Δ-CONVERGENCE THEOREM FOR TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPING IN UNIFORMLY CONVEX HYPERBOLIC SPACES

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Abstract. Recently, Chang, et al introduce the concept of total asymptotically nonexpansive mapping which contain the asymptotically nonexpansive mapping. The purpose of the paper is to analyze a three-step iterative scheme for total asymptotically nonexpansive mapping in uniformly convex hyperbolic spaces. Meanwhile, we obtain a Δ-convergence theorem of the three-step iterative scheme for total asymptotically nonexpansive mapping in CAT(0) spaces. Ours results obtained in this paper extend and improve some previous known results.

Keywords and phrases: total asymptotically nonexpansive mapping, three-step iterations, uniformly convex hyperbolic spaces, Δ-convergence theorem, CAT(0) spaces.

2010 Mathematics Subject Classification: 47H09, 47H10, 47H15.

1. Introduction

Throughout this paper, (M, d) will stand for a metric space. For any $x, y \in M$, let $[x, y]$ be an isometric image of the real line interval $[0, d(x, y)]$. Suppose that there exists a family $\mathcal{F}$ of metric segments (or geodesic) such that any two points x, y in M are endpoints of a unique metric segment $[x, y] \in \mathcal{F}$. We shall denote by $(1-\beta)x \oplus \beta y$ the unique point z of $[x, y]$ which satisfies

$$d(x, z) = \beta d(x, y) \quad \text{and} \quad d(z, y) = (1-\beta)d(x, y).$$

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Such metric spaces are usually called convex metric spaces (see [10]). The (M, d) is said to be a geodesic space if every two points of M are joined by a geodesic. For a convex metric spaces (M, d), if

$$d\left(\frac{1}{2}p \oplus \frac{1}{2}x,\ \frac{1}{2}p \oplus \frac{1}{2}y\right) \le \frac{1}{2}d(x, y),$$

for all p, x, y in M, then M is said to be a hyperbolic metric space, for more details about hyperbolic metric space see ([9]). A subset C of a hyperbolic metric space M is convex if $[x, y] \subset C$ whenever x, y are in C. Obviously, normed linear spaces are hyperbolic spaces. One can consider, as nonlinear examples, the Hadamard manifolds ([2]), the Hilbert open unit ball equipped with the hyperbolic metric ([5]), and the CAT(0)spaces. Recall that a geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality ([13]):

$$d\left(z, \frac{1}{2}x \oplus \frac{1}{2}y\right)^2 \le \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2.$$

In particular, if x, y, z are points in a CAT(0) space and $t \in [0, 1]$, then

$$d((1-t)x \oplus ty, z) \le (1-t)d(x, z) + td(y, z).$$

For more details about CAT(0) spaces, see [1].

In 2011, Khamsi and Khan (see [7]) introduced uniformly convex hyperbolic metric space.

Definition 1.1 Let (M, d) be a hyperbolic metric space, M is said to be uniformly convex, if for any $a \in M$, for every $r > 0$, and for each $\varepsilon > 0$,

$$\delta(r, \varepsilon) = \inf\left\{1 - \frac{1}{r}d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right) :\ d(x, a) \le r,\ d(y, a) \le r,\ d(x, y) \ge r\varepsilon\right\} > 0.$$

If (M, d) is uniformly convex hyperbolic spaces, then for every $s \ge 0$, $\varepsilon > 0$, there exists $\eta(s, \varepsilon) > 0$ depending on s and ε such that

$$\delta(r, \varepsilon) > \eta(s, \varepsilon) > 0, \quad \text{for any } r > s.$$

CAT(0) spaces are uniformly convex hyperbolic spaces with $\delta(r, \varepsilon) = 1 - \sqrt{1 - \varepsilon^2/4}$. Uniformly convex hyperbolic spaces can be viewed as 2-uniformly convex (see [7]).

2. Preliminaries

Let us start by making some basic definitions.

Definition 2.1. ([4]) Let C be bounded subset of X, a mapping $T : X \to X$ is called asymptotically nonexpansive, if there exists a sequence $\{k_n\}$ of positive real numbers with $k_n \to 1$ as $n \to \infty$ for which

$$d(T^n x, T^n y) \le k_n d(x, y), \quad \text{for all } x,\ y \in X.$$

Definition 2.2. T is said to be uniformly L-Lipschitzian, if there exists a constant $L > 0$ such that

$$d(T^n x, T^n y) \leq L d(x, y), \quad \forall n \geq 1, x, y \in X.$$

Chang et al. (see [3]) recently introduce the concept of total asymptotically nonexpansive mappings and prove the demiclosed principle for this kind of mappings in CAT(0) spaces.

Definition 2.3. A mapping $T : X \to X$ is said to be $(\{\mu_n\}, \{\nu_n\}, \zeta)-$total asymptotically nonexpansive, if there exist nonnegative sequences $\{\mu_n\}, \{\nu_n\}$ with $\mu_n \to 0$, $\nu_n \to 0$ and a strictly increasing continuous function $\zeta : [0, +\infty) \to [0, +\infty)$ with $\zeta(0) = 0$ such that

$$d(T^n x, T^n y) \leq d(x, y) + \nu_n \zeta(d(x, y)) + \mu_n \quad \forall n \geq 1, x, y \in X.$$

Rermark 2.4. From the above definition, it is to know that, each nonexpansive mapping is a asymptotically nonexpansive mapping with sequence $\{k_n = 1\}$, and each asymptotically nonexpansive mapping is a $(\{\mu_n\}, \{\nu_n\}, \zeta)-$total asymptotically nonexpansive mapping with $\mu_n = 0, \nu_n = k_n - 1, \forall n \geq 1$ and $\zeta(t) = t, t \geq 0$.

Let $\{x_n\}$ be a bounded sequence in M and $C \subset M$ be a nonempty subset of M. The asymptotic radius of $\{x_n\}$ with respect to C is defined by

$$r(C, \{x_n\}) = \inf \left\{ \limsup_{n\to\infty} d(x, x_n) : \ x \in C \right\}.$$

The asymptotic radius of $\{x_n\}$, denoted by $r(\{x_n\})$, is the asymptotic radius of $\{x_n\}$ with respect to M. The asymptotic center of $\{x_n\}$ with respect to C is defined by

$$A(C, \{x_n\}) = \left\{ z \in C : \ \limsup_{n\to\infty} d(z, x_n) = r(\{C, x_n\}) \right\}.$$

When $C = M$, we call the asymptotic center of $\{x_n\}$ and use the notation $A(\{x_n\})$ for $A(C, \{x_n\})$.

Definition 2.5. ([11]) A sequence $\{x_n\}$ in (M, d) is said to Δ-converge to $x \in M$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = x$ and call x the Δ-limit of $\{x_n\}$.

Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence ([8]). The proof of following lemma is implicit in the proof of Theorem 3.5 in [3].

Lemma 2.6. *Let C be a closed convex subset of a complete CAT(0) space M, and $T : C \to C$ be a uniformly L-Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)-$total asymptotically nonexpansive mapping. Suppose that $\{x_n\}$ is a bounded sequence in C*

such that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and* $d(x_n, p)$ *converges for each* $p \in F(T)$*, then* $\omega_w(x_n) \subset F(T)$*. Here* $\omega_w(x_n) = \cup A(\{u_n\})$*, the union is taken over all subsequences* $\{u_n\}$ *of* $\{x_n\}$*. Moreover,* $\omega_w(x_n)$ *consists of exactly one point.*

In 2002, Xu and Noor [14] introduced and analyzed a three-step iterative schemes for solving the nonlinear equation $Tx = x$ for asymptotically nonexpansive mappings in Banach space. It has been shown that the three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations ([6]). Inspired and motivated by these facts, we analyze a three-step iterative scheme for mappings of total asymptotically nonexpansive in uniformly convex hyperbolic spaces.

Algorithm 1. Let C be a nonempty closed subset of a hyperbolic metric space (M, d) and $T : C \to C$ be a uniformly $L-$ Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)-$total asymptotically nonexpansive mapping. For a given $x_1 \in C$, compute sequences $\{z_n\}$,$\{y_n\}$,$\{x_n\}$ by the iterative schemes

$$\begin{aligned} z_n &= a_n T^n x_n \oplus (1 - a_n)x_n \\ y_n &= b_n T^n z_n \oplus (1 - b_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n \oplus (1 - \alpha_n)x_n \end{aligned} \tag{2.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are real numbers in $[0, 1]$. Let $a_n = 0$ or $a_n = 0$ and $b_n = 0$, we get Ishikawa-type and Krasnoselski-Mann iteration as special cases.

$$\begin{aligned} y_n &= b_n T^n x_n \oplus (1 - b_n)x_n \\ x_{n+1} &= \alpha_n T^n y_n \oplus (1 - \alpha_n)x_n; \end{aligned} \tag{2.2}$$

$$x_{n+1} = \alpha_n T^n x_n \oplus (1 - \alpha_n)x_n; \tag{2.3}$$

For a suitable choice of $\{\alpha_n\}$ and $\{b_n\}$, we obtain a Δ-convergence theorem of the three-step iterative scheme for uniformly $L-$ Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)-$total asymptotically nonexpansive mapping in CAT(0) spaces.

3. Main results

The following Lemma is trivial (see [12]).

Lemma 3.1. *Let* $\{a_n\}$*,* $\{\lambda_n\}$ *and* $\{c_n\}$ *be the sequences of nonnegative numbers such that*

$$a_{n+1} \le (1 + \lambda_n)a_n + c_n, \quad \forall\ n \ge 1.$$

If $\sum_{n=1}^{\infty} \lambda_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$*, then* $\lim_{n\to\infty} a_n$ *exists. If there exists a subsequence of* $\{a_n\}$ *which converges to 0, then* $\lim_{n\to\infty} a_n = 0$.

Lemma 3.2. *Let (M,d) be uniformly convex hyperbolic spaces and C be a bounded closed nonempty convex subset of M. Let $T : C \to C$ be a uniformly L-Lipschitzian and $(\{\mu_n\},\{\nu_n\},\zeta)-$total asymptotically nonexpansive mapping. If $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are real numbers in $[0,1]$. Suppose that $x_1 \in C$ and $\{x_n\}$ is given by (2.1).*

(i) *If $\sum\limits_{n=1}^{\infty} \nu_n < \infty$; $\sum_{n=1}^{\infty} \mu_n < \infty$;*

(ii) *there exists a constant $M^* > 0$ such that $\zeta(r) \le M^* r$, $r \ge 0$.*

Then $\lim\limits_{n\to\infty} d(x_n,p)$ exists for each $p \in F(T)$.

Proof. Let $p \in F(T)$. From (2.1), we have

$$\begin{aligned} d(z_n,p) &= d(a_nT^nx_n \oplus (1-a_n)x_n, p)\\ &\le a_nd(T^nx_n,p) + (1-a_n)d(x_n,p)\\ &= a_nd(T^nx_n,T^np) + (1-a_n)d(x_n,p)\\ &\le a_n(d(x_n,p) + \nu_n\zeta(d(x_n,p)) + \mu_n) + (1-a_n)d(x_n,p)\\ &\le (1+\nu_nM^*)d(x_n,p) + \mu_n \end{aligned} \tag{3.1}$$

$$\begin{aligned} d(y_n,p) &= d(b_nT^nz_n \oplus (1-b_n)x_n, p)\\ &\le b_nd(T^nz_n,p) + (1-b_n)d(x_n,p)\\ &= b_nd(T^nz_n,T^np) + (1-b_n)d(x_n,p)\\ &\le b_n(d(z_n,p) + \nu_n\zeta(d(z_n,p)) + \mu_n) + (1-b_n)d(x_n,p)\\ &\le b_n((1+\nu_nM^*)d(x_n,p) + \mu_n + \nu_nM^*d(z_n,p) + \mu_n)\\ &\qquad +(1-b_n)d(x_n,p)\\ &\le (1+2\nu_nM^* + (\nu_nM^*)^2)d(x_n,p) + (\nu_nM^*+2)\mu_n \end{aligned} \tag{3.2}$$

From (3.2), we get

$$\begin{aligned} d(x_{n+1},p) &= d(\alpha_nT^ny_n \oplus (1-\alpha_n)x_n, p)\\ &\le \alpha_nd(T^ny_n,p) + (1-\alpha_n)d(x_n,p)\\ &\le \alpha_n(d(y_n,p) + \nu_n\zeta(d(y_n,p)) + \mu_n) + (1-\alpha_n)d(x_n,p)\\ &\le (1+3\nu_nM^* + 3(\nu_nM^*)^2 + (\nu_nM^*)^3)d(x_n,p)\\ &\quad +(3+3\nu_nM^* + (\nu_nM^*)^2)\mu_n. \end{aligned} \tag{3.3}$$

In Lemma 3.1, take $a_n = d(x_n,p)$, $\lambda_n = 3\nu_nM^* + 3(\nu_nM^*)^2 + (\nu_nM^*)^3$ and $c_n = (3+3\nu_nM^* + (\nu_nM^*)^2)\mu_n$, then all conditions in Lemma 3.1 are satisfied. The conclusion is obtained from Lemma 3.1 immediately. ∎

Lemma 3.3. *Let (M,d) be uniformly convex hyperbolic spaces and C be a bounded closed nonempty convex subset of M. Let $T : C \to C$ be a uniformly L-Lipschitzian and $(\{\mu_n\},\{\nu_n\},\zeta)-$total asymptotically nonexpansive mapping. If $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are real numbers in $[0,1]$ satisfying*

(i) $0 < \liminf_n \alpha_n \le \limsup_n \alpha_n < 1$;

(ii) $0 < \liminf_n b_n \le \limsup_n b_n < 1$;

(iii) $\sum_{n=1}^{\infty} \nu_n < \infty$; $\sum_{n=1}^{\infty} \mu_n < \infty$;

(iv) *there exists a constant* $M^* > 0$ *such that* $\zeta(r) \le M^* r$, $r \ge 0$.

Suppose that $x_1 \in C$ *and* $\{x_n\}$ *is given by* (2.1). *Then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Proof. Take a $p \in F(T)$, in view of Lemma 3.2, we can let $\lim_{n\to\infty} d(x_n, p) = r$ for some $r \in \mathbb{R}$. If $r = 0$, then we immediately obtain

$$d(x_n, Tx_n) \le d(x_n, p) + d(Tx_n, p) = d(x_n, p) + d(Tx_n, Tp) \le (1 + L)d(x_n, p),$$

by the uniformly L-Lipschitzian of T, then $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

If $r > 0$, then we shall prove that

$$\lim_{n\to\infty} d(T^n y_n, p) = \lim_{n\to\infty} d(T^n z_n, p) = r, \tag{3.4}$$

by showing that for any increasing sequence $\{n_i\}$ of positive integers for which the limits in (3.4) exist, and it follows that the limit is r. From condition (i), we may assume that the corresponding subsequence $\{\alpha_{n_i}\}$ converges to some α; we shall have $\alpha > 0$ because $\{\alpha_{n_i}\}$ is assumed to be bounded away from 0. From conditions (i) and (iii), we have

$$\begin{aligned} r &= \lim_{n\to\infty} d(x_n, p) = \lim_{i\to\infty} d(x_{n_i+1}, p) \\ &= \lim_{i\to\infty} d\big(\alpha_{n_i} T^{n_i} y_{n_i} \oplus (1 - \alpha_{n_i}) x_{n_i}, p\big) \\ &\le \lim_{i\to\infty} \big[\alpha_{n_i} d(T^{n_i} y_{n_i}, p) + (1 - \alpha_{n_i}) d(x_{n_i}, p)\big] \\ &\le \lim_{i\to\infty} \big[\alpha_{n_i}(d(y_{n_i}, p) + \nu_n \zeta(d(y_{n_i}, p)) + \mu_n) + (1 - \alpha_{n_i}) d(x_{n_i}, p)\big] \\ &\le \alpha \limsup_{i\to\infty} d(x_{n_i}, p) + (1 - \alpha) r \\ &= r. \end{aligned}$$

It follows that $\lim_{n\to\infty} d(T^n y_n, p) = r$. From condition (ii), we may assume that the corresponding subsequence $\{b_{n_i}\}$ converges to some b; we shall have $b > 0$ because $\{b_{n_i}\}$ is assumed to be bounded away from 0. From (3.1) and conditions (ii), (iii), we have

$$
\begin{aligned}
r &= \lim_{n\to\infty} d(x_n, p) = \lim_{i\to\infty} d(x_{n_i+1}, p) \\
&= \lim_{i\to\infty} d\big(\alpha_{n_i} T^{n_i} y_{n_i} \oplus (1-\alpha_{n_i}) x_{n_i}, p\big) \\
&\le \lim_{i\to\infty} \big[\alpha_{n_i} d(T^{n_i} y_{n_i}, T^{n_i} p) + (1-\alpha_{n_i}) d(x_{n_i}, p)\big] \\
&\le \lim_{i\to\infty} \big[\alpha_{n_i} (d(y_{n_i}, p) + \nu_{n_i} \zeta(d(y_{n_i}, p)) + \mu_{n_i}) + (1-\alpha_{n_i}) d(x_{n_i}, p)\big] \\
&\le \alpha \lim_{i\to\infty} (b_{n_i} (d(T^{n_i} z_{n_i}, p) + (1-b_{n_i}) d(x_{n_i}, p) + \nu_n M^* d(y_{n_i}, p) + \mu_{n_i}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (1-\alpha) d(x_{n_i}, p) \\
&= \alpha \lim_{i\to\infty} (b_{n_i} d(T^n z_{n_i}, T^{n_i} p) + (1-b_{n_i}) d(x_{n_i}, p)) + (1-\alpha) d(x_{n_i}, p) \\
&\le \alpha (b \lim_{i\to\infty} (d(z_{n_i}, p) + \nu_{n_i} \zeta(d(z_{n_i}, p)) + \mu_{n_i}) + (1-b) d(x_{n_i}, p)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (1-\alpha) d(x_{n_i}, p) \\
&\le \alpha (b \lim_{i\to\infty} (d(x_{n_i}, p) + (1-b) d(x_{n_i}, p)) + (1-\alpha) d(x_{n_i}, p) \\
&= r.
\end{aligned}
$$

It follows that $\lim_{n\to\infty} d(T^n z_n, p) = r$ holds. In addition, it is easy to see that

$$
\lim_{n\to\infty} d\left(\frac{x_n \oplus T^n x_n}{2}, p\right) = r.
$$

In the sequel, we shall prove

$$
\lim_{n\to\infty} d(T^n y_n, x_n) = \lim_{n\to\infty} d(T^n z_n, x_n) = 0.
$$

Assume by contradiction that $\{T^n y_n\}$ does not converge $\{x_n\}$, and so we can find $\varepsilon_0 > 0$ and $\{n_k\} \subset \mathbb{N}$ such that

$$
d(T^{n_k} y_{n_k}, x_{n_k}) \ge \varepsilon_0. \tag{3.5}
$$

We can assume $\varepsilon_0 \in (0, 2]$, then $\dfrac{\varepsilon_0}{r+1} \in (0, 2]$. Since M is uniformly convex, there exits $\theta \in (0, 1]$ such that

$$
1 - \delta\left(r+\theta, \frac{\varepsilon_0}{r+1}\right) \le 1 - \eta\left(r, \frac{\varepsilon_0}{r+1}\right) \le \frac{r-\theta}{r+\theta} \tag{3.6}
$$

By (3.4), for the above $\theta > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$
d(x_{n_k}, p) \le r + \theta, d(T^{n_k} y_{n_k}, p) \le r + \theta \quad \forall k \ge N_0.
$$

For $k \ge N_0$, we also have that

$$
d\left(T^{n_k} y_{n_k}, x_{n_k}\right) \ge \varepsilon_0 = (r+\theta)\frac{\varepsilon_0}{r+\theta} \ge (r+\theta)\frac{\varepsilon_0}{r+1}.
$$

Now, applying the fact that M is uniformly convex and (3.6), we get that

$$\begin{aligned} d\left(\frac{x_{n_k} \oplus T^{n_k} x_{n_k}}{2}, p\right) &\leq \left(1 - \delta\left(r + \theta, \frac{\varepsilon_0}{r+1}\right)\right)(r+\theta) \\ &< r - \theta. \end{aligned}$$

Let $n_k \to \infty$, we obtain that

$$r = \lim_{n_k \to \infty} d\left(\frac{x_{n_k} \oplus T^{n_k} x_{n_k}}{2}, p\right) \leq r - \theta.$$

Hence, we get a contradiction, and therefore

$$\lim_{n \to \infty} d(T^n y_n, x_n) = 0. \tag{3.7}$$

This is equivalent to

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = 0. \tag{3.8}$$

Using the same way, we can prove that

$$\lim_{n \to \infty} d(T^n z_n, x_n) = 0 \tag{3.9}$$

This is equivalent to $\lim_{n \to \infty} d(y_n, x_n) = 0$ and meanwhile we get

$$\lim_{n \to \infty} d(y_{n+1}, y_n) = 0 \tag{3.10}$$

Thus

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} y_{n+1}) \\ &+ d(T^{n+1} y_{n+1}, T^{n+1} y_n) + d(T(T^n y_n), Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1} y_{n+1}) \\ &+ Ld(y_{n+1}, y_n) + Ld(T^n y_n, x_n) \end{aligned}$$

By (3.7), (3.8), (3.10) and the uniformly L-Lipschitzian of T, we conclude that

$$\lim_{n \to \infty} d(x_n, Tx_n) = 0. \qquad \blacksquare$$

Theorem 3.4. *Let (M, d) be a completely $CAT(0)$ spaces and C be a bounded closed nonempty convex subset of M. Let $T : C \to C$ be a uniformly L-Lipschitzian and $(\{\mu_n\}, \{\nu_n\}, \zeta)$-total asymptotically nonexpansive mapping. If $F(T) \neq \emptyset$, $\{a_n\}$, $\{b_n\}$, $\{\alpha_n\}$ are real numbers in $[0, 1]$ satisfying*

(i) $0 < \liminf_n \alpha_n \leq \limsup_n \alpha_n < 1$;

(ii) $0 < \liminf_n b_n \leq \limsup_n b_n < 1$;

(iii) $\sum_{n=1}^{\infty} \nu_n < \infty; \sum_{n=1}^{\infty} \mu_n < \infty;$

(iv) *there exists a constant* $M^* > 0$ *such that* $\zeta(r) \leq M^* r$, $r \geq 0$.

Supposing that $x_1 \in C$ *and* $\{x_n\}$ *is given by* (2.1), Δ*-converges to a fixed point of* T.

Proof. Since CAT(0) spaces are uniformly convex hyperbolic spaces, by Lemma 3.2, $\{d(x_n, p)\}$ is convergent for each $p \in F(T)$. By Lemma 3.3, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. By Lemma 2.6, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $F(T)$.■

Remark 3.5.

1. Since normed linear spaces are hyperbolic spaces and total asymptotically nonexpansive mapping which contain the asymptotically nonexpansive mapping. Therefore, Lemma 3.3 extend Lemma 2.2 of Xu and Noor [14].

2. Let $a_n = 0$ or $a_n = 0$ and $b_n = 0$, we get the Ishikawa-type and the Krasnoselski-Mann iteration as special cases, therefore Lemma 3.3 extends Theorem 5.2 of U. Kohlenbach (see [9]) and Lemma 5.4, Theorem 5.7 in [11]. Theorem 3.4 gives some new Δ-convergence theorem different from Theorem 3.5 ([3]) in CAT(0) spaces.

Acknowledgement. This work was supported by the Natural Science Foundation Project of CQCSTC under Grant no. cstc2014jcyjA00022 and by the Scientific and Technological Research Program of Chongqing Three Gorges University of China (14ZD09).

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Accepted: 04.08.2014