EXISTENCE AND UNIQUENESS THEOREM FOR A SOLUTION OF FUZZY IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we prove the existence and uniqueness of a solution of the fuzzy impulsive differential equation $x^{'}(t) = f(t, x(t)), x(t_0) = x_0, \Delta x(t_k) = I_k(x(t_k))$ by using the method of successive approximation. We also consider the ϵ-approximate solution for the above fuzzy impulsive differential equation.

Keywords: Fuzzy differential equation, Levelwise continuous, Fuzzy impulsive equation, Fuzzy integral, Fuzzy ϵ-approximate solution.

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1. Introduction

Knowledge about differential equations is often incomplete or vague. For example, initial conditions or the values of functional relationships may not be known precisely. In such a situation, the usage of fuzzy differential equations (FDEs) is a natural way to model dynamical systems under possibilistic uncertainty. FDEs is a very important topic from the theoretical point of view (see e.g. [8] and references therein) as well as of their applications, for example, in modelling hydraulic [2], in population models [3, 12], in modelling of a three-phase induction

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motor [26]. The study of fuzzy differential equations forms a suitable setting for modelling dynamical systems.

Some authors have studied fuzzy differential equations. FDEs were first formulated by Kaleva [7]. He discussed the properties of differentiable fuzzy set value mappings and give the existence and uniqueness theorem for a solution of the fuzzy differential equation. Seikkala [23] defined the fuzzy derivatives which is generalization of the Hukuhara derivative in [19]. Since then there appeared a lot of papers concerning different approaches to the theory of FDEs. A rich collection of results from the theory of FDEs is contained in the monographs of Lakshmikantham and Mohapatra [8]. Park [17] studied the existence and uniqueness theorem for fuzzy differential equations. For the cauchy problem $x^{'} = f(t, x), x(t_0) = x_0$, the local existence theorems are proved in [25], and the existence theorems under compactness-type conditions are investigated in [24] when the fuzzy valued mapping f satisfies the generalized Lipschitz condition. There appeared a lot of papers concerning different approaches to the theory of FDEs (see, e.g., [14], [15], [16]).

On the other hand, the theory of impulsive differential equations or implicit impulsive integro-differential equations has been emerging as an important area of investigation in recent years and has been developed very rapidly due to the fact that such equations find a wide range of applications modeling adequately many real processes observed in physics, chemistry, biology and engineering. Correspondingly, applications of the theory of impulsive differential equations to different areas were considered by many authors (see, e.g, [5], [11], [13]). There are not too many papers on impulsive fuzzy differential equations, but some basic results on impulsive fuzzy differential equations can be found in [4], [6], [9], [20], [21]. For the monographs of the theory of impulsive diffferential equations, we can refer the books of Bainov and Simenov [1], Lakshmikantham et.al [10], Samoilenko and Perestyuk [22].

Motivated and inspired by the above works, In this paper, we prove the existence and uniqueness theorem of a solution to the fuzzy impulsive differential equation,

$$\begin{aligned} x'(t) &= f(t, x(t)), \\ x(t_0) &= x_0, \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2..., m \end{aligned} \tag{1.1}$$

where $f : I \times E^d \to E^d$ is levelwise continuous and satisfies a generalized Lipschitz condition, $x_0 \in E^d$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at $t = t_k$ respectively. Under some hypotheses, we also consider the ϵ-approximate solution for the above fuzzy differential equation.

The paper is organized as follows. In Section 2, we collect the fundamental notions and facts which will be used in the rest of the article. In Section 3, we prove the existence and uniqueness theorem of a solution to the fuzzy impulsive differential equation (1.1).

2. Preliminaries

In this section, our aim is to give a background of the fuzzy set space, and an overview of properties used by us, of integration and differentiation of fuzzy set-valued mappings.

Let A, B be nonempty compact subsets of $\mathbb{R}^d$. The Hausdorff metric is defined as follows

$$d_H(A, B) = \max\{d_H^*(A, B), d_H^*(B, A)\},$$

where

$$d_H^*(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|$$

and $\|.\|$ denotes usual Euclidean norm in $\mathbb{R}^d$.

We have $d_H^*(A, B) = 0$ if and only if $A \subset B$ and $d_H^*(A, B) \leq d^*(A, C) + d_H^*(C, B)$ for nonempty compact subsets A, B, C of $\mathbb{R}^d$.

Let $\mathcal{K}(\mathbb{R}^d)$ denote a family of all nonempty compact convex subsets of $\mathbb{R}^d$ and define addition and scalar multiplication in $\mathcal{K}(\mathbb{R}^d)$ as usual, i.e, for $A, B \in \mathcal{K}(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$.

$$A + B = \{a + b | a \in A, b \in B\}, \qquad \lambda A = \{\lambda a | a \in A\}.$$

Denote

$$E^d = \{u : \mathbb{R}^d \to [0, 1] \mid \quad u \text{ satisfies (i)-(iv) below}\},$$

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}^d$ such that $u(x_0) = 1$,

(ii) u is fuzzy convex, that is, $u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}^d$ and $0 \leq \lambda \leq 1$,

(iii) u is upper semicontinuous,

(iv) $[u]^0 = \text{cl}\{x \in \mathbb{R}^d : u(x) > 0\}$ is compact, where cl denotes the closure in $(\mathbb{R}^d, \|.\|)$.

For $\alpha \in (0, 1]$, denote $[u]^\alpha = \{x \in \mathbb{R}^d | u(x) \geq \alpha\}$. We will call this set an α-cut (α-level set) of u. For $u \in E^d$ one has that $[u]^\alpha \in \mathcal{K}(\mathbb{R}^d)$ for every $\alpha \in (0, 1]$.

If $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a function then according to Zadeh's extension principle we can extend g to $E^d \times E^d \to E^d$ by the formula

$$g(u, v)(z) = \sup_{z = g(x,y)} min\{u(x), v(y)\}.$$

It is well known that if g is continuous then $[g(u, v)]^\alpha = g([u]^\alpha, [v]^\alpha)$ for all $u, v \in E^d$, $\alpha \in [0, 1]$. Especially, for addition and a scalar multiplication in fuzzy number space E^d, we have:

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha, \quad [\lambda u]^\alpha = \lambda [u]^\alpha,$$

where $u, v \in E^d$, $\lambda \in \mathbb{R}$ and $\alpha \in [0, 1]$.

Define $D : E^d \times E^d \to [0, \infty)$ by the expression

$$D(u, v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha),$$

where d_H is the Hausdorff metric in $\mathcal{K}(\mathbb{R}^d)$. It is easy to see that D is a metric in E^d. In fact (E^d, D) is a complete metric space, and for every $u, v, w, z \in E^d$, $\lambda \in \mathbb{R}$, one has $D(u + w, v + w) = D(u, v), D(u + v, w + z) \leq D(u, w) + D(v, z)$, $D(\lambda u, \lambda v) = |\lambda| D(u, v)$ (see, e.g., [18]).

We define $\widehat{\theta} \in E^d$ as $\widehat{\theta} = \chi_{\{0\}}$, where for $x \in \mathbb{R}^d$ we have

$$\begin{aligned} \chi_{\{x\}}(y) &= 1 \quad \text{if } y = x \text{ and} \\ \chi_{\{x\}}(y) &= 0 \quad \text{if } y \neq x. \end{aligned}$$

Let $[a, b] \subset \mathbb{R}$ be a compact interval, $-\infty < a, b < +\infty$. A fuzzy valued mapping $F : [a, b] \to E^d$ is strongly measurable if for all $\alpha \in [0, 1]$ the set-valued mapping $[F(.)]^\alpha : [a, b] \to \mathcal{K}(\mathbb{R}^d)$ is measurable, i.e., the set $\{t \in [a, b] \mid [F(t)]^\alpha \cap C \neq \emptyset\}$ for each closed set $C \subset \mathbb{R}^d$ is Lebesgue measurable. A fuzzy mapping $F : [a, b] \to E^d$ is called integrably bounded if there exists an integrable function $h : [a, b] \to \mathbb{R}$ such that $\|x\| \leq h(t)$ for all $x \in [F(t)]^0$.

Definition 2.1. (Puri and Ralescu [18]). Let $F : [a, b] \to E^d$. The integral of F over $[a, b]$, denoted by $\int_a^b F(t)dt$, is defined levelwise by the expression

$$\begin{aligned} \left[\int_a^b F(t)dt\right]^\alpha &= \int_a^b [F(t)]^\alpha dt \\ &= \left\{\int_a^b f(t)dt \mid f : [a, b] \to \mathbb{R}^d \text{ is measurable selection for } [F(.)]^\alpha\right\}, \end{aligned}$$

for all $\alpha \in (0, 1]$.

By virtue of Remark 4.1 in [7], we have that

$$\left[\int_a^b F(t)dt\right]^0 = \int_a^b [F(t)]^0 dt.$$

We recall (see [7]) some properties of integrability for fuzzy mappings.

1. Let $F, G : [a, b] \to E^d$ be integrable and $\lambda \in \mathbb{R}$. Then
   (i) $\int_a^b \big(F(t) + G(t)\big)dt = \int_a^b F(t)dt + \int_a^b G(t)dt$,
   (ii) $\int_a^b \lambda F(t)dt = \lambda \int_a^b F(t)dt$,
   (iii) $D(F, G)$ is integrable,

(iv) $D\Big(\int_a^b F(t)dt, \int_a^b G(t)dt\Big) \le \int_a^b D\big(F(t), G(t)\big)dt.$

2. If $F : [a, b] \to E^d$ is continuous then it is integrable.

3. If $F : [a, b] \to E^d$ is integrable and $c \in [a, b]$, then
$\int_a^b F(t)dt = \int_a^c F(t)dt + \int_c^b F(t)dt.$

Let $u, v \in E^d$. If there exists $w \in E^d$ such that $u = v + w$, then we call w the H-difference of u and v and we denote it by $u \ominus v$. Note that $u \ominus v \neq u + (-1)v$.

Definition 2.2. (Puri and Ralescu [19]) A mapping $F : [a, b] \to E^d$ is differentiable at $t_0 \in [a, b]$ if there exists $F^{'}(t_0) \in E^d$ such that the limits

$$\lim_{h\to 0^+} \frac{1}{h}(F(t_0 + h) \ominus F(t_0)),$$
$$\lim_{h\to 0^+} \frac{1}{h}(F(t_0) \ominus F(t_0 - h))$$

exist and equal to $F^{'}(t_0)$. The limits are taken in the metric space (E^d, D), and at the boundary points we consider only the one-sided derivatives.

Definition 2.3. The integral of a fuzzy mapping $F : [a, b] \to E^d$ is defined levelwise by

$$\Big[\int_a^b F(t)dt\Big]^\alpha = \int_a^b F_\alpha(t)dt,$$

i.e., the set of all $\int_a^b f(t)dt$ such that $f : [a, b] \to \mathbb{R}^d$ is a measurable selection for F_α for all $\alpha \in [0, 1]$.

Definition 2.4. A strongly measurable and integrably bounded mapping $F : [a, b] \to E^d$ is said to be integrable over $[a, b]$ if $\displaystyle\int_a^b F(t)dt \in E^d$.

Note that if $F : [a, b] \to E^d$ is strongly measurable and integrably bounded, then F is integrable. Further if $F : [a, b] \to E^d$ is continuous, then it is integrable.

Definition 2.5. A mapping $F : [a, b] \to E^d$ is called differentiable at $t_0 \in [a, b]$ if for any $\alpha \in [0, 1]$, the set-valued mapping $F_\alpha(t) = [F(t)]^\alpha$ is Hukuhara differentiable at point t_0 with $DF_\alpha(t_0)$ and the family $\{DF_\alpha(t_0) : \alpha \in [0, 1]\}$ define a fuzzy number $F(t_0) \in E^d$. If $F : [a, b] \to E^d$ is differentiable at $t_0 \in [a, b]$, then we say that $F'(t_0)$ is the fuzzy derivative of $F(t)$ at the point t_0.

Theorem 2.1. *Let $F : [a, b] \to E^d$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$. Then f_α and g_α are differentiable and*

$$[F'(t)]^\alpha = [f'_\alpha(t), g'_\alpha(t)].$$

Theorem 2.2. *Let $F : [a, b] \to E^d$ be differentiable and assume that the derivative F' is integrable over $[a, b]$. Then, for each $s \in [a, b]$, we have*

$$F(s) = F(a) + \int_a^s F'(t)dt.$$

Definition 2.6. A mapping $f : [a, b] \times E^d \to E^d$ is called levelwise continuous at a point $(t_0, x_0) \in [a, b] \times E^d$ provided for any fixed $\alpha \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists $\delta(\epsilon, \alpha) > 0$ such that

$$d([f(t, x)]^\alpha, [f(t_0, x_0)]^\alpha) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha)$ for all $t \in [a, b], x \in E^d$.

Definition 2.7. A mapping $f : [a, b] \times E^d \to E^d$ is called levelwise continuous at a point $(t_0, x_0) \in [a, b] \times E^d$ provided for any fixed $\alpha \in [0, 1]$ and arbitrary $\epsilon > 0$, there exists $\delta(\epsilon, \alpha) > 0$ such that

$$d([f(t, x)]^\alpha, [f(t_0, x_0)]^\alpha) < \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, \alpha)$ and $d([x]^\alpha, [x_0]^\alpha) < \delta(\epsilon, \alpha)$ for all $t \in [a, b], x \in E^d$.

3. Existence and uniqueness results

Assume that $f : I \times E^d \to E^d$ is levelwise continuous, where the interval $I = \{t : |t - t_0| \leq \delta \leq a\}$.

Consider the fuzzy differential equation (1.1). We denote $J_0 = I \times B(x_0, b)$, where $a > 0$, $b > 0$, $x_0 \in E^d$,

$$B(x_0, b) = \{x \in E^d \mid D(x, x_0) \leq b\}. \tag{3.1}$$

Definition 3.1. A mapping $x : I \to E^d$ is a solution to the problem (1.1) if it is levelwise continuous and satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s))ds + \sum_{0<t<t_k} I_k(x(t_k)),\ k = 1, 2, ..., m,\ \forall t \in I. \tag{3.2}$$

According to the method of successive approximation, let us consider the sequence $\{x_n(t)\}$ such that

$$x_n(t) = x_0 + \int_{t_0}^t f(s, x_{n-1}(s))ds + \sum_{0<t<t_k} I_k(x(t_k)), n = 1, 2, ..., \tag{3.3}$$

where $x_0(t) = x_0, t \in I$.

Theorem 3.1. *Assume that*

(i) *a mapping $f : J_0 \to E^d$ is levelwise continuous,*

(ii) *for any pair $(t,x),(t,y) \in J_0$ we have*

$$d([f(t,x)]^\alpha,[f(t,y)]^\alpha) \leq Ld([x]^\alpha,[y]^\alpha), \tag{3.4}$$

where $L > 0$ is a given constant and for any $\alpha \in [0,1]$.

(iii) *There exists a constant κ and χ such that*

(a) $d\Big([I_k(x(t_k))]^\alpha,[I_k(y(t_k))]\Big)^\alpha \leq \kappa,$

(b) $d\big([I_k(x(t_k))]^\alpha,0\big) \leq \chi.$

Then there exists a unique solution $x = x(t)$ of (1.1) defined on the interval

$$|t-t_0| \leq \delta = min\left\{a,\frac{b}{M}\right\}, \tag{3.5}$$

where $M = D(f(t,x),\hat{0}), \hat{0} \in E^d$ such that $\hat{0}(t) = 1$ for $t = 0$ and 0 otherwise and for any $(t,x) \in J_0$.

Moreover, there exists a fuzzy set valued mapping $x : I \to E^d$ such that $D(x_n(t),x(t)) \to 0$ on $|t-t_0| \leq \delta$ as $n \to \infty$.

Proof. Let $t \in I$, from (3.3), it follows that, for $n = 1$,

$$x_1(t) = x_0 + \int_{t_0}^t f(s,x_0)ds + \sum_{0<t<t_k} I_k(x(t_k)) \tag{3.6}$$

which proves that $x(t)$ is levelwise continuous on $|t-t_0| \leq a$ and, hence on $|t-t_0| \leq \delta$.

Moreover, for any $\alpha \in [0,1]$ we have

$$\begin{aligned} d([x_1(t)]^\alpha,[x_0]^\alpha) &= d\left(\left[\int_{t_0}^t f(s,x_0)ds\right]^\alpha,0\right) + d\big([I_k(x(t_k))]^\alpha,0\big) \\ &\leq \int_{t_0}^t d([f(s,x_0)]^\alpha,0)ds + d\left([I_k(x(t_k))]^\alpha,0\right) \end{aligned} \tag{3.7}$$

and by the definition of D, we get

$$D(x_1(t),x_0) \leq M|t-t_0| + \chi \leq M\delta + \chi = b + \chi \tag{3.8}$$

Now, assume that $x_{n-1}(t)$ is levelwise continuous on $|t-t_0| \leq \delta$, and that

$$D(x_{n-1}(t),x_0) \leq M|t-t_0| + \chi \leq M\delta + \chi = b + \chi \tag{3.9}$$

From (3.3), we deduce that $x_n(t)$ is levelwise continuous on $|t-t_0| \leq \delta$ and that

$$D(x_n(t),x_0) \leq M|t-t_0| + \chi \leq M\delta + \chi = b + \chi \tag{3.10}$$

Consequently, we conclude that $\{x_n(t)\}$ consists of levelwise continuous mappings on $|t-t_0| \leq \delta$, and

$$(t, x_n(t)) \in J_0, \quad |t-t_0| \leq \delta,\ n = 1, 2, 3... \tag{3.11}$$

Let us prove that there exists a fuzzy set valued mapping $x : I \to E^d$ such that $D(x_n(t), x(t)) \to 0$ uniformly on $|t-t_0| \leq \delta$ as $n \to \infty$.

For n=2, from (3.3),

$$x_2(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))ds + \sum_{0<t<t_k} I_k(x(t_k)). \tag{3.12}$$

From (3.6) and (3.12) , we have

$$\begin{aligned} d([x_2(t)]^\alpha, [x_1(t)]^\alpha) &= d\left(\left[\int_{t_0}^{t} f(s, x_1(s))ds\right]^\alpha, \left[\int_{t_0}^{t} f(s, x_0(s))ds\right]^\alpha\right) \\ &\quad + d([I_k(x_2(t_k))]^\alpha, [I_k(x_1(t_k))]^\alpha) \\ &\leq \int_{t_0}^{t} d([f(s, x_1(s))]^\alpha, [f(s, x_0)]^\alpha)ds \\ &\quad + d([I_k(x_2(t_k))]^\alpha, [I_k(x_1(t_k))]^\alpha) \end{aligned} \tag{3.13}$$

for any $\alpha \in [0, 1]$. According to condition (3.4), we obtain

$$\begin{aligned} d([x_2(t)]^\alpha, [x_1(t)]^\alpha) &\leq \int_{t_0}^{t} Ld([x_1(s)]^\alpha, [x_0]^\alpha)ds \\ &\quad + d([I_k(x_2(t_k))]^\alpha, [I_k(x_1(t_k))]^\alpha) \end{aligned} \tag{3.14}$$

and by the definition of D, we obtain

$$D(x_2(t), x_1(t)) \leq L \int_{t_0}^{t} D(x_1(s), x_0(s))ds + \kappa. \tag{3.15}$$

Now, we can apply the first inequality (3.8) in the right hand side of (3.15) to get

$$D(x_2(t), x_1(t)) \leq ML\frac{|t-t_0|^2}{2!} + \kappa \leq ML\frac{\delta^2}{2!} + \kappa. \tag{3.16}$$

Starting from (3.8) and (3.16), assume that

$$D(x_n(t), x_{n-1}(t)) \leq ML^{n-1}\frac{|t-t_0|^n}{n!} + \kappa \leq ML^{n-1}\frac{\delta^n}{n!} + \kappa, \tag{3.17}$$

and let us prove that such an inequality holds for $D(x_{n+1}(t), x_n(t))$.

Indeed, from (3.3) and condition (3.4), it follows that

$$
\begin{aligned}
d([x_{n+1}(t)]^\alpha, [x_n(t)]^\alpha) &= d\left(\left[\int_{t_0}^t f(s, x_n(s))ds\right]^\alpha, \left[\int_{t_0}^t f(s, x_{n-1}(s))ds\right]^\alpha\right) \\
&\quad + d([I_k(x(t_k))]^\alpha, [I_k(y(t_k))]^\alpha) \\
&\le \int_{t_0}^t d([f(s, x_n(s))]^\alpha, [f(s, x_{n-1}(s))]^\alpha)ds \\
&\quad + d([I_k(x_2(t_k))]^\alpha, [I_k(x_1(t_k))]^\alpha) \\
&\le \int_{t_0}^t Ld([x_n(s)]^\alpha, [x_{n-1}(s)]^\alpha)ds \\
&\quad + d([I_k(x_2(t_k))]^\alpha, [I_k(x_1(t_k))]^\alpha)
\end{aligned} \tag{3.18}
$$

for any $\alpha \in [0, 1]$ and from the definition of D, we have

$$
D(x_{n+1}(t), x_n(t)) \le L \int_{t_0}^t D([x_n(s)]^\alpha, [x_{n-1}(s)]^\alpha)ds + \kappa. \tag{3.19}
$$

According to (3.17), we get

$$
\begin{aligned}
D(x_{n+1}(t), x_n(t)) &\le ML^n \int_{t_0}^t \frac{|s - t_0|^n}{n!} ds + \kappa \\
&= ML^n \frac{|t - t_0|^{n+1}}{(n+1)!} + \kappa \le ML^n \frac{\delta^{n+1}}{(n+1)!} + \kappa.
\end{aligned} \tag{3.20}
$$

Consequently, inequality (3.17) holds for $n = 1, 2....$ We can also write

$$
D(x_n(t), x_{n-1}(t)) \le \frac{M}{L} . \frac{(L\delta)^n}{n!} + \kappa \tag{3.21}
$$

for $n = 1, 2...$, and $|t - t_0| \le \delta$.

Let us mention now that

$$
x_n(t) = x_0 + [x_1(t) - x_0] + ... + [x_n(t) - x_{n-1}(t)], \tag{3.22}
$$

which implies that the sequence $\{x_n(t)\}$ and the series

$$
x_0 + \sum_{n-1}^{\infty} [x_n(t) - x_{n-1}(t)] \tag{3.23}
$$

have the same convergence properties.

From (3.21), according to the convergence criterion of Weierstrass, it follows that the series having the general term $x_n(t) - x_{n-1}(t)$, so $D(x_n(t), x_{n-1}(t)) \to 0$ uniformly on $|t - t_0| \le \delta$ as $n \to \infty$.

Hence, there exists a fuzzy set -valued mapping $x : I \to E^d$ such that $D(x_n(t), x(t)) \to 0$ uniformly on $|t - t_0| \le \delta$ as $n \to \infty$.

From (3.4), we get

$$d\Big([f(t,x_n(t))]^\alpha,[f(t,x(t))]^\alpha\Big)\le Ld\Big([x_n(t)]^\alpha,[x(t)]^\alpha\Big) \tag{3.24}$$

for any $\alpha\in[0,1]$. By the definition of D,

$$D\Big(f(t,x_n(t)),f(t,x(t))\Big)\le LD(x_n(t)),x(t))\to 0 \tag{3.25}$$

uniformly on $|t-t_0|\le\delta$ as $n\to\infty$.

Taking (3.25) in to account, from (3.3), we obtain, for $n\to\infty$,

$$x(t)=x_0+\int_{t_0}^t f(s,x(s))ds+\sum_{0<t<t_k} I_k(x(t_k)). \tag{3.26}$$

Consequently, there is at least one levelwise continuous solution of (1.1).

We want to prove now that this solution is unique, that is, from

$$y(t)=x_0+\int_{t_0}^t f(s,y(s))ds+\sum_{0<t<t_k} I_k(y(t_k)) \tag{3.27}$$

on $|t-t_0|\le\delta$, it follows that $D\Big(x(t),y(t)\Big)\equiv 0$. Indeed, from (3.3) and (3.27), we obtain

$$\begin{aligned} d([y(t)]^\alpha,[x_n(t)]^\alpha) &= d\left(\left[\int_{t_0}^t f(s,y(s))ds\right]^\alpha,\left[\int_{t_0}^t f(s,x_{n-1}(s))ds\right]^\alpha\right)\\ &\quad + d([I_k(y(t_k))]^\alpha,[I_k(x_n(t_k))]^\alpha)\\ &\le \int_{t_0}^t d([f(s,y(s))ds]^\alpha,[f(s,x_{n-1}(s))ds]^\alpha)ds\\ &\quad + d([I_k(y(t_k))]^\alpha,[I_k(x_n(t_k))]^\alpha)\\ &\le \int_{t_0}^t Ld([y(s)]^\alpha,[x_{n-1}(s)]^\alpha)ds\\ &\quad + d([I_k(y(t_k))]^\alpha,[I_k(x_n(t_k))]^\alpha)\end{aligned} \tag{3.28}$$

for any $\alpha\in[0,1], n=1,2....$

By the definition of D, we obtain

$$D\Big(y(t),x_n(t)\Big)\le L\int_{t_0}^t D\Big(y(s),x_{n-1}(s)\Big)ds+\kappa,\quad n=1,2..., \tag{3.29}$$

But $D(y(t),x_0)\le b$ on $|t-t_0|\le\delta$, $y(t)$ being a solution of (3.27). It follows from (3.29) that

$$D\Big(y(t),x_1(t)\Big)\le bL|t-t_0|+\kappa \tag{3.30}$$

on $|t - t_0| \leq \delta$. Now, assume that

$$D(y(t), x_n(t)) \leq bL^n \frac{|t - t_0|^n}{n!} + \kappa \tag{3.31}$$

on the interval $|t - t_0| \leq \delta$. From

$$D(y(t), x_{n+1}(t)) \leq L \int_{t_0}^{t} D(y(s), x_n(s))ds + \kappa \tag{3.32}$$

and (3.31), one obtains

$$D\big(y(t), x_{n+1}(t)\big) \leq bL^{n+1} \frac{|t - t_0|^{n+1}}{(n+1)!} + \kappa \tag{3.33}$$

Consequently, (3.31) holds for any n, which leads to the conclusion

$$D(y(t), x_n(t)) = D(x(t), x_n(t)) \to 0 \tag{3.34}$$

on the interval $|t - t_0| \leq \delta$ as $n \to \infty$. This proves the uniqueness of the solution for (1.1). ∎

Definition 3.2. A mapping $x : L \to E^d$ is an ϵ-approximate solution of (1.1) if the following properties hold

(a) $x(t)$ is levelwise continuous on $|t - t_0| \leq \delta$,

(b) the derivative $x'(t)$ exists and it is levelwise continuous,

(c) for all t for which $x'(t)$ is defined,we have

$$D\Big(x'(t), f(t, x(t))\Big) < \epsilon. \tag{3.35}$$

Theorem 3.2. *A mapping $f : J_0 \to E^d$ is levelwise continuous, and let $\epsilon > 0$ be arbitrary.Then there exists at least one ϵ-approximate solution of* (1.1)*, defined on $|t - t_0| \leq \delta = min\{a, b/M\}$, where $M = D(f(t, x), \hat{0}), \hat{0} \in E^d$ and for any $(t, x) \in J_0$.*

Proof. In as much as a mapping $f : J_0 \to E^d$ is a levelwise continuous on a compact set J_0, it follows that $f(t, x)$ is uniformly levelwise continuous.

Consequently, for any $\alpha \in [0, 1]$, we can find $\delta > 0$ such that

$$d([f(t, x)]^\alpha, [f(s, y)]^\alpha) < \epsilon.$$

Now, we construct the approximate solution for $t \in [t_0, t_0+\delta]$, the construction being completely similar for $t \in [t_0 - \delta, t_0]$.

Let us consider a division

$$t_0 < t_1 < ... < t_n = t_0 + \delta \tag{3.36}$$

of $[t_0, t_0+\delta]$ such that

$$\max_k (t_k - t_{k-1}) < \lambda = min\left\{\delta, \frac{\delta}{M}\right\}. \tag{3.37}$$

We define a mapping $x : I \to E^d$ as follows

$$x(t_0) = x_0, \tag{3.38}$$

$$x(t) = x(t_k) + f(t_k, x(t_k))(t - t_k) \tag{3.39}$$

on $t_k < t < t_{k+1}, k = 0, 1, ...n-1$. It is obvious that a mapping $x : I \to E^d$ satisfies the first two properties from the definition of an ϵ-approximate solution.

Now, we want to prove that the last property is also fulfilled.

Indeed, $x'(t) = f(t_k, x(t_k))$ on (t_k, t_{k+1}) and for any $\alpha \in [0,1]$,

$$d\Big([x'(t)]^\alpha, [f(t, x(t))]^\alpha\Big) = d\Big(\big[f\big(t_k, x(t_k)\big)\big]^\alpha, \big[f(t, x(t))\big]^\alpha\Big) < \epsilon \tag{3.40}$$

since $|t - t_k| < \lambda \leq \delta$,

$$d\Big([x(t)]^\alpha, [x(t_k)]^\alpha\Big) \leq d\Big([f\big(t_k, x(t_k)\big)]^\alpha, 0\Big)|t - t_k| < M\lambda \leq \delta. \tag{3.41}$$

Thus, by the definition of D, we have

$$D\Big(x'(t), f(t, x(t))\Big) < \epsilon \tag{3.42}$$

on $|t - t_0| < \delta$ and $(t, x) \in J_0$. Since I_k is a bounded function, we know that the theorem (3.2) holds. ∎

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