PRIME SUBMODULES IN EXTENDED BCK-MODULE

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Abstract. In this paper, by considering the notion of BCK-module, we define the concept of extended BCK-module which is a generalization of BCK-module and we state and prove some related results. Specially, we define the notions of prime submodule and torsion free module and we investigate some important results. Finally, we define the concept of radical of any submodule in extended BCK-modules and we characterize the elements of it.

Keywords: BCK-algebra, BCK-module, extended BCK-module, prime submodule of BCK-module.

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1. Introduction

The notion of BCK-algebra was formulated first in 1966 by Imai and Iseki. This notion is originated from two different ways. One of the motivations is based on set theory. Another motivation is from classical and non-classical propositional calculus. The notion of BCK-module was introduced in 1994 [2] as an action of a BCK-algebra over a commutative group by M. Aslam, A.B. Thaheem and H.A.S. Abujaabal. The idea was further explored in 1994 by F. Kôpka and F. Chovanec [8]. The concept of BCK-module was extended by R. A. Borzooei, J. Shohani and M. Jafari in 2011 [4]. Now, we introduce a different extended BCK-module that we can obtain some interesting results by it. Since the notion of prime-submodule is fundamental notion in modules theory, in this paper we introduce and investigate it on BCK-modules and we obtain some results as mentioned in the abstract.

2. Preliminaries

Definition 2.1. [9] A BCK-algebra is a structure $X = (X, *, 0)$ of type $(2, 0)$ such that:

$(BCK1)$ $((x * y) * (x * z)) * (z * y) = 0$,

$(BCK2)$ $(x * (x * y)) * y = 0$,

$(BCK3)$ $x * x = 0$,

$(BCK4)$ $0 * x = 0$,

$(BCK5)$ $x * y = y * x = 0$ implies that $x = y$,for all $x, y, z \in X$.

The relation $x \leq y$ which is defined by $x * y = 0$ is a partial order with 0 as least element. In any BCK-algebra X, for all $x, y, z \in X$, we have

$(BCK6)$ $x * y \leq x$, $(x * y) * z = (x * z) * y$.

Definition 2.2. [9] Let $(X, *, 0)$ be a BCK-algebra. Then

(i) $\emptyset \neq X_0 \subseteq X$ is called to be a *subalgebra* of X, if for any $x, y \in X_0$, $x*y \in X_0$,

(ii) $\emptyset \neq I \subseteq X$ is called an *ideal* of X, if $0 \in I$ and for any $x, y \in X$, $x * y \in I$ and $y \in I$, implies that $x \in I$. Specially, generated ideal by x is defined by $(x] = \{y \in X : y * x = 0\}$, for any $x \in X$,

(iii) X is called *bounded*, if there exists $1 \in X$ such that $x \leq 1$, for any $x \in X$. In this case, we set $Nx = 1 * x$,

(iv) X is said to be *commutative*, if $y * (y * x) = x * (x * y)$, for all $x, y \in X$,

(v) proper ideal I of X, is called *prime ideal* if X is commutative and $a \wedge b \in I$ implies that $a \in I$ or $b \in I$, for any $a, b \in X$,

(iv) X is said to be *implicative* if $x * (y * x) = x$, for all $x, y \in X$.

Note. In a BCK-algebra X, we let $x \wedge y = y * (y * x)$ and in a bounded BCK-algebra X, we let $x \vee y = N(Nx \wedge Ny)$, for all $x, y \in X$. Moreover, in bounded commutative BCK-algebra, $x \wedge y$ is the least upper bound and $x \vee y$ is the grate lower bound of x, y, for any $x, y \in X$ and so $(L, \vee, \wedge)$ is a bounded lattice.

Lemma 2.3. [9] *Let X be a bounded implicative BCK-algebra. Then for all $x, y, z \in X$,*

(i) $x \wedge y = x * Ny$,

(ii) $x * (x \wedge y) = x * y$,

(iii) $x \wedge (y * z) = (x \wedge y) * (x \wedge z)$,

(iv) $(x * y) + (y * x) = x + y$, *where* $x + y = (x * y) \vee (y * x)$,

(v) $(x + y) \wedge z = (x \wedge z) + (y \wedge z)$,

(vi) $x + x = 0$ *and so* $x = -x$,

(vii) $x + 0 = 0 + x = x$.

Let A be an ideal of BCK-algebra X. For any $x, y \in X$, we define $x \sim y$ if and only if $x * y \in A$ and $y * x \in A$. So $\sim$ is an equivalence relation on X. Denote the equivalence class containing x by C_x and $\frac{X}{A} = \{C_x : x \in X\}$. Then $\left(\frac{X}{A}, *, C_0\right)$ is a BCK-algebra (quotient BCK-algebra), where $C_x \star C_y = C_{x*y}$, for all $x, y \in X$. Moreover, the relation " $\leq$ " which is defined by, $C_x \leq C_y$ if and only if $x * y \in A$, is a partial order relation. If X is bounded and commutative, then $\frac{X}{A}$ is bounded and commutative, too. Let $(X, *, 0)$ and $(Y, *', 0)$ be two BCK-algebras. A mapping $f : X \to Y$ is called a *homomorphism* if $f(0) = 0$ and $f(x * y) = f(x) *' f(y)$, for any $x, y \in X$ (see [9]).

Definition 2.4. [1] Let X be a BCK-algebra, M be an abelian group under "+" and $(x, m) \to x.m$ be a mapping of $X \times M \longrightarrow M$ such that,

$(XM1)$ $(x \wedge y).m = x.(y.m)$,

$(XM2)$ $x.(m + n) = x.m + x.n$,

$(XM3)$ $0.m = 0$, for all $x, y \in X$ and $m, n \in M$.

Then M is called a *BCK-module* or briefly X-module. If X is bounded and for any $m \in M$, $1.m = m$, then M is called a *unitary X module*.

Definition 2.5. [1] A map $f : M \to N$, where M and N are X-modules, is an X-homomorphism if the following hold:

(i) $f(m + n) = f(m) + f(n)$, for all $m, n \in M$,

(ii) $f(x.m) = x.f(m)$, for all $m \in M$ and $x \in X$.

Proposition 2.6. [3] *Let M and N be two BCK-modules over commutative BCK-algebra X and $Hom(M, N) = \{f : f \text{ is a homomorphism from } M \text{ into } N\}$. Then $(Hom(M, N), +)$ forms an abelian group where $(f + g)(m) = f(m) + g(m)$, for any $f, g \in Hom(M, N)$ and $m \in M$. Moreover by operation $\bullet : X \times Hom(M, N) \longrightarrow Hom(M, N)$, $Hom(M, N)$ is an X-module, where $x \bullet f(m) = x.f(m)$.*

Theorem 2.7. [4] *Let X be a bounded implicative BCK-algebra. Then $(X, +)$, is an abelian group and X is an X-module, where $x + y = (x * y) \vee (y * x)$.*

Note. From now on, in this paper X is a BCK-algebra and M is an abelian group.

3. Extended BCK-Modules

Definition 3.8. Let operation $. : X \times M \longrightarrow M$ satisfies the following axioms:

$(XM1)$ $(x \wedge y).m = x.(y.m)$,

$(XM2)$ $x.(m + n) = x.m + x.n$,

$(XM3)$ $0.m = 0$,

$(XM4)$ $(x * y).m = x.m - y.m$, where $x * y \neq 0$, for $x \neq y$, for all $x, y \in X$ and $m, n \in M$. Then M is called an *extended BCK-module* or briefly X^E-module. If X is bounded and $1.m = m$, for any $m \in M$, then M is called a *unitary X^E-module.*

Example 3.9. Let X be a bounded implicative BCK-algebra such that "$\leq$" is totally ordered and operations "$+,.$":$X \times X \longrightarrow X$ are defined by, $x + y = (x * y) \vee (y * x)$, $x.y = x \wedge y$, for all $x, y \in X$. Then X is an X^E-module. By Theorem 2.7, it is enough to show that $(x * y).z = x.z - y.z$, for any $x, y, z \in X$, where $x * y \neq 0$ for $x \neq y$. If $x = y$, then the proof is clear. Now, let $x * y \neq 0$, for $x \neq y$. Since $x * y \neq 0$, $x \nleqslant y$ and so $y \leqslant x$ and this means that $y * x = 0$. Therefore,

$$\begin{aligned}
(x * y).z &= (x * y) \wedge z, \\
&= (x * y + 0) \wedge z, \quad \text{by Lemma 2.3(vii)} , \\
&= (x * y + y * x) \wedge z, \quad \text{since } y * x = 0, \\
&= (x + y) \wedge z, \quad \text{by Lemma 2.3(iv)}, \\
&= (x \wedge z) + (y \wedge z), \quad \text{by Lemma 2.3(v)}, \\
&= x.z + y.z, \\
&= x.z - y.z, \quad \text{by Lemma 2.3(vi)}.
\end{aligned}$$

Example 3.10. (i) Let X be a bounded commutative BCK-algebra such that $(X, .)$ be an X^E-module and A be an ideal of X. Then $\left(\dfrac{X}{A}, +'\right)$ is an abelian group, where $C_x +' C_y = C_{x+y}$ and $x + y = x * y \vee y * x$, for any $x, y \in X$. Moreover, if operation $\bullet : X \times \dfrac{X}{A} \longrightarrow \dfrac{X}{A}$ is defined by $x \bullet C_y = C_{x.y}$, for any $x, y \in X$, then $\dfrac{X}{A}$ is an X^E-module.

(ii) Let $X = \{0, x\}$ and operation "$*$" on X is defined by $0*x = 0*0 = x*x = 0$ and $x*0 = x$. Then $(X, *, 0)$ is a BCK-algebra. Now, let operation $. : X \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined by $x.n = n$ and $0.n = 0$, for any $n \in \mathbb{Z}$. We claim that $\mathbb{Z}$ is an X^E-module. It is clear that $(x \wedge 0).n = 0.n = 0$, $x.(0.n) = x.0 = 0$ and so $(x \wedge 0).n = x.(0.n)$. Similarly $(x \wedge x).n = x.(x.n)$ and $(0 \wedge x).n = 0.(x.n)$. Then $(XM1)$ holds. The proof of $(XM2)$ and $(XM3)$ is clear. Moreover, since $(x * 0).n = n = x.n - 0.n$ and $(x * x).n = 0 = x.n - x.n$, $(XM4)$ holds.

(iii) It is easy to see that BCK-algebra $(X, *, 0)$ in (ii) is bounded with unit x. Moreover, $(X, +)$ is an abelian group, where $a + b = (a * b) \vee (b * a)$, for any $a, b \in X$. Now, let operation $. : X \times X \longrightarrow X$ is defined by $a.b = a \wedge b$, for any $a, b \in X$. Then $(X, +)$ is an X^E-module.

(iv) Let $X = \{0, a, b, 1\}$ and operation "$*$" on X is defined by

$*$	0	a	b	1
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
1	1	b	a	0

Then $(X, *, 0)$ is a bounded BCK-algebra. Let $M = \{0, a\} \subseteq X$. Then $(M, +)$ is an abelian group, where $x + y = (x * y) \vee (y * x)$, for any $x, y \in M$. We define the operation $. : X \times M \to M$ by

$$x.y = \begin{cases} a, & \text{if } x = b \text{ or } 1 \text{ and } y = a \\ 0, & \text{otherwise} \end{cases}$$

Then M is an X^E-module.

(v) Let $(X, *, 0)$ be a bounded BCK-algebra with unit 1, $1 \neq a \in X$ and $1 * a = 1$ or a. Now, if $Y = \{0, a, 1\}$, then Y is a subalgebra of X and so it is a BCK-algebra. Moreover, let $M = \{0, 1\} \subseteq X$. Then $(M, +)$ is an abelian group, where $x + y = (x * y) \vee (y * x)$, for any $x, y \in M$. Now, let the operation $. : Y \times M \to M$ is defined by $y.m = y \wedge m$ for any $y \in Y$ and $m \in M$. Then M is a Y^E-module.

(vi) Let $X = \{0, 1, 2, 3, 4\}$ and the operation " $*$ " is defined by

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

Then $(X, *, 0)$ is a bounded BCK-algebra. Let $Y = \{0, 1, 4\}$ and $M = \{0, 2, 3, 4\}$. It is clear that Y is a subalgebra of X and so is BCK-algebra. It is easy to show that $(M, +)$ is an abelian group, where $x + y = (x * y) \vee (y * x)$, for any $x, y \in M$. Now, we define the operation $. : Y \times M \to M$ by $y.m = y \wedge m$, for any $y \in Y$ and $m \in M$. Then M is a Y^E-module.

(vii) Let $X = \{P, \{2\}, \{1, 2\}\}$ be a subset of BCK-algebra [7, Example 2.8]. Then it is easy to see that $(X, \odot, P)$ is a BCK-algebra. If operation $. : X \times \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined by $\{2\}.n = n$ and $\{1, 2\}.n = P.n = 0$, for any $n \in \mathbb{Z}$, then $\mathbb{Z}$ is an X^E-module.

Theorem 3.11. *Every X^E-module is an X-module.*

Proof. The proof is clear. ∎

Example 3.12. Let X be a nonempty set. Then $(\mathcal{P}(X), -)$ is a bounded implicative BCK-algebra and $\mathbb{Z}$ is a $\mathcal{P}(X)$-module with operation $. : \mathcal{P}(X) \times \mathbb{Z} \longrightarrow \mathbb{Z}$ such that $A.n = \mu(A)n$, for any $A \subseteq X$, where for $a \in X$,

$$\mu(A) = \begin{cases} 0, & \text{if } a \notin A \\ 1, & \text{if } a \in A \end{cases}$$

But $\mathbb{Z}$ is not a $\mathcal{P}(X)^E$-module. Since for $A, B \in \mathcal{P}(X)$ such that $a \notin A, a \in B$, we have

$$(A - B).n = \mu(A - B)n = 0 \neq -n = 0 - n = A.n - B.n$$

and so $(XM4)$ is not true.

Definition 3.13. A map $f : M \to N$, where M and N are X^E-modules, is called an X^E-*homomorphism*, if the following hold:

(i) $f(m + n) = f(m) + f(n)$,

(ii) $f(x.m) = x.f(m)$, for all $m, n \in M$ and $x \in X$.

Proposition 3.14. *Let M, N be two X^E-modules and*

$$Hom(M, N) = \{f : f : M \to N \ is\ an\ X^E - homomorphism\}.$$

Then $(Hom(M, N)$ is an X^E-module by the operation which is defined in Proposition 2.6.

Proof. By Proposition 2.6, it is enough to show that $(x * y) \bullet f(m) = x \bullet f(m) - y \bullet f(m)$, for any $x, y \in X$, where $x * y \neq 0$, and $x \neq y$. Now, since N is an X^E-module, we have

$$(x * y) \bullet f(m) = (x * y).f(m) = x.f(m) - y.f(m) = x \bullet f(m) - y \bullet f(m). \quad \blacksquare$$

Theorem 3.15. *Let X be a bounded implicative BCK-algebra. Then, by the assumption of Example* 3.9, $\left(\sum_{i \in I} X, +'\right)$ *is an abelian group, where* $\{x_i\}_{i \in I} +' \{y_i\}_{i \in I} = \{x_i + y_i\}_{i \in I}$, *for any* $\{x_i\}_{i \in I}, \{y_i\}_{i \in I} \in \sum_{i \in I} X$. *Moreover, if the operation* $. : X \times \sum_{i \in I} X \longrightarrow \sum_{i \in I} X$ *is defined by* $x.\{x_i\} = \{x \wedge x_i\}$, *for any* $x, x_i \in X$, $i \in \mathbb{N}$, *then* $\sum_{i \in I} X$ *is an X^E-module.*

Proof. Since by Theorem 2.7, $(X, +)$ is an abelian group, then it is clear that $\left(\sum_{i \in I} X, +'\right)$ is an abelian group.

Now, for any $x, y, x_i, y_i \in X$ and $i \in \mathbb{N}$, we have:

$(XM1)$: $(x \wedge y).\{x_i\} = \{(x \wedge y) \wedge x_i\} = \{x \wedge (y \wedge x_i)\} = x.\{y \wedge x_i\} = x.(y.\{x_i\})$.
$(XM2)$: By Lemma 2.3(v),

$$\begin{aligned} x.(\{x_i\} +' \{y_i\}) &= x.\{x_i + y_i\} = \{x \wedge (x_i + y_i)\} = \{x \wedge x_i + x \wedge y_i\} \\ &= \{x \wedge x_i\} +' \{x \wedge y_i\} = x.\{x_i\} +' x.\{y_i\}. \end{aligned}$$

$(XM3)$: $0.\{x_i\} = \{0 \wedge x_i\} = \{0\}$.
$(XM4)$: Let $x * y \neq 0$ for $x \neq y$. Then by Lemma 2.3(v) and (vi),

$$\begin{aligned} x.\{x_i\} -' y.\{x_i\} &= \{x \wedge x_i\} -' \{y \wedge x_i\} \\ &= \{x \wedge x_i\} +' \{y \wedge x_i\} \\ &= \{x \wedge x_i + y \wedge x_i\} \\ &= \{(x + y) \wedge x_i\} \\ &= (x + y).\{x_i\} \\ &= (x * y + y * x).\{x_i\} \\ &= (x * y + 0).\{x_i\} = (x * y).\{x_i\}. \end{aligned}$$

■

Theorem 3.16. *Let X be BCK-algebra in Theorem 3.15, and A be an ideal in X. Then $\left(\sum_{i \in I} \frac{X}{A}, \bar{+}\right)$ is an abelian group, where $\{C_{x_i}\}\bar{+}\{C_{y_i}\} = \{C_{x_i + y_i}\}$ and $x_i + y_i = x_i * y_i \vee y_i * x_i$, for any $x_i, y_i \in X$ and $i \in I$. Moreover, if we define $\bullet : X \times \sum_{i \in I} \frac{X}{A} \longrightarrow \sum_{i \in I} \frac{X}{A}$ by $x \bullet \{C_{x_i}\} = \{C_{x \wedge x_i}\}$, then $\left(\sum_{i \in I} \frac{X}{A}\right)$ is an X^E-module.*

Proof. It is easy to show that $\left(\sum_{i \in I} \frac{X}{A}, \bar{+}\right)$ is an abelian group and $\sum_{i \in I} \frac{X}{A}$ is an X^E-module. ■

Theorem 3.17. *Let $(X, *)$ and $(Y, \star)$ be two BCK-algebras, M be a Y^E-module and $\Phi : X \longrightarrow Y$ be a BCK-homomorphism such that $x \neq 0$ implies that $\phi(x) \neq 0$, for any $x \in X$. If operation $\bullet : X \times M \longrightarrow M$ is defined by $x \bullet m = \phi(x).m$, for any $x \in X$ and $m \in M$, then M is an X^E-module.*

Proof. Let M be a Y^E-module and $\Phi : X \longrightarrow Y$ be a BCK-homomorphism such that $x \neq 0$ implies that $\phi(x) \neq 0$, for any $x \in X$. Then for any $x, y \in X$ and $m, n \in M$, we have:
$(XM1)_X$: By $(XM1)_Y$, we have

$$\begin{aligned} (x \wedge y) \bullet m &= \phi(x \wedge y).m = \phi(y * (y * x)).m = (\phi(y) \star (\phi(y) \star \phi(x))).m, \\ &= (\phi(x) \wedge \phi(y)).m = \phi(x).(\phi(y).m) = x \bullet (y \bullet m). \end{aligned}$$

$(XM2)_X$: By $(XM2)_Y$, we have

$$x \bullet (m + n) = \phi(x).(m + n) = \phi(x).m + \phi(x).n = x \bullet m + x \bullet n$$

$(XM3)_X$: $0 \bullet m = \phi(0).m = 0.m = 0$
$(XM4)_X$: By $(XM4)_Y$, where $x * y \neq 0$, for $x \neq y$ we have

$$(x * y) \bullet m = \phi(x * y).m = (\phi(x) \star \phi(y)).m = \phi(x).m - \phi(y).m = x \bullet m - y \bullet m.$$ ■

Theorem 3.18. *Let X be a bounded commutative BCK-algebra, $(X,+)$ be an X^E-module and A be an ideal of X. Then $\frac{X}{A}$ is an X^E-module.*

Proof. Let X be a bounded commutative BCK-algebra, $(X,+)$ be an X^E-module and A be an ideal of X. It is easy to show that $\left(\dfrac{X}{A},+'\right)$ is an abelian group, where $C_x +' C_y = C_{x+y}$ and $x+y=(x*y)\vee(y*x)$, for any $x,y\in X$. Let operation $\bullet : X\times \dfrac{X}{A}\longrightarrow \dfrac{X}{A}$ is defined by $x\bullet C_y = C_{x.y}$, for any $x,y\in X$. Then we show that $\dfrac{X}{A}$ is an X^E-module. For $x,x',y\in X$,

$(XM1)_{\frac{X}{A}}$: By $(XM1)$, $(x\wedge x')\bullet C_y = C_{(x\wedge x').y} = C_{x.(x'.y)} = x\bullet(x'\bullet C_y)$

$(XM2)_{\frac{X}{A}}$: By $(XM2)$,

$$x\bullet(C_x+'C_y) = x\bullet C_{y+y'} = C_{x.(y+y')} = C_{x.y+x.y'} = C_{x.y}+'C_{x.y'} = x\bullet C_y +' x\bullet C_{y'}$$

$(XM3)_{\frac{X}{A}}$: By $(XM3)$, $0\bullet C_x = C_0$

$(XM4)_{\frac{X}{A}}$: Let $x*y\neq 0$, for $x\neq y$. By $(XM4)$,

$$(x*y)\bullet C_{y'} = C_{(x*y).y'} = C_{x.y'-y.y'} = C_{x.y'} -' C_{y.y'} = x\bullet C_{y'} -' y\bullet C_{y'}. \qquad \blacksquare$$

3. Prime submodules in X^E-modules

Definition 3.1. A subgroup N of X^E-module M is a *submodule* of M if for any $x\in X$ and any $n\in N$, $x.n\in N$.

Example 3.2. (i) By considering the Example 3.10 (ii), $2\mathbb{Z}$ is a submodule of $\mathbb{Z}$.

(ii) Let X be a bounded implicative BCK-algebra with the assumption of Example 3.9, and $M_r=\{x\in X : x\leq r\}$, where $r\in X$. Then M_r is a submodule of X. First we show that M is a subgroup of X. Let $m,n\in M_r$. By assumption, $m*n=0$ or $n*m=0$. W.L.G, $n*m=0$. Hence, by Lemma 2.3(vi), $m-n=m+n=(m*n)\vee(n*m)=(m*n)\vee 0=m*n$. On the other hand by $(BCK6)$, $m*n\leq m$ and $m\leq r$. Hence $m*n\leq r$ and so $m-n\in M_r$. It means that M_r is a subgroup of X. Now, we will show that $x.m\in M_r$, for any $x\in X$ and $m\in M_r$. By $(BCK4)$ and $(BCK6)$, we have

$$(x.m)*r=(x\wedge m)*r=(m*(m*x))*r=(m*r)*(m*x)=0*(m*x)=0$$

Hence, $x.m\leq r$ and so $x.m\in M_r$. Therefore, M_r is a submodule of X.

Lemma 3.3. *Let M be an X^E-module and N be a submodule of M. Then $\dfrac{M}{N}$ is an X^E-module.*

Proof. Let N be a submodule of M and operation $\bullet : X \times \frac{M}{N} \longrightarrow \frac{M}{N}$ is defined by $x \bullet (m+N) = x.m+N$, for any $x \in X$ and $m \in M$. Let $x = y$ and $m+N = m'+N$. Then $m - m' \in N$. Since N is a submodule of M, $x.(m - m') = x.m - x.m' \in N$ and so $x \bullet m + N = x \bullet m' + N$. It means that " $\bullet$ " is well defined. For any $x, y \in X$ and $m, m' \in M$,

$(XM1)_{\frac{M}{N}}$: By $(XM1)$,

$$(x \wedge y) \bullet (m+N) = (x \wedge y).m+N = x.(y.m)+N = x \bullet (y.m+N) = x \bullet (y \bullet (m+N))$$

$(XM2)_{\frac{M}{N}}$: By $(XM2)$,

$$\begin{aligned} x \bullet (m + N + m' + N) &= x.(m + m') + N = x.m + x.m' + N \\ &= x.m + N + x.m' + N = x \bullet (m + N) + x \bullet (m' + N) \end{aligned}$$

$(XM3)_{\frac{M}{N}}$: By $(XM3)$, $0 \bullet (m + N) = 0.m + N = N$

$(XM4)_{\frac{M}{N}}$: Let $x * y \neq 0$, for $x \neq y$. By $(XM4)$,

$$\begin{aligned} (x * y) \bullet (m + N) &= (x * y).m + N = (x.m - y.m) + N = x.m + N - y.m + N \\ &= x \bullet (m + N) - y \bullet (m + N). \end{aligned}$$ ∎

Theorem 3.4. *Let M, M' be X^E-modules, $\phi : M \longrightarrow M'$ be an X^E-homomorphism and N be a submodule of M such that $\phi(N) = 0$. Then there exists an X^E-homomorphism from $\frac{M}{N}$ to M'.*

Proof. We define $\bar{\phi} : \frac{M}{N} \longrightarrow M'$ by $\bar{\phi}(m + N) = \phi(m)$. It is easy to show that $\bar{\phi}$ is well defined and it is an X^E-homomorphism. ∎

Theorem 3.5. *Let M, M' be X^E-modules and $\phi : M \longrightarrow M'$ be an X^E-homomorphism. Then*

(i) *$Ker\phi$ and $Img\phi$ are submodules of M and M', respectively,*

(ii) $\frac{M}{Ker\phi} \simeq Img\phi$.

Proof. (i) The proof is clear.

(ii) We know that, $\phi : M \longrightarrow Img\phi$ is an epimorphism. Now, in Theorem 3.4, it is enough to consider $N = Ker\phi$. ∎

Theorem 3.6. *Let M be an X^E-module and N, K are submodules of M. Then*

(i) *$N + K = \{n + k : n \in N, k \in K\}$ and $N \cap K$ are X^E-modules,*

(ii) $\frac{K}{N \cap K} \simeq \frac{N + K}{N}$,

(iii) $\dfrac{K}{N}$ *is a submodule of* $\frac{M}{N}$ *and* $\dfrac{\frac{M}{N}}{\frac{K}{N}} \simeq \dfrac{M}{K}$, *where* $N \subseteq K$.

Proof. (i) $N + K$ is an X-module (*see*[3]). Now, let $x * y \neq 0$, where $x \neq y$, for any $x, y \in X$ and $n + k \in N + K$. Then,

$$(x*y).(n+k) = (x*y).n + (x*y).k = x.n - y.n + x.k - y.k = x.(n+k) - y.(n+k)$$

and so we have $(XM4)_{N+K}$. Therefore, $N+K$ is an X^E-module. Moreover, $N \cap K$ is an X-module (see [3]) and it is not difficult to verify the condition $(XM4)_{N\cap K}$. Hence $N \cap K$ is an X^E-module, too.

(ii), (iii) The proofs are easy. ∎

Theorem 3.7. *Let X be a bounded commutative BCK-algebra such that $x*y \neq x$, where $x \neq y$ for $x, y \neq 0$, M be an X^E-module and K be a proper submodule of M. Then $(K : M) = \{x \in X : x.M \subseteq K\}$ is a prime ideal of X.*

Proof. First, we show that $(K : M)$ is an ideal of X. If $(K : M) = X$, then $1.M \subseteq K$ and so $M \subseteq K$, which is a contradiction. Since K is a subgroup of M, $0.m = 0 \in K$, for any $m \in M$ and so $0 \in (K : M)$. Now, for any $x, y \in X$, let $x * y \in (K : M) = \{x \in X : x.M \subseteq K\}$ and $y \in (K : M)$. Then $(x * y).m \in K$ and $y.m \in K$, for any $m \in M$. If $x = y$ or $x = 0$, then it is clear that $x.m \in M$. So let $x \neq y$. If $x * y = 0$, then $x * (x * y) = x$ and so $x \wedge y = x$. Since $y.m \in K$, $x.(y.m) \in K$, for any $m \in M$ and K is a submodule of M and so $x.m = (x \wedge y).m \in K$. If $x * y \neq 0$, then by $(XM4)$, $x.m - y.m = (x * y)m \in K$. Since $(K, +)$ is a subgroup of M and $y.m \in K$, we have $x.m \in K$. Therefore, $(K : M)$ is an ideal of X.

Now, we prove that $(K : M)$ is prime. Let $x \wedge y \in (K : M)$, for $x, y \in X$. Then for any $m \in M$, $(x \wedge y).m \in K$ and so $(y * (y * x)).m \in K$. Now if $x = y$, then $x.m = (x * 0).m = (x * (x * x)).m = (x \wedge y).m \in K$. If $x = 0$ or $y = 0$, then it is clear that $x \in (K : M)$ or $y \in (K : M)$. If $x \neq y$, $x, y \neq 0$ and $x * y = 0$, then $x.m = (x * 0).m = (x * (x * y)).m = (y \wedge x).m = (x \wedge y).m \in K$, for any $m \in M$. If $x \neq y$, $x, y \neq 0$, $x * y \neq 0$, $x \neq x * y$ and $x * (x * y) \neq 0$. Then by $(XM4)$, $y.m = x.m - (x.m - y.m) = x.m - (x*y).m = x*(x*y).m = (y \wedge x).m = (x \wedge y).m \in K$, for any $m \in M$. Finally, if $x \neq y$, $x, y \neq 0$, $x * y \neq 0$, $x \neq x * y$ and $x * (x * y) = 0$, then by $(BCK6)$, we have $(x * y) * x = 0$ and so $x = x * y$, which is a contradiction. Therefore, $(K : M)$ is a prime ideal of X. ∎

Proposition 3.8. *Let M be an X^E-module. If for any $x, y \in X$, $x \neq y$ implies that $x * y \neq 0$, then $Ann_X(M) = \{x \in X : x.m = 0, \forall m \in M\}$ is an ideal of X.*

Proof. It is clear that $0 \in Ann_X(M)$. Now, let $x * y, y \in Ann_X(M)$ and $x \neq y$, for any $x, y \in X$. If $x = 0$, then it is clear that $x \in Ann_X(M)$. Now, let $x \neq 0$. Then by $(XM4)$, $x.m = x.m - 0 = x.m - y.m = (x * y).m = 0$, for any $m \in M$, and so $x \in Ann_X(M)$. Therefore, $Ann_X(M)$ is an ideal of X. ∎

Theorem 3.9. *Let M be an X^E-module and I be an ideal of X such that $I \subseteq Ann_X(M)$. If the operation $\bullet : X/I \times M \longrightarrow M$ is defined by $c_x \bullet m = x.m$, for any $x, y \in X$ and $m \in M$, then M is an $(X/I)^E$-module.*

Proof. Let $\bullet : X/I \times M \longrightarrow M$ is defined by $c_x \bullet m = x.m$, for any $x \in X$ and $m \in M$. First we prove that $\bullet$ is well defined. Let $c_x = c_y$, $m = n$ and $x \neq y$, for all $x, y \in X$ and $m, n \in M$. Then $x * y \in I$ and $y * x \in I$. If $x * y = y * x = 0$, then by $(BCK5)$, $x = y$, which is a contradiction. If $x * y \neq 0$ or $y * x \neq 0$, since $I \subseteq Ann_X(M)$, by $(XM4)$, $0 = (x * y).m = x.m - y.m$ or $0 = (y * x).m = y.m - x.m$ and so $x.m = y.m$. Now, since $m = n$, $x.m = y.n$. Hence, $\bullet$ is well defined. Now, we will show that M is an $(X/I)^E$-module.

$(XM1)_{X/I}$: We have $c_x \wedge c_y = c_y \star (c_y \star c_x) = c_{y*(y*x)}$, then by $(XM1)_X$,

$$(c_x \wedge c_y) \bullet m = (y * (y * x).m = (x \wedge y).m = x.(y.m) = c_x \bullet (c_y \bullet m)$$

$(XM2)_{X/I}$: By $(XM2)_X$,

$$c_x \bullet (m + n) = x.(m + n) = x.m + x.n = c_x \bullet m + c_x \bullet n.$$

$(XM3)_{X/I}$: $c_0 \bullet m = 0.m = 0$.

$(XM4)_{X/I}$: Let $c_x \star c_y \neq c_0$, for $c_x \neq c_y$. Hence $c_{x*y} \neq c_0$. Since $0 * (x * y) = 0 \in I$, $(x * y) * 0 = x * y \notin I$ and so $x * y \neq 0$. Therefore, by $(XM4)_X$,

$$(c_x \star c_y) \bullet m = c_{x*y} \bullet m = (x * y).m = x.m - y.m = c_x \bullet m - c_y \bullet m. \quad \blacksquare$$

Notion. For X^E-module M, $Y \subseteq X$ and submodule N of M, we consider

$$Y.M = YM = \{x.m : x \in Y, m \in M\}.$$

Lemma 3.10. *Let X be a commutative BCK-algebra, M be an X^E-module, N be a submodule of M and I be an ideal of X. Then*

$$I.M + N = \left\{ \sum_{i=1}^{n} t_i.m_i + n : t_i \in I,\ m_i \in M,\ n \in N \right\}$$

is a submodule of M.

Proof. Let N be a submodule of M and I be an ideal of X. It is clear that " + " is an associative operation in $I.M + N$ and $0 \in I.M + N$. Moreover, by $(XM4)$,

$$\sum_{i=1}^{n} t_i.m_i + n - \left(\sum_{i=1}^{n} t_i.m_i + n \right) = \sum_{i=1}^{n} (t_i * t_i).m_i = 0,$$

for any $\sum_{i=1}^{n} t_i.m_i + n \in I.M + N$. Hence, every element in $I.M + N$ has an inverse element and so $I.M + N$ is a subgroup of M. Now, by $(XM1)$ and $(XM2)$,

$$\begin{aligned} x.\left(\sum_{i=1}^{n} t_i.m_i + n \right) &= \sum_{i=1}^{n} x.(t_i.m_i) + x.n = \sum_{i=1}^{n} (x \wedge t_i).m_i + x.n \\ &= \sum_{i=1}^{n} (t_i \wedge x).m_i + x.n = \sum_{i=1}^{n} t_i.(x.m_i) + x.n \in I.M + N, \end{aligned}$$

for any $\sum_{i=1}^{n} t_i.m_i + n \in I.M + N$ and $x \in X$. Therefore, $I.M + N$ is a submodule of M. ∎

Theorem 3.11. *Let X be a bounded BCK-algebra, I be a proper ideal of X and M be an X^E-module. Then M/IM is an $(X/I)^E$-module.*

Proof. Let I be a proper ideal of X and M be an X^E-module. By Lemma 3.10, IM is a submodule of M. Now, we define $\bullet : X/I \times M/IM \longrightarrow M/IM$ by $c_x \bullet m + IM = x.m + IM$, for any $x \in X$ and $m \in M$. Since $I \bullet (M/IM) = \{x \bullet (m + IM) : x \in I, m \in M\} = \{x.m + IM : x \in I, m \in M\} = IM$, then $I \subseteq ann_X(M/IM)$. By Lemma 3.9, " $\bullet$ " is well defined. Now, we show that M/IM is an $(X/I)^E$-module, for any $x, y \in X$ and $m, n \in M$.

$(XM1)_{X/I}$: Since $(c_x \wedge c_y) = c_y \star (c_y \star c_x) = c_{y*(y*x)}$, by $(XM1)_X$,

$$\begin{aligned}(c_x \wedge c_y) \bullet (m + IM) &= c_{y*(y*x)} \bullet (m + IM) = (y * (y * x)).m + IM,\\ &= (x \wedge y).m + IM = x.(y.m) + IM,\\ &= c_x \bullet (y.m + IM) = c_x \bullet (c_y \bullet (m + IM))\end{aligned}$$

$(XM2)_{X/I}$: By $(XM2)_X$,

$$\begin{aligned}c_x \bullet ((m + IM) + (n + IM)) &= c_x \bullet (m + n + IM) = x.(m + n) + IM\\ &= (x.m + x.n) + IM, = x.m + IM + x.n + IM\\ &= c_x \bullet (m + IM) + c_x \bullet (n + IM)\end{aligned}$$

$(XM3)_{X/I}$: $c_0 \bullet (m + IM) = 0.m + IM = 0 + IM = IM = 0_{M/IM}$
$(XM4)_{X/I}$: If $c_x = c_y$, then by $(XM4)_X$,

$$\begin{aligned}(c_x \star c_y) \bullet (m + IM) &= c_{x*x} \bullet (m + IM) = c_0 \bullet (m + IM)\\ &= 0.m + IM = (x * x).m + IM,\\ &= x.m + IM - x.m + IM,\\ &= c_x \bullet (m + IM) - c_x \bullet (m + IM)\end{aligned}$$

Now, let $c_x \star c_y \neq 0$ where $c_x \neq c_y$. Then $c_{x*y} \neq c_0$ i.e., $(x * y) * 0 = x * y \notin I$ and so $x * y \neq 0$. Hence, by $(XM4)_X$,

$$\begin{aligned}(c_x \star c_y) \bullet (m + IM) &= c_{x*y} \bullet (m + IM)=(x * y).m + IM=(x.m - y.m) + IM,\\ &= x.m + IM - y.m + IM=c_x \bullet (m + IM) - c_y \bullet (m + IM)\end{aligned}$$

Therefore, M/IM is an $(X/I)^E$-module. ∎

Definition 3.12. Let M be an X^E-module and N be a submodule of M. Then N is called a *prime* submodule of M, if $N \neq M$ and for any $x \in X$, $x.m \in N$ implies that $m \in N$ or $x \in (N : M)$.

Example 3.13. By considering the Example 3.10 (ii), $2\mathbb{Z}$ is a prime submodule of $\mathbb{Z}$. It is clear that $2\mathbb{Z}$ is a subgroup of $\mathbb{Z}$. Now, let $x.n \in 2\mathbb{Z}$. If $x \neq 0$, $x.n = n$, then $n \in 2\mathbb{Z}$. If $x = 0$, then $x.n = 0.n = 0$ and so $0 \in (2\mathbb{Z} : \mathbb{Z})$. Hence, $2\mathbb{Z}$ is a prime submodule of $\mathbb{Z}$.

Theorem 3.14. *Let X be a commutative BCK-algebra, M be an X^E-module and $N \neq M$ be a submodule of M. Then N is a prime submodule of M if and only if for any ideal I in X and for any submodule D of M, $ID \subseteq N$ implies that $I \subseteq (N : M)$ or $D \subseteq N$.*

Proof. ($\Rightarrow$) Let N be a prime submodule of M, I be an ideal in X and D be a submodule of M such that $ID \subseteq N$. We show that $I \subseteq (N : M)$ or $D \subseteq N$. Let $I \nsubseteq (N : M)$ and $D \nsubseteq N$. Then there exist $x \in X$ and $d \in D$ such that $x.M \nsubseteq N$ and $d \notin N$. On the other hand, $ID \subseteq N$ implies that $x.d \in N$. Since N is a prime submodule of M, $x.M \subseteq N$, which is a contradiction.

($\Leftarrow$) Let $x \in X$ and $m \in M$ such that $x.m \in N$ and $m \notin N$. Let $I = (x] = \{y \in X : y * x = 0\}$ and $D = \prec m \succ = \{y'.m : y' \in X\}$. For any $y \in I$, we have

$$y.m = (y * 0).m = y * (y * x).m = (x \wedge y).m = (y \wedge x).m = y.(x.m) \in N$$

So $ID = \{y.(y'.m) : y, y' \in X\} = \{y'.(y.m) : y, y' \in X\} \subseteq N$ and so $I \subseteq (N : M)$ or $D \subseteq N$. Since $m \notin N$, $I \subseteq (N : M)$ and this implies that $x.M \subseteq N$. Therefore, N is a prime submodule of M. ∎

Proposition 3.15. *Let M be an X^E-module and N be a submodule of M. Then P is a prime submodule of M if and only if $\frac{P}{N}$ is a prime submodule of $\frac{M}{N}$, where $N \subseteq P$.*

Proof. By Lemma 3.3, the proof is easy. ∎

Definition 3.16. Let M be an X^E-module. M is called *torsion* free if $x.m = 0$ implies that $m = 0$ or $x = 0$, for any $x \in X$ and $m \in M$.

Example 3.17. (i) In Example 3.10(ii), $\mathbb{Z}$ is a torsion free.

(ii) In Example 3.10(iv), M is not a torsion free. Because, $a.a = 0$ but $a \neq 0$.

Theorem 3.18. *Let X be bounded, M be a unitary X^E-module and K be a submodule of M. Then K is a prime submodule of M if and only if $P = (K : M)$ is a prime ideal of X and $\frac{M}{K}$ is a torsion free $\left(\frac{X}{P}\right)^E$-module, where $\left(\frac{X}{P}, \star, P\right)$ is a quotient BCK-algebra.*

Proof. ($\Rightarrow$) Let K be a prime submodule of M. By Theorem 3.7, $P = (K : M)$ is an ideal of X. If $X = (K : M)$, then $1 \in P$ and so $M = K$, which is a contradiction. Now, let $x \wedge y \in P$, for any $x, y \in X$. Then for any $m \in M$, $(x \wedge y).m \in K$ and so by $(XM1)$, $x.(y.m) \in K$. Since K is a prime submodule of M, we have $y.m \in K$ or $x \in (K : M)$. It means that $y \in (K : M)$ or

$x \in (K : M)$. Hence, $(K : M)$ is a prime ideal. Now, we show that $\frac{M}{K}$ is a torsion free $\left(\frac{X}{P}\right)^E$-module. Let the operation $\bullet : \frac{X}{P} \times \frac{M}{K} \longrightarrow \frac{M}{K}$ is defined by $c_x \bullet (m+K) = x.m + K$, for any $x \in X$, $m \in M$. Similar to the proof of Theorem 3.9, $\bullet$ is well defined. Finally, for any $c_x \in \frac{X}{P}$ and $m+K \in \frac{M}{K}$, we will show that, $c_x \bullet (m+K) = K$ implies that $c_x = c_0$ or $m+K = K$. Let $c_x \bullet (m+K) = K$, for any $x \in X$ and $m \in M$. Then $x.m+K = K$ and so $x.m \in K$. Since K is a prime submodule of M, $m \in K$ or $x \in (K : M)$. If $m \in K$, then $m + K = K$. Now, if $x \in (K : M) = P$, then $c_x = c_0 = P$ (because $x * 0 = x \in P$ and $0 * x = 0 \in P$). Therefore, $\frac{M}{K}$ is a torsion free.

($\Leftarrow$) Let P be a prime ideal in X and $\frac{M}{K}$ is a torsion free $\left(\frac{X}{P}\right)^E$-module. First we show that $K \subsetneqq M$. Since, if $K = M$, $P = (K : M) = (M : M) = X$, which is a contradiction. Now, let $x.m \in K$, for any $x \in X$, $m \in M$. Hence $x.m + K = K$ and so $c_x \bullet (m + K) = K$. Since $\frac{M}{K}$ is torsion free, $c_x = c_0 = P$ or $m + K = K$. This means that $x \in P$ or $m \in K$. Therefore, K is a prime submodule of M. ∎

Theorem 3.19. *Let X be a bounded commutative BCK-algebra, M be a unitary X^E-module, N be a submodule of M and P be a prime ideal of X. Then $K(N,P) = \{m \in M : c.m \in P.M + N, \exists c \in X - P\}$ is a submodule of M and $P.M + N \subseteq K(N,P)$.*

Proof. First, we show that $K(N,P)$ is a subgroup of M. Let $m, n \in K(N,P)$. Then there exists $c, c' \in X - P$ such that $c.m, c'.n \in P.M + N$. Let $t = c \wedge c'$. Then

$$\begin{aligned}
t.(m-n) &= (c \wedge c').(m-n), \\
&= c.(c'.(m-n)) \quad \text{by } (XM1), \\
&= c.(c'.m - c'.n), \quad \text{by } (XM2), \\
&= c.(c'.m) - c.(c'.n) \quad \text{by } (XM2), \\
&= (c \wedge c').m - c.(c'.n), \quad \text{by } (XM1), \\
&= (c' \wedge c).m - c.(c'.n), \\
&= c'.(c.m) - c.(c'.n) \in P.M + N, \quad \text{by Lemma 3.10.}
\end{aligned}$$

and so $m - n \in K(N,P)$, which means that $K(N,P)$ is a subgroup. Now, let $x \in X$ and $m \in K(N,P)$. Since $m \in K(N,P)$, there exists $c \in X - P$ such that $c.m \in P.M + N$. Now, by Lemma 3.10,

$$c.(x.m) = (c \wedge x).m = (x \wedge c).m = x.(c.m) \in P.M + N$$

Hence, $x.m \in K(N,P)$ and so $K(N,P)$ is a submodule of M. Finally, for any $m \in P.M+N$, if we let $c = 1$ then $c.m = 1.m = m \in P.M+N$, then $m \in K(N,P)$

(note that $1 \in X - P$, otherwise for any $x \in X$, $x * 1 = 0 \in P$ and $1 \in P$ results in $x \in P$ i.e., $X = P$, which is impossible). Therefore, $P.M + N \subseteq K(N, P)$. ∎

Theorem 3.20. *Let X be a bounded commutative BCK-algebra, M be a unitary X^E-module, N be a submodule of M and P be a prime ideal of X. Then $K(N, P) = M$ or $K(N, P)$ is a prime submodule of M such that*

$$P = (K(N, P) : M).$$

Proof. Let $K(N, P) \neq M$. We will show that $K(N, P)$ is a prime submodule of M and $P = (K(N, P) : M)$. By Theorem 3.19, $K(N, P)$ is a submodule of M. Let $x.m \in K(N, P)$, for any $x \in X$, $m \in M$. Then there exists $c \in X - P$ such that $c.(x.m) \in P.M + N$. We will show that $m \in K(N, P)$ or $x \in (K(N, P) : M)$. If $x \in P$, then $x.M \subseteq P.M + N \subseteq K(N, P)$ and so $x.M \subseteq K(N, P)$. Hence $x \in (K(N, P) : M)$. If $x \notin P$, then $x \in X - P$. Since P is a prime ideal of X, $c \wedge x \in X - P$. because, if $c \wedge x \in P$, then $c \in P$ or $x \in P$, which is a contradiction. On the other hand, $c.(x.m) \in P.M + N$ and so $(c \wedge x).m \in P.M + N$. Hence, $m \in K(N, P)$. Therefore, $K(N, P)$ is a prime submodule of M. Now, we will prove that $P = (K(N, P) : M)$. Let $p \in P$. Then for any $m \in M$, $p.m \in P.M + N$. Let $c = 1$. Then $c.(p.m) \in P.M + N$ and so $p.m \in K(N, P)$, which implies that $P.M \subseteq K(N, P)$. Hence, $P \subseteq (K(N, P) : M)$. Now, let $q \in (K(N, P) : M)$ such that $q \notin P$. Since $q.M \subseteq K(N, P)$, $q.m \in K(N, P)$, for any $m \in M$. Hence there exists $c \in X - P$ such that $c.(q.t) \in P.M + N$ and so $(c \wedge q).t \in P.M + N$. Now, since P is prime, $c \wedge q \notin P$ i.e., $c \wedge q \in X - P$ and so $t \in K(N, P)$. Hence, $M = K(N, P)$, which is a contradiction. Then $q \in P$ and so $(K(N, P) : M) \subseteq P$. Therefore, $P = (K(N, P) : M)$. ∎

Definition 3.21. Let M be an X^E-module and N be a submodule of M. The intersection of all prime submodules of M, including N, is called *radical* of N and it is shown by $rad_M(N)$. If there exists no prime submodule of M consisting of N, then we let $rad_M(N) = M$.

Theorem 3.22. *Let X be a bounded commutative BCK-algebra and M be an X^E-module. Then for any submodule N of M,*

$$rad_M(N) = \bigcap\{K(N, P) : P \textit{ is a prime ideal of } X\}.$$

Proof. Let $T = \bigcap\{K(N, P) : P \textit{ is a prime ideal of } X\}$ and $m \in T$. Let L be a prime submodule of M including of N. Hence, by Theorem 3.7, $Q = (L : M)$ is a prime ideal of X. Since for any prime ideal P of X, $m \in K(N, P)$, $m \in K(N, Q)$ and so there exists $c \in X - Q$ such that $c.m \in Q.M + N = (L : M).M + N \subseteq L + L \subseteq L$. Since L is a prime submodule of M and $c \notin Q = (L : M)$, $m \in L$. Hence $T \subseteq rad_M(N)$. Now, let $m \in rad_M(N)$. Hence, $m \in L$, where L is any prime submodule of M consisting of N and P be a prime ideal of X. If $K(N, P) = M$, then the proof is complete. Let $K(N, P) \neq M$. By Theorem 3.20, $K(N, P)$ is a prime submodule of M and $P = (K(N, P) : M)$. Now, we show that $N \subseteq K(N, P)$. By Theorem 3.19, we have $P.M + N \subseteq K(N, P)$ and so $N \subseteq K(N, P)$. Since $m \in rad_M(N)$, then $m \in K(N, P)$. Hence $m \in T$ and so $rad_M(N) \subseteq T$. Therefore, $rad_M(N) = T$. ∎

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