

MIDPOINT DERIVATIVE-BASED TRAPEZOID RULE FOR THE RIEMANN-STIELTJES INTEGRAL

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Abstract. In this paper, the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented which uses derivative value at the midpoint. This kind of quadrature rule obtains an increase of two orders of precision over the trapezoid rule for the Riemann-Stieltjes integral and the error term is investigated. At last, the rationality of the generalization of midpoint derivative-based trapezoid rule for Riemann-Stieltjes integral is demonstrated.

Keywords: midpoint derivative; trapezoid rules; Riemann-Stieltjes integral; error term.

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1. Introduction

In mathematics, the Riemann-Stieltjes integral is a kind of generalization of the Riemann integral, named after Bernhard Riemann and Thomas Stieltjes. It is Stieltjes [1] that first give the definition of this integral in 1894. It serves as an instructive and useful precursor of the Lebesgue integral. It is known that the Riemann-Stieltjes integral has wide applications in the field of probability theory [2-3], stochastic process [4] and functional analysis [5], especially in original

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formulation of F. Riesz's theorem [2], [5] and the spectral theorem for self-adjoint operators in a Hilbert space [2], [5].

Definite integration is one of the most important and basic concepts in mathematics. And it has numerous applications in fields such as physics and engineering. In several practical problems, we need to calculate integrals. As is known to all, as for $I = \int_a^b f(x)dx$, once the primitive function $F(x)$ of integrand $f(x)$ is known, the definite integral of $f(x)$ over the interval $[a, b]$ is given by Newton-Leibniz formula, that is,

$$\int_a^b f(x)dx = F(b) - F(a).$$

However, the explicit primitive function $F(x)$ is not available or its primitive function is not easy to obtain, such as $e^{\pm x^2}$, $\sin x^2$, $\frac{\sin x}{x}$, etc. Moreover, the integrand $f(x)$ is only available at certain points x_i , $i = 1, 2, \dots, n$. How to get high-precision numerical integration formulas becomes one of the challenges in fields of mathematics [6].

The methods of quadrature are usually based on the interpolation polynomials and can be written in the following form:

$$(1.1) \quad \int_a^b f(x)dx \approx \sum_{i=0}^n w_i f(x_i),$$

where there are $n+1$ distinct integration points at x_0, x_1, \dots, x_n within the interval $[a, b]$ and $n+1$ weights w_i . If the integration points are uniformly distributed over the interval, so $x_i = x_0 + ih$ in which $h = \frac{b-a}{n}$.

These w_i can be derived in several different ways [7]-[9]. One is interpolate $f(x)$ at the $n+1$ points x_0, x_1, \dots, x_n , using the Lagrange polynomials and then integrating the foresaid polynomials to obtain (1.1).

The other is based on the precision of a quadrature formula. Select the w_i , $i = 1, 2, \dots, n$, so that the error

$$(1.2) \quad R_n(f) = \int_a^b f(x)dx - \sum_{i=0}^n w_i f(x_i),$$

is exactly zero for $f(x) = x^j$, $j = 1, 2, \dots, n$. Using the method of undetermined coefficients, this approach generates a system of $n+1$ linear equations for weights w_i . Since the monomials $1, x, \dots, x^n$ are linearly independent, the linear system of equations has a unique solution.

The trapezoidal rule is the most well known numerical integration rules of this type. Trapezoidal rule for classical Riemann integral is

$$(1.3) \quad \int_a^b f(x)dx = \frac{b-a}{2} (f(a) + f(b)) - \frac{(b-a)^3}{12} f''(\xi),$$

where $\xi \in (a, b)$.

In spite of the many accurate and efficient methods for numerical integration being available in [7]-[9], recently Mercer [10] has obtained trapezoid rule for Riemann-Stieltjes integral which engender a generalization of Hadamard’s integral inequality. Then he develops Midpoint and Simpson’s rules for Riemann-Stieltjes integral [11] by using the concept of relative convexity. Burg [12] has proposed derivative-based closed Newton-Cotes numerical quadrature which uses both the function value and the derivative value on uniformly spaced intervals. Zhao and Li have proposed midpoint derivative-based closed Newton-Cotes quadrature [13] and numerical superiority has been shown. Then, the derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented by Zhao and Zhang [14], which uses derivative values at the endpoints. Recently, Simos and his partners have made a contribution to the Newton-Cotes formula for the Riemann integral and its applications [15]-[17], especially the connection between closed Newton-Cotes, trigonometrically-fitted differential methods, symplectic integrators and efficient solution of the Schrodinger equation [17].

Motivation for the research presented here lies in construction of midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral, which is generalization of the results in [13]. In this paper, the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented. These new scheme is investigated in Section 2. In Section 3, the error term is presented. Finally, conclusions are drawn in Section 4.

2. Midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral

In this section, by adding the derivative at the midpoint, midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented.

Theorem 2.1 *Suppose that $f'(t)$ and $g(t)$ are continuous on $[a, b]$ and $g(t)$ is increasing there. The midpoint derivative-based Trapezoid rule for the Riemann-Stieltjes integral is*

$$\begin{aligned}
 \int_a^b f(t)dg \approx T &= \left(\frac{1}{b-a} \int_a^b g(t)dt - g(a) \right) f(a) \\
 &+ \left(g(b) - \frac{1}{b-a} \int_a^b g(t)dt \right) f(b) \\
 &+ \left(\int_a^b \int_a^t g(x)dxdt - \frac{b-a}{2} \int_a^b g(t)dt \right) f''(c),
 \end{aligned}
 \tag{2.1}$$

where $c = \frac{(-2b^2 + a^2 - ab) \int_a^b g(t)dt + 6b \int_a^b \int_a^t g(x)dxdt - 6 \int_a^b \int_a^t \int_a^y g(x)dx dy dt}{6 \int_a^b \int_a^t g(x)dxdt - 3(b-a) \int_a^b g(t)dt}$.

Proof. Looking for the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral, we seek numbers a_0, a_1, c_0, c such that

$$\int_a^b f(t)dg \approx a_0f(a) + a_1f(b) + c_0f''(c)$$

is equality for $f(t) = 1, t, t^2, t^3$. That is,

$$\left\{ \begin{array}{l} \int_a^b 1 dg = a_0 + a_1; \\ \int_a^b t dg = a_0 a + a_1 b; \\ \int_a^b t^2 dg = a_0 a^2 + a_1 b^2 + 2c_0; \\ \int_a^b t^3 dg = a_0 a^3 + a_1 b^3 + 6c_0 c. \end{array} \right.$$

Therefore,

$$(2.2) \quad \left\{ \begin{array}{l} a_0 + a_1 = g(b) - g(a); \\ a_0 a + a_1 b = b g(b) - a g(a) - \int_a^b g(t) dt; \\ a_0 a^2 + a_1 b^2 + 2c_0 = b^2 g(b) - a^2 g(a) - 2b \int_a^b g(t) dt \\ \quad + 2 \int_a^b \int_a^t g(x) dx dt; \\ a_0 a^3 + a_1 b^3 + 6c_0 c = b^3 g(b) - a^3 g(a) - 3b^2 \int_a^b g(t) dt \\ \quad + 6b \int_a^b \int_a^t g(x) dx dt - 6 \int_a^b \int_a^t \int_a^y g(x) dx dy dt. \end{array} \right.$$

Solving simultaneous equations (2.2) for a_0, a_1, c_0, c , we obtain

$$\left\{ \begin{array}{l} a_0 = \frac{1}{b-a} \int_a^b g(t) dt - g(a); \\ a_1 = g(b) - \frac{1}{b-a} \int_a^b g(t) dt; \\ c_0 = \int_a^b \int_a^t g(x) dx dt - \frac{b-a}{2} \int_a^b g(t) dt; \\ c = \frac{(-2b^2 + a^2 - ab) \int_a^b g(t) dt + 6b \int_a^b \int_a^t g(x) dx dt - 6 \int_a^b \int_a^t \int_a^y g(x) dx dy dt}{6 \int_a^b \int_a^t g(x) dx dt - 3(b-a) \int_a^b g(t) dt}. \end{array} \right.$$

So we have the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral as desired. ■

Corollary 2.2 *The precision of the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is 3.*

Proof. From the construction of a_0, a_1, c_0, c , we obtain that the derivative-based trapezoidal rule for the Riemann-Stieltjes integral has degree of precision not less than 3.

In Section 3, by applying Theorem 3.1, we can easily see that the quadrature is not equality for $f(t) = t^4$. So the precision of this method is 3. ■

3. The error term for Riemann-Stieltjes midpoint derivative-based trapezoid rule

In this section, the error term of the derivative-based trapezoid rule for the Riemann-Stieltjes is investigated. The error term can be found in mainly 3 different ways [8, 9].

Here, we use the concept of precision to calculate the error term, where the error term is related to the difference between the quadrature formula for the monomial $\frac{x^{p+1}}{(p+1)!}$, and the exact value $\frac{1}{(p+1)!} \int_a^b x^{p+1} dx$ where p is the precision of the quadrature formula.

Theorem 3.1 *Suppose that $f'(t)$ and $g(t)$ are continuous on $[a, b]$ and $g(t)$ is increasing there. The derivative-based trapezoid rule for the Riemann-Stieltjes integral with the error term is*

$$\begin{aligned}
 \int_a^b f(t) dg \approx T &= \left(\frac{1}{b-a} \int_a^b g(t) dt - g(a) \right) f(a) + \left(g(b) - \frac{1}{b-a} \int_a^b g(t) dt \right) f(b) \\
 &+ \left(\int_a^b \int_a^t g(x) dx dt - \frac{b-a}{2} \int_a^b g(t) dt \right) f''(c) \\
 (3.1) \quad &+ \left(\frac{a^3 + ab^2 + a^2b - 3b^3 + 6(b-a)c^2}{24} \int_a^b g(t) dt + \frac{b^2 - c^2}{2} \int_a^b \int_a^t g(x) dx dt \right. \\
 &\left. - b \int_a^b \int_a^t \int_a^y g(x) dx dy dt + \int_a^b \int_a^t \int_a^z \int_a^y g(x) dx dy dz dt \right) f^{(4)}(\xi) g'(\eta),
 \end{aligned}$$

where $c = \frac{(-2b^2 + a^2 - ab) \int_a^b g(t) dt + 6b \int_a^b \int_a^t g(x) dx dt - 6 \int_a^b \int_a^t \int_a^y g(x) dx dy dt}{6 \int_a^b \int_a^t g(x) dx dt - 3(b-a) \int_a^b g(t) dt}$,
 $\xi, \eta \in (a, b)$.

And the error term $R[f]$ of this method is

$$\begin{aligned}
 (3.2) \quad &\left(\frac{a^3 + ab^2 + a^2b - 3b^3 + 6(b-a)c^2}{24} \int_a^b g(t) dt + \frac{b^2 - c^2}{2} \int_a^b \int_a^t g(x) dx dt \right. \\
 &\left. - b \int_a^b \int_a^t \int_a^y g(x) dx dy dt + \int_a^b \int_a^t \int_a^z \int_a^y g(x) dx dy dz dt \right) f^{(4)}(\xi) g'(\eta).
 \end{aligned}$$

Proof. Let $f(t) = \frac{t^4}{4!}$. So

$$(3.3) \quad \begin{aligned} \frac{1}{4!} \int_a^b t^4 dg &= \frac{1}{24} (b^4 g(b) - a^4 g(a)) - \frac{b^3}{6} \int_a^b g(t) dt + \frac{b^2}{2} \int_a^b \int_a^t g(x) dx \\ &\quad - b \int_a^b \int_a^t \int_a^y g(x) dx dy dt + \int_a^b \int_a^t \int_a^z \int_a^y g(x) dx dy dz dt. \end{aligned}$$

By Theorem 2.1, we have

$$(3.4) \quad \begin{aligned} \int_a^b f(t) dg &\approx T = \left(\frac{1}{b-a} \int_a^b g(t) dt - g(a) \right) f(a) \\ &\quad + \left(g(b) - \frac{1}{b-a} \int_a^b g(t) dt \right) f(b) \\ &\quad + \left(\int_a^b \int_a^t g(x) dx dt - \frac{b-a}{2} \int_a^b g(t) dt \right) f''(c). \end{aligned}$$

By (8)-(9), we obtain

$$\begin{aligned} \frac{1}{4!} \int_a^b t^4 dg - T &= \left(\frac{a^3 + ab^2 + a^2b - 3b^3 + 6(b-a)c^2}{24} \int_a^b g(t) dt \right. \\ &\quad + \frac{b^2 - c^2}{2} \int_a^b \int_a^t g(x) dx dt - b \int_a^b \int_a^t \int_a^y g(x) dx dy dt \\ &\quad \left. + \int_a^b \int_a^t \int_a^z \int_a^y g(x) dx dy dz dt \right) f^{(4)}(\xi) g'(\eta). \end{aligned}$$

This implies that

$$\begin{aligned} R[f] &= \left(\frac{a^3 + ab^2 + a^2b - 3b^3 + 6(b-a)c^2}{24} \int_a^b g(t) dt + \frac{b^2 - c^2}{2} \int_a^b \int_a^t g(x) dx dt \right. \\ &\quad \left. - b \int_a^b \int_a^t \int_a^y g(x) dx dy dt + \int_a^b \int_a^t \int_a^z \int_a^y g(x) dx dy dz dt \right) f^{(4)}(\xi) g'(\eta). \end{aligned}$$

■

Corollary 3.2 *Conditions are the same as in Theorem 3.1. When $g(t) = t$, (3.1) reduces to the midpoint derivative-based trapezoid rule (see [13]) for the classical Riemann integral.*

Proof. It is easy to obtain

$$\begin{aligned} \int_a^b t dt &= \frac{1}{2}b^2 - \frac{1}{2}a^2, \\ \int_a^b \int_a^t x dx dt &= \frac{1}{6}b^3 - \frac{1}{2}a^2b + \frac{1}{3}a^3, \\ \int_a^b \int_a^t \int_a^y x dx dy dt &= \frac{1}{24}b^4 - \frac{1}{4}a^2b^2 + \frac{1}{3}a^3b - \frac{1}{8}a^4, \\ \int_a^b \int_a^t \int_a^z \int_a^y x dx dy dz dt &= \frac{1}{120}b^5 - \frac{1}{12}a^2b^3 + \frac{1}{6}a^3b^2 - \frac{1}{8}a^3b + \frac{1}{30}a^5. \end{aligned}$$

Therefore,

$$c = \frac{(-2b^2 + a^2 - ab) \int_a^b t dt + 6b \int_a^b \int_a^t x dx dt - 6 \int_a^b \int_a^t \int_a^y x dx dy dt}{6 \int_a^b \int_a^t x dx dt - 3(b-a) \int_a^b t dt} = \frac{a+b}{2}.$$

By Theorem 3.1, we have

$$\begin{aligned} \int_a^b f(t) dg &= \int_a^b f(t) dt \\ &= \left(\frac{1}{b-a} \int_a^b t dt - g(a) \right) f(a) + \left(g(b) - \frac{1}{b-a} \int_a^b t dt \right) f(b) \\ &\quad + \left(\int_a^b \int_a^t x dx dt - \frac{b-a}{2} \int_a^b t dt \right) f''\left(\frac{a+b}{2}\right) \\ &\quad + \left(\frac{a^3 + ab^2 + a^2b - 3b^3}{24} \int_a^b t dt + \frac{b^2}{2} \int_a^b \int_a^t x dx dt \right. \\ &\quad \left. - b \int_a^b \int_a^t \int_a^y x dx dy dt + \int_a^b \int_a^t \int_a^z \int_a^y x dx dy dz dt \right. \\ &\quad \left. - \left(\frac{a+b}{2} \right)^2 \left(\frac{1}{2} \int_a^b \int_a^t x dx dt - \frac{b-a}{4} \int_a^b t dt \right) \right) f^{(4)}(\xi) \\ &= \frac{b-a}{2} (f(a) + f(b)) - \frac{(b-a)^3}{12} f''\left(\frac{a+b}{2}\right) + \frac{(b-a)^5}{480} f^{(4)}(\xi). \quad \blacksquare \end{aligned}$$

Remark 3.3 From Corollary 3.1, we know that the results in Theorem 3.1 possess the reducibility. When $g(t) = t$, formula (3.1) reduces to the derivative-based trapezoid rule for the classical Riemann integral. So Theorem 3.1 is a reasonable generalization of the results in [13].

4. Conclusions

We briefly summarize our main conclusions in this paper as follows.

1. The midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented which uses derivative value at the midpoint.
2. This kind of quadrature rule obtains an increase of two orders of precision over the trapezoid rule for the Riemann-Stieltjes integral.
3. The error term for Riemann-Stieltjes midpoint derivative-based trapezoid rule is investigated. And the rationality of the generalization of midpoint derivative-based trapezoid rule for Riemann-Stieltjes integral is demonstrated.

The derivative-based Simpson’s rules for the Riemann-Stieltjes integral will be achieved by further research.

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