$ are functions such that $F(\cdot, t_1, ..., t_n)$, $G(\cdot, t_1, ..., t_n)$ are measurable in Ω for all $(t_1, ..., t_n) \in R^n$ and $F(x, \cdot)$, $G(x, \cdot)$ are continuously differentiable in R^n for a.e. $x \in \Omega$. F_{u_i} is the partial derivative of F with respect to u_i, $1 \leq i \leq n$, so does G_{u_i}. $M_i : R^+ \to R$, $i = 1, 2, ..., n$ are continuous functions, which satisfy the bounded conditions as follows.
(M) There are two positive constants m_0, m_1 such that

$$
(1.2)\qquad m_0 \leq M_i(t) \leq m_1, \quad \forall t \geq 0, \quad i = 1, 2, ..., n.
$$

In what follows, $|\Omega|$ denotes the Lebesgue measure of Ω, X denotes the Cartesian product of Sobolev spaces $W_0^{1,p_1}(\Omega), \ldots, W_0^{1,p_{n-1}}(\Omega)$ and $W_0^{1,p_n}(\Omega)$, i.e., $X = W_0^{1,p_1}(\Omega) \times \cdots \times W_0^{1,p_n}(\Omega)$. The space X is endowed with the norm

$$
\|(u_1, \ldots, u_n)\| = \sum_{i=1}^{n} \|u_i\|_{p_i}, \quad \|u_i\|_p = \left(\int_\Omega |\nabla u_i|^{p_i}\right)^{1/p_i}, \ 1 \leq i \leq n.
$$

Let

$$
(1.3)\qquad C = \max\left\{ \sup_{u_i\in W_0^{1,p_i}(\Omega)\setminus\{0\}} \frac{\max_{x\in\bar\Omega}\{|u_i(x)|^{p_i}\}}{\|u_i\|_{p_i}^{p_i}} \right\}.
$$

Since $p_i > N$, $W_0^{1,p_i}(\Omega) \to C^0(\bar\Omega)$, $1 \leq i \leq n$, are compact, and one has $C < +\infty$. As usual, a weak solution of system (1.1) is any $(u_1, ..., u_n) \in X$ such that

$$
(1.4)\qquad \begin{aligned}
&\sum_{i=1}^{n}\left[M_i\left(\int_\Omega |\nabla u_i|^{p_i}\right)\right]^{p_i-1}\int_\Omega |\nabla u_i|^{p_i-2}\nabla u_i \nabla \xi_i\\
&\quad -\sum_{i=1}^{n}\lambda\int_\Omega F_{u_i}(x, u_1, ..., u_n)\,\xi_i dx - \sum_{i=1}^{n}\lambda\int_\Omega G_{u_i}(x, u_1, ..., u_n)\,\xi_i dx = 0
\end{aligned}
$$

for all $(\xi_1, ..., \xi_n) \in X$.

The system (1.1) is related to the stationary version of a model, the so-called Kirchhoff equation which was introduced by [1]. More precisely, Kirchhoff proposed the following mathematical model.

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.5}$$

which generalizes the D'Alembert's wave equation involving free vibrations of elastic strings, where ρ is the mass density, P_0 is the initial tension, h is the area of the cross-section, E is the Young modulus of the material, and L is the length of the string.

Later, (1.5) was developed to the following result

$$u_{tt}-M\left(\int_\Omega|\nabla u|^2\right)\Delta u=f(x,u) \text{ in } \Omega, \tag{1.6}$$

where $M : R^+ \to R$ is a given function. After that, some authors studied the following problem

$$-M\left(\int_\Omega|\nabla u|^2\right)\Delta u=f(x,u) \text{ in } \Omega,\ u=0 \text{ on } \partial\Omega, \tag{1.7}$$

which is the stationary counterpart of (1.6). By using variational methods and other techniques, many results of (1.7) were obtained, please refer to [2]-[12] and the references therein. In particular, Alves et al. [2, Theorem 4] assumed that M satisfies bounded condition (M) and $f(x,t)$ satisfies the following condition.

$$0<\upsilon F(x,t)\le f(x,t)t, \text{ for all } |t|\ge R, x\in\Omega \text{ for some } \upsilon>2 \text{ and } R>0, \tag{AR}$$

where $F(x,t)=\int_0^t f(x,s)ds$. One positive solutions for (1.7) was obtained.

In [13], applying Ekeland's Variational Principle, the authors established the existence of a weak solution for boundary problem involving the nonlocal elliptic system of p-Kirchhoff type

$$\begin{cases} -\left[M_1\left(\int_\Omega|\nabla u|^p\right)\right]^{p-1}\Delta_p u=f(u,\upsilon)+\rho_1(x), & \text{in } \Omega,\\ -\left[M_2\left(\int_\Omega|\nabla \upsilon|^p\right)\right]^{p-1}\Delta_p \upsilon=g(,u,\upsilon)+\rho_2(x), & \text{in } \Omega,\\ \dfrac{\partial u}{\partial\eta}=\dfrac{\partial \upsilon}{\partial\eta}=0, & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where η is the unit exterior vector on $\partial\Omega$, and $M_i, \rho_i (i=1,2)$, f, g satisfy suitable assumptions.

In [14], when $\mu=0, n=2$ in (1.1), Cheng et al. studied the existence of two solutions and three solutions of the following nonlocal elliptic system

$$\begin{cases} -\left[M_1\left(\int_\Omega|\nabla u|^p\right)\right]^{p-1}\Delta_p u=\lambda F_u(x,u,\upsilon), & \text{in } \Omega,\\ -\left[M_2\left(\int_\Omega|\nabla \upsilon|^q\right)\right]^{q-1}\Delta_q \upsilon=\lambda F_\upsilon(x,u,\upsilon), & \text{in } \Omega,\\ u=\upsilon=0, & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

In [15], when $n = 2$ in (1.1), Chen et al. proved the existence of three solutions of the following problem

$$\begin{cases} -\left[M_1\left(\int_\Omega |\nabla u|^p\right)\right]^{p-1} \Delta_p u = \lambda F_u(x,u,v) + \mu G_u(x,u,v), & \text{in } \Omega, \\ -\left[M_2\left(\int_\Omega |\nabla v|^q\right)\right]^{q-1} \Delta_q v = \lambda F_v(x,u,v) + \mu G_v(x,u,v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

In this paper, our objective is to prove the existence of three solutions of problem (1.1) by applying three critical points theorem introduced by Ricceri [16]. Our result, under some suitable conditions, ensures the existence of an open interval $\Lambda \subset [0, +\infty)$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, problem (1.1) admits at least three weak solutions whose norms in X are less than ρ. The purpose of the present paper is to generalize the main result of [15] to the general case.

Now, for every $x_0 \in \Omega$ and choosing R_1, R_2 with $R_2 > R_1 > 0$, such that $B(x_0, R_2) \subseteq \Omega$, where $B(x,R) = \{y \in R^N : |y - x| < R\}$, let

$$\alpha_i = \alpha_i(N, p_i, R_1, R_2) = \frac{C^{1/p_i}(R_2^N - R_1^N)^{1/p_i}}{R_2 - R_1}\left(\frac{\pi^{N/2}}{\Gamma(1+N/2)}\right)^{1/p_i}, \quad 1 \le i \le n, \tag{1.11}$$

where Γ is the Gamma function. Moreover, assume that a, c are positive constants, define

$$\begin{aligned} y(x) &= \frac{a}{R_2 - R_1}\left(R_2 - \left\{\sum_{i=1}^N (x^i - x_0^i)^2\right\}^{1/2}\right), \ \forall x \in B(x_0,R_2)\backslash B(x_0,R_1), \\ A(c) &= \left\{(t_1, \ldots, t_n) \in R^n : \sum_{i=1}^n |t_i|^{p_i} \le c\right\}, \\ M^+ &= \max\left\{\frac{m_1^{p_i-1}}{p_i},\ i = 1, ..., n\right\},\ M_- = \min\left\{\frac{m_0^{p_i-1}}{p_i},\ i = 1, ..., n\right\}. \end{aligned}$$

Our main result is the following theorem.

Theorem 1.1 *Let $R_2 > R_1 > 0$, such that $B(x_0, R_2) \subseteq \Omega$. Assume that there exist $n+2$ positive constants a, b, γ_i for $1 \le i \le n$, with $\gamma_i < p_i$, $\sum_{i=1}^n (a\alpha_i)^{p_i} > bM^+/M_-$, and a function $\alpha(x) \in L^\infty(\Omega)$ such that*

(j1) *$F(x, t_1, ..., t_n) \ge 0$, for a.e. $x \in \Omega\backslash B(x_0, R_1)$ and all $(t_1, ..., t_n) \in [0, a] \times \cdots \times [0, a]$;*

(j2) $\sum_{i=1}^n (a\alpha_i)^{p_i} |\Omega| \sup_{(x,t_1,...,t_n)\in\Omega\times A(bM^+/M_-)} F(x, t_1, ..., t_n) < b \int_{B(x_0,R_1)} F(x, a, ..., a)dx;$

(j3) $F(x,t_1,...,t_n) \le \alpha(x)\left(1+\sum_{i=1}^{n}|t_i|^{\gamma_i}\right)$ *for a.e.* $x\in\Omega$ *and all* $(t_1,...,t_n)\in R^n$*;*

(j4) $F(x,0,...,0)=0$*, for a.e.* $x\in\Omega$*.*

Then there exist an open interval $\Lambda\subseteq[0,\infty)$ *and a positive real number* ρ *with the following property:*

for each $\lambda\in\Lambda$ *and for Carathéodory functions* $G_{u_i}:\Omega\times R^n\mapsto R$ *satisfying*

(j5) $\sup_{\{|t_i|\le\xi,1\le i\le n\}}\left(\sum_{i=1}^{n}|G_{u_i}(\cdot,s,t)|\right)\in L^1(\Omega)$ *for all* $\xi>0$*,*

there exists $\delta>0$ *such that, for each* $\mu\in[0,\delta]$*, problem* (1.1) *has at least three weak solutions* $w_i=(u_{i1},...,u_{in})\in X$ $(i=1,2,3)$ *whose norms* $\|w_i\|$ *are less than* ρ*.*

2. Proof of the main result

Our analysis is based on the following modified form of Ricceri's three critical points theorem (Theorem 1 in [16]) and Proposition 3.1 of [17], which is our mainly tool in proving our main result.

Theorem 2.1 ([16], Theorem 1) *Let* X *be a reflexive real Banach space and* $\Phi: X\mapsto R$ *be a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on* X^* *and* Φ *is bounded on each bounded subset of* X*;* $\Psi: X\mapsto R$ *is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact;* $I\subseteq R$ *an interval. Suppose that*

$$\lim_{\|x\|\to+\infty}(\Phi(x)+\lambda\Psi(x))=+\infty$$

for all $\lambda\in I$*, and that there exists* $h\in R$ *such that*

$$\sup_{\lambda\in I}\inf_{x\in X}(\Phi(x)+\lambda(\Psi(x)+h))<\inf_{x\in X}\sup_{\lambda\in I}(\Phi(x)+\lambda(\Psi(x)+h)). \tag{2.1}$$

Then, there exists an open interval $\Lambda\subseteq I$ *and a positive real number* ρ *with the following property: for every* $\lambda\in\Lambda$ *and every* C^1 *functional* $J: X\mapsto R$ *with compact derivative, there exists* $\delta>0$ *such that, for each* $\mu\in[0,\delta]$ *the equation*

$$\Phi'(x)+\lambda\Psi'(x)+\mu J'(x)=0$$

has at least three solutions in X *whose norms are less than* ρ*.*

Proposition 2.1 ([17], Proposition 3.1) *Suppose that X is a non-empty set and Φ, Ψ are two real functions on X. Assume that there exist $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = -\Psi(x_0) = 0, \quad \Phi(x_1) > 1, \quad \sup_{x\in\Phi^{-1}([-\infty,r])} -\Psi(x) < r\frac{-\Psi(x_1)}{\Phi(x_1)}.$$

Then, for each h satisfying

$$\sup_{x\in\Phi^{-1}([-\infty,r])} -\Psi(x) < h < r\frac{-\Psi(x_1)}{\Phi(x_1)}$$

one has

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x) + \lambda(\Psi(x)+h)) < \inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x) + \lambda(\Psi(x)+h)).$$

Before giving the proof of Theorem 1.1, we define a functional and give a lemma.

The functional $H : X \to R$ is defined by

$$\begin{aligned} H(u_1, ..., u_n) &= \Phi(u_1, ..., u_n) + \lambda J(u_1, ..., u_n) + \mu\psi(u_1, ..., u_n) \\ &= \sum_{i=1}^{n}\frac{1}{p_i}\widehat{M_i}\left(\int_\Omega |\nabla u_i|^{p_i}\right) - \lambda\int_\Omega F(x, u_1, ..., u_n)dx - \mu\int_\Omega G(x, u_1, ..., u_n)dx \end{aligned} \tag{2.2}$$

for all $(u_1, ..., u_n) \in X$, and where

$$\widehat{M_i}(t) = \int_0^t [M_i(s)]^{p_i-1}\,ds, \quad 1\leq i\leq n, \quad \text{for all } t\geq 0. \tag{2.3}$$

By the conditions (M) and (j3), it is easy to see that $H \in C^1(X, R)$ and a critical point of H corresponds to a weak solution of the system (1.1).

Lemma 2.2 *Suppose that there exist two positive constants a, b with $\sum\limits_{i=1}^{n}(a\alpha_i)^p > bM^+/M_-$, such that*

- (j1) $F(x, t_1, ..., t_n)\geq 0$, *for a.e.* $x\in\Omega\backslash B(x_0, R_1)$ *and all* $(t_1, ..., t_n)\in[0, a]\times\cdots\times[0, a]$;
- (j2) $\sum\limits_{i=1}^{n}(a\alpha_i)^{p_i}|\Omega| \sup\limits_{(x,t_1,...,t_n)\in\Omega\times A(bM^+/M_-)} F(x, t_1, ..., t_n) < b\int\limits_{B(x_0,R_1)} F(x, a, ..., a)dx.$

Then there exist $r > 0$ and $u_{i0} \in W_0^{1,p_i}(\Omega)$, $1\leq i\leq n$, such that

$$\Phi\left(u_{10}, ..., u_{n0}\right) > r$$

and

$$|\Omega|\sup_{(x,t_1,...,t_n)\in\Omega\times A(bM^+/M_-)} F(x, t_1, \dots, t_n) \leq \frac{bM^+}{C}\frac{\int_\Omega F(x, u_{10}, ..., u_{n0})dx}{\Phi(u_{10}, ..., u_{n0})}.$$

Proof. Let

$$w_0(x) = \begin{cases} 0, & x \in \bar{\Omega}\backslash B(x_0, R_2), \\ \dfrac{a}{R_2 - R_1}\left(R_2 - \left\{\sum_{i=1}^{N}\left(x^i - x_0^i\right)\right\}^{1/2}\right), & x \in B(x_0, R_2)\backslash B(x_0, R_1), \\ a, & x \in B(x_0, R_1). \end{cases}$$

and $u_{10}(x) = \cdots = u_{n0}(x) = w_0(x)$. It is obvious to verify $(u_{10}, ..., u_{n0}) \in X$, and in particular, we have

$$\|u_{i0}\|_{p_i}^{p_i} = (R_2^N - R_1^N)\frac{\pi^{N/2}}{\Gamma(1+N/2)}\left(\frac{a}{R_2 - R_1}\right)^{p_i}, \quad 1 \leq i \leq n. \tag{2.4}$$

Hence, it follows from (1.11) and (2.4) that

$$\|u_{i0}\|_{p_i}^{p_i} = \|w_0\|_{p_i}^{p_i} = \frac{(a\alpha_i)^{p_i}}{C}, 1 \leq i \leq n. \tag{2.5}$$

Under condition (M), by a direct computation, one has

$$M_-\left(\sum_{i=1}^{n}\|u_i\|_{p_i}^{p_i}\right) \leq \Phi\left(u_1, \ldots, u_n\right) \leq M^+\left(\sum_{i=1}^{n}\|u_i\|_{p_i}^{p_i}\right). \tag{2.6}$$

Put $r = \dfrac{bM^+}{C}$, and using the assumption of Lemma 2.2

$$\sum_{i=1}^{n}(a\alpha_i)^{p_i} > bM^+/M_-,$$

it follows from (2.5)and (2.6) that

$$\Phi\left(u_{10}, \ldots, u_{n0}\right) \geq M_-\left(\sum_{i=1}^{n}\|u_{i0}\|_{p_i}^{p_i}\right) = \frac{M_-}{C}\sum_{i=1}^{n}(a\alpha_i)^{p_i} > \frac{M_-}{C}\frac{bM^+}{M_-} = r.$$

Since, $0 \leq u_{i0} \leq a$, $1 \leq i \leq n$, for each $x \in \Omega$, condition (j1) ensures that

$$\int_{\Omega\backslash B(x_0,R_2)} F(x, u_{10}, \ldots, u_{n0})dx + \int_{B(x_0,R_2)\backslash B(x_0,R_1)} F(x, u_{10}, \ldots, u_{n0})dx \geq 0.$$

Hence, from condition (j2), we get

$$
\begin{aligned}
&|\Omega| \sup_{(x,t_1,\ldots,t_n)\in\Omega\times A(bM^+/M_-)} F(x,t_1,\ldots,t_n) < \frac{b}{\sum\limits_{i=1}^{n}(a\alpha_1)^p}\int_{B(x_0,R_1)} F(x,a,...,a)dx \\
&= \frac{bM^+}{C}\frac{\int_{B(x_0,R_1)} F(x,a,\ldots,a)dx}{M^+\sum\limits_{i=1}^{n}(a\alpha_1)^p/C} \\
&\le \frac{bM^+}{C}\frac{\int_{\Omega\setminus B(x_0,R_1)} F(x,u_{10},...,u_{n0})dx + \int_{B(x_0,R_1)} F(x,u_{10},...,u_{n0})dx}{M^+\left(\sum\limits_{i=1}^{n}\|u_{i0}\|_{p_i}^{p_i}\right)} \\
&\le \frac{bM^+}{C}\frac{\int_{\Omega} F(x,u_{10},...,u_{n0})dx}{\Psi(u_{10},...,u_{n0})}.
\end{aligned}
$$

Next, we can give the proof of our main result.

Proof of Theorem 1.1. For each $(u_1,\ldots,u_n)\in X$, $1\le i\le n$, assume that

$$
\begin{aligned}
\Phi(u_1,...,u_n) &= \sum_{i=1}^{n}\frac{\widehat{M_i}(||u_i||_{p_i}^{p_i})}{p_i}, \\
\Psi(u_1,...,u_n) &= -\int_{\Omega} F(x,u_1,...,u_n)dx, \\
J(u,v) &= -\int_{\Omega} G(x,u_1,...,u_n)dx.
\end{aligned}
$$

based on conditions of Theorem 1.1, it is easy to know that Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional. Additionally from Lemma 2.2 the Gâteaux derivative of Φ has a continuous inverse on X^*. Ψ and J are continuously Gâteaux differential functionals whose Gâteaux derivatives are compact. Obviously, Φ is bounded on each bounded subset of X. In particular, for each $(u_1,\ldots,u_n),(\xi_1,\ldots,\xi_n)\in X$, we have

$$
\begin{aligned}
\left\langle\Phi'(u_1,...,u_n),(\xi_1,...,\xi_n)\right\rangle &= \sum_{i=1}^{n}\left[M_i\left(\int_{\Omega}|\nabla u_i|^{p_i}\right)\right]^{p_i-1}\int_{\Omega}|\nabla u_i|^{p_i-2}\nabla u_i\nabla\xi_i \\
\left\langle\Psi'(u_1,...,u_n),(\xi_1,...,\xi_n)\right\rangle &= -\sum_{i=1}^{n}\int_{\Omega} F_{u_i}(x,u_1,...,u_n)\xi_i dx, \\
\left\langle J'(u_1,...,u_n),(\xi_1,...,\xi_n)\right\rangle &= -\sum_{i=1}^{n}\int_{\Omega} G_{u_i}(x,u_1,...,u_n)\xi_i dx.
\end{aligned}
$$

Hence, it follows from (1.4) that the weak solutions of problem (1.1) are exactly the solutions of the following equation

$$
\Phi'(u_1,...,u_n)+\lambda\Psi'(u_1,...,u_n)+\mu J'(u_1,...,u_n)=0\ .
$$

Thanks to (j3), for each $\lambda > 0$, one has

$$\lim_{\|(u,v)\|\to+\infty} (\lambda\Phi(u_1,\ldots,u_n) + \mu\Psi(u_1,\ldots,u_n)) = +\infty, \tag{2.7}$$

and so the first condition of Theorem 2.1 holds.

By Lemma 2.2, there exists $(u_{10},...,u_{n0}) \in X$ such that

$$\begin{aligned} \Phi(u_{10},...,u_{n0}) &= \sum_{i=1}^{n} \frac{\widehat{M}_i(||u_{i0}||_{p_i}^{p_i})}{p_i} \\ &\geq M_- \left(\sum_{i=1}^{n} ||u_{i0}||_{p_i}^{p_i} \right) = \frac{M_-}{C} \sum_{i=1}^{n} (a\alpha_i)^{p_i} \\ &> \frac{M_-}{C}\frac{bM^+}{M_-} = \frac{bM^+}{C} > 0 = \Phi\,(0,...,0) \end{aligned} \tag{2.8}$$

and

$$|\Omega| \sup_{(x,t_1,\ldots,t_n)\in\Omega\times A(bM^+/M_-)} F(x,t_1,\ldots,t_n) \leq \frac{bM^+}{C}\frac{\int_\Omega F(x,u_{10},\ldots,u_{n0})dx}{\Phi\,(u_{10},\ldots,u_{n0})}. \tag{2.9}$$

From (1.3), we have

$$\max_{x\in\bar{\Omega}} \{|u_i(x)|^{p_i}\} \leq C\,\|u\|_{p_i}^{p_i}, \quad 1 \leq i \leq n,$$

for each $(u_1,\ldots,u_n) \in X$. One has

$$\max_{x\in\bar{\Omega}} \left\{ \sum_{i=1}^{n} \frac{|u_i\,(x)|^{p_i}}{p_i}, 1 \leq i \leq n \right\} \leq C \left\{ \sum_{i=1}^{n} \frac{\|u\|_{p_i}^{p_i}}{p_i}, 1 \leq i \leq n \right\}, \tag{2.10}$$

for each $(u_1,\ldots,u_n) \in X$.

Suppose that $r = \dfrac{bM^+}{C}$, for each $(u_1,...,u_n) \in X$ such that

$$\Phi(u_1,...,u_n) = \sum_{i=1}^{n} \frac{\widehat{M}_i(||u_i||_{p_i}^{p_i})}{p_i} \leq r.$$

Thanks to (2.10), we get

$$\sum_{i=1}^{n} |u_i(x)|^{p_i} \leq C \sum_{i=1}^{n} \|u_i\|_{p_i}^{p_i} \leq \frac{Cr}{M_-} = \frac{C}{M_-}\frac{bM^+}{C} = \frac{bM^+}{M_-}. \tag{2.11}$$

Then, from (2.9) and (2.11), we obtain

$$\begin{aligned}\sup_{(u_1,\dots,u_n)\in\Phi^{-1}(-\infty,r)}(-\Psi(u_1,\dots,u_n)) &= \sup_{\{(u_1,\dots,u_n)|\Phi(u_1,\dots,u_n)\le r\}}\int_\Omega F(x,u_1,\dots,u_n)dx\\ &\le \sup_{\left\{(u_1,\dots,u_n)|\sum_{i=1}^{n}|u_i(x)|^{p_i}\le bM^+/M_-\right\}}\int_\Omega F(x,u_1,\dots,u_n)dx\\ &\le \int_\Omega \sup_{(t_1,\dots,t_n)\in A(bM^+/M_-)} F(x,t_1,\dots,t_n)dx\\ &\le |\Omega| \sup_{(x,t_1,\dots,t_n)\in\Omega\times A(bM^+/M_-)} F(x,t_1,\dots,t_n)\\ &\le \frac{bM^+}{C}\frac{\int_\Omega F(x,u_{10},\dots,u_{n0})dx}{\Phi(u_{10},\dots,u_{n0})}\\ &= r\frac{-\Psi(u_{10},\dots,u_{n0})}{\Phi(u_{10},\dots,u_{n0})}.\end{aligned}$$

Consequently we have

$$\sup_{\{(u_1,\dots,u_n)|\Phi(u_1,\dots,u_n)\le r\}}(-\Psi(u_1,\dots,u_n)) < r\frac{-\Psi(u_{10},\dots,u_{n0})}{\Phi(u_{10},\dots,u_{n0})}. \tag{2.12}$$

Fix h such that

$$\sup_{\{(u_1,\dots,u_n)|\Phi(u_1,\dots,u_n)\le r\}}(-\Psi(u_1,\dots,u_n)) < h < r\frac{-\Psi\left(u_{10},\dots,u_{n0}\right)}{\Phi\left(u_{10},\dots,u_{n0}\right)},$$

by (2.8), (2.12) and Proposition 2.1, with $(u_{11},\dots,v_{n1}) = (0,\dots,0)$ and $(u_1^*,\dots,u_n^*) = (u_{10},\dots,u_{n0})$, we have

$$\sup_{\lambda\ge0}\inf_{x\in X}\left(\Phi\left(x\right)+\lambda\left(h+\Psi\left(x\right)\right)\right) < \inf_{x\in X}\sup_{\lambda\ge0}\left(\Phi\left(x\right)+\lambda\left(h+\Psi\left(x\right)\right)\right), \tag{2.13}$$

and so the condition (2.1) of Theorem 2.1 holds.

Now, all the conditions of Theorem 2.1 hold. Hence, applying Theorem 2.1, our conclusion is obtained.

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