# SOME STRUCTURAL PROPERTIES OF HYPER KS-SEMIGROUPS

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**Abstract.** This study introduces a new class of algebra related to hyper BCK-algebras and semihypergroups, called hyper KS-semigroups. It presents some characterizations of a hyper KS-semigroup with respect to its hyper subKS-semigroups, hyper KS-ideals, and reflexive hyper KS-ideals and their relationships. A quotient structure is constructed from a hyper KS-semigroup via a reflexive hyper KS-ideal and some properties are established. This paper also shows some properties of hyper KS-semigroups homomorphism and specifically, the three isomorphism theorems for hyper KS-semigroups are proved. Moreover, this paper shows that the hyper product of any nonempty finite family of hyper KS-semigroups is also a hyper KS-semigroup and investigates some related properties.

**Keywords:** hyper KS-semigroup, hyper subKS-semigroup, hyper KS-ideal, reflexive hyper KS-ideal, hyper P-ideal.

**2000 Mathematics Subject Classification:** 20N20, 06F35.

## 1. Introduction

In 1966, Y. Imai and K. Iseki introduced a class of algebra with one binary operation called BCK-algebra in their paper entitled "On Axiom systems of Propositional Calculi XIV" [7]. This algebra is a generalization of the concept of set-theoretic difference and propositional calculi. On the other hand, K. H. Kim [11]

[1]Supported by a grant from DOST-ASTHRDP.

introduced a new class of algebra called KS-semigroups, which is a combination of BCK-algebra and semigroup. He characterized the KS-semigroups from its ideal up to the first isomorphism theorem. Cawi, in her masteral thesis [3], proved the second and third isomorphism theorems for KS-semigroups and gave other characterizations parallel to ring theory.

Hyperstructure theory (also called multivalued algebras) was introduced by F. Marty at the 8th congress of Scandinavian Mathematicians in 1934. Recall that in a classical algebraic structure, the image of two elements of a set is an element of the set, while in an algebraic hyperstructure, the image of two elements is a set. This theory has been studied by many mathematicians and several books have been written in this topic, for example see [4], [5], [6]. It is considered as a generalization of classical algebraic structures. In this paper, we apply the theory of hyperstructures to KS-semigroups and provide some characterizations.

## 2. Preliminaries

Let $H$ be a non-empty set endowed with a *hyper operation* " $*$ " that is, " $*$ " is a function from $H \times H$ to $P^*(H) = P(H) \setminus \{\varnothing\}$. For two nonempty subsets $A$ and $B$ of $H$, $A * B = \bigcup_{a \in A, b \in B} a * b$. We shall use $x * y$ instead of $x * \{y\}, \{x\} * y$, or $\{x\} * \{y\}$. When $A$ is a nonempty subset of H and $x \in H$, we agree to write $A * x$ instead of $A * \{x\}$. Similarly, we write $x * A$ for $\{x\} * A$. In effect, $A * x = \bigcup_{a \in A} a * x$ and $x * A = \bigcup_{a \in A} x * a$.

**Definition 2.1** [8] A *semihypergroup* is a hypergroupoid $(H, \cdot)$ such that for all $x, y, z \in H$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, that is, $\bigcup_{u \in x \cdot y} u \cdot z = \bigcup_{v \in y \cdot z} x \cdot v$.

**Definition 2.2** [9] A *hyper BCK-algebra* is a nonempty set $H$ endowed with a hyperoperation " $*$ " and a constant 0 satisfying the following axioms: for all $x, y, z \in H$,

(H1) $(x * z) * (y * z) < x * y$,

(H2) $(x * y) * z = (x * z) * y$,

(H3) $x * H < x$,

(H4) $x < y$ and $y < x$ imply $x = y$,

where (a) $x < y$ is defined by $0 \in x * y$, and (b) for every $A, B \subseteq H$, $A < B$ is defined as follows: for all $a \in A$, there exists $b \in B$ such that $a < b$. In such case, we call " $<$ " the *hyper order* in $H$.

**Theorem 2.3** [9] *In any hyper BCK-algebra* $(H, *, 0)$, *the following hold: for any* $x, y, z \in H$ *and for any nonempty subsets* $A, B$ *of* $H$, (a1) $x < x$ *and* (a2) $A \subseteq B$ *implies* $A < B$.

**Definition 2.4** [9] Let $I$ be a nonempty subset of a hyper BCK-algebra $(H, *, 0)$. Then $I$ is said to be a *hyper BCK-ideal* of $H$ if $0 \in I$ and $x * y < I$ and $y \in I$ imply $x \in I$ for all $x, y \in H$. If additionally $x * x \subseteq I$ for all $x \in H$, then $I$ is said to be a *reflexive hyper BCK-ideal* of $H$.

**Lemma 2.5** [10] *If* $I$ *is a reflexive hyper BCK-ideal of a hyper BCK-algebra* $H$, *then* $(x * y) \cap I \neq \varnothing$ *implies* $x * y \subseteq I$ *for all* $x, y \in H$.

**Theorem 2.6** [12] *Let* $f : (H_1, *_1, 0_1) \to (H_2, *_2, 0_2)$ *be a homomorphism of hyper BCK-algebras. Then* $H_1/Kerf \cong Imf$.

**Theorem 2.7** [12] *Let* $I$ *be a reflexive hyper BCK-ideal and* $J$ *be a hyper BCK-ideal of* $H$ *such that* $I \subseteq J$. *Then* $(H/I)/(J/I) \cong H/J$.

## 3. Hyper SubKS-semigroups, Hyper KS-Ideals and Hyper P-ideals

In this section, we introduce the concept of hyper KS-semigroup and we study the notions of hyper subKS-semigroups, hyper KS-ideals and hyper $P$-ideals of a hyper KS-semigroup and investigate some of their properties.

**Definition 3.1** A *hyper KS-semigroup* is a nonempty set $H$ together with two hyperoperations "$*$" and "$\cdot$" and a constant 0 satisfying the following conditions:

(i) $(H, *, 0)$ is a hyper BCK-algebra.

(ii) $(H, \cdot)$ is a semihypergroup having zero as a bilaterally absorbing element, that is, $x \cdot 0 = 0 \cdot x = \{0\}$ for all $x \in H$; and

(iii) "$\cdot$" is left and right distributive over "$*$", that is, for any $x, y, z \in H$

$$x \cdot (y * z) = (x \cdot y) * (x \cdot z) \text{ and } (x * y) \cdot z = (x \cdot z) * (y \cdot z).$$

From now on, a hyper KS-semigroup $(H, *, \cdot, 0)$ shall be denoted by $H$ and for all $x, y \in H$, we agree to write $x \cdot y$ as $xy$.

**Example 3.2** Let $H = \{0, 1, 2\}$. Define the operation "$*$" and "$\cdot$" by the Cayley's table shown below. Then by routine calculations, $H$ is a hyper KS-semigroup.

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0,1\}$ | $\{0,1\}$ |
| 2 | $\{2\}$ | $\{1,2\}$ | $\{0,1,2\}$ |

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{0\}$ | $\{1\}$ | $\{0,1\}$ |
| 2 | $\{0\}$ | $\{0,1\}$ | $\{0,1,2\}$ |

**Lemma 3.3** *Let $A, B$ and $C$ be nonempty subsets of a hyper KS-semigroup $H$. Then $A < B$ and $B \subseteq C$ imply $A < C$.*

**Definition 3.4** Let $I$ be a nonempty subset of a hyper KS-semigroup $H$. Then $I$ is said to be a *hyper subKS-semigroup* of $H$ if for all $x, y \in I$, $x * y \subseteq I$ and $xy \subseteq I$. $I$ is a *hyper left* (resp. *hyper right*) *stable* if $xa \subseteq I$ (resp. $ax \subseteq I$) for all $x \in H$ and for all $a \in I$. $I$ is a *hyper stable* if $I$ is both hyper left and right stable. $I$ is a *hyper left* (resp. *hyper right*) *KS-ideal* if $I$ is a hyper left (resp. hyper right) stable and for any $x, y \in H$, $x * y < I$ and $y \in I$ imply that $x \in I$. $I$ is a *hyper KS-ideal* if $I$ is both a hyper left and a hyper right KS-ideal. $I$ is said to be *reflexive* if for all $x \in H, x * x \subseteq I$ and $xx \subseteq I$.

**Example 3.5** Consider the hyper KS-semigroup $H = \{0, 1, 2\}$ in Example 3.2 and $I = \{0, 1\}$. By routine calculations, for all $x, y \in I$ such that $x * y < I$ and $y \in I$ imply $x \in I$ and for all $a \in I$, $xa, ax \subseteq I$. Hence, $I$ is a hyper KS-ideal of $H$.

The following remark follows from Definition 3.4.

**Remark 3.6** For any hyper KS-semigroup $H$, the following hold.

(i) $\{0\}$ and $H$ are hyper KS-ideals of $H$.

(ii) $0 \in I$ for any hyper KS-ideal $I$ of $H$.

(iii) Every hyper KS-ideal is a hyper BCK-ideal.

**Proposition 3.7** *Every hyper KS-ideal of a hyper KS-semigroup is a hyper subKS-semigroup.*

The converse of Proposition 3.7 (ii) may not be true in general. Consider the following example.

**Example 3.8** Let $H = \{0, 1, 2\}$. Define the operation "$*$" and "$\cdot$" by the Cayley's table shown below. Then by routine calculations, $H$ is a hyper KS-semigroup.

| $*$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{1\}$ | $\{0,1\}$ | $\{0,1\}$ |
| 2 | $\{2\}$ | $\{1,2\}$ | $\{0,1,2\}$ |

| $\cdot$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\{0\}$ | $\{0\}$ | $\{0\}$ |
| 1 | $\{0\}$ | $\{1\}$ | $\{0,1,2\}$ |
| 2 | $\{0\}$ | $\{0,1\}$ | $\{0,2\}$ |

Consider $I = \{0, 1\}$. Observe that $I$ is a hyper subKS-semigroup but $I$ is not hyper right stable since $1 \cdot 2 = \{0, 1, 2\} \nsubseteq I$.

**Lemma 3.9** *Let $A$ and $B$ be nonempty subsets of a hyper KS-semigroup $H$ and $I$ be a hyper KS-ideal of $H$. Then $B \subseteq I$ implies $A \cdot B \subseteq I$ and $B \cdot A \subseteq I$.*

**Theorem 3.10** *Let $\{A_i : i \in \mathscr{I}\}$ be a nonempty collection of subsets of a hyper KS-semigroup $H$.*

(i) *If* $A_i$ *is a hyper KS-ideal of* $H$ *for all* $i \in \mathscr{I}$*, then so is* $\bigcap_{i\in\mathscr{I}} A_i$.

(ii) *If* $A_j$ *is a hyper subKS-semigroup where* $j \in \mathscr{I}$ *and* $A_{i_0}$ *is a hyper KS-ideal of* $H$ *for some* $i_0 \in \mathscr{I}$*, then* $\bigcap_{i\in\mathscr{I}} A_i$ *is a hyper KS-ideal of* $A_j$*,* $j \neq i_0$*. Moreover, if* $A_{i_0}$ *is reflexive in* $H$*, then* $\bigcap_{i\in\mathscr{I}} A_i$ *is reflexive in* $A_j$.

**Proof.** Let $\{A_i : i \in \mathscr{I}\}$ be a nonempty collection of subsets of $H$.

(i) Suppose that $A_i$ is a hyper KS-ideal of $H$ for all $i \in \mathscr{I}$. Then $0 \in A_i$ for all $i \in \mathscr{I}$ and so $0 \in \bigcap_{i\in\mathscr{I}} A_i$. Thus, $\bigcap_{i\in\mathscr{I}} A_i \neq \varnothing$. Clearly, $\bigcap_{i\in\mathscr{I}} A_i \subseteq A_i$ for all $i \in \mathscr{I}$. Let $a \in \bigcap_{i\in\mathscr{I}} A_i$ and $x \in H$. Then $a \in A_i$ for all $i \in \mathscr{I}$. Since $A_i$ is hyper stable in $H$ for all for all $i \in \mathscr{I}$, it follows that $xa, ax \subseteq A_i$ for all $i \in \mathscr{I}$. Thus, $xa, ax \subseteq \bigcap_{i\in\mathscr{I}} A_i$ and so $\bigcap_{i\in\mathscr{I}} A_i$ is hyper stable in $H$. Suppose that $x, y \in H$ such that $x * y < \bigcap_{i\in\mathscr{I}} A_i$ and $y \in \bigcap_{i\in\mathscr{I}} A_i$. Since $\bigcap_{i\in\mathscr{I}} A_i \subseteq A_i$ for all $i \in \mathscr{I}$, it follows by Lemma 3.3 that $x * y < A_i$ for all $i \in \mathscr{I}$. Also, $y \in A_i$ for all $i \in \mathscr{I}$. Hence, $A_i$ hyper KS-ideals for all $i \in \mathscr{I}$ imply $x \in A_i$ for all $i \in \mathscr{I}$. Therefore, $\bigcap_{i\in\mathscr{I}} A_i$ is a hyper KS-ideal of $H$.

(ii) Suppose that $A_j$ is a hyper subKS-semigroup where $j \in \mathscr{I}$ and $A_{i_0}$ is a hyper KS-ideal of $H$ for some $i_0 \in \mathscr{I}$. Since $0 \in A_i$ for all $i \in \mathscr{I}$, $0 \in \bigcap_{i\in\mathscr{I}} A_i$ and so $\bigcap_{i\in\mathscr{I}} A_i \neq \varnothing$. Clearly, $\bigcap_{i\in\mathscr{I}} A_i \subseteq A_i$ for all $i \in \mathscr{I}$. Let $x \in A_j$ where $j \neq i_0$ and $a \in \bigcap_{i\in\mathscr{I}} A_i$. Then $a \in A_i$ for all $i \in \mathscr{I}$. In particular, $a \in A_j$. Since $A_j$ is a hyper subKS-semigroup, $xa, ax \subseteq A_j$. Also, since $A_{i_0}$ is hyper stable in $H$, it follows that $xa, ax \subseteq A_{i_0}$. Thus, $xa, ax \subseteq A_i$ for all $i \in \mathscr{I}$ and so $xa, ax \subseteq \bigcap_{i\in\mathscr{I}} A_i$. Hence, $\bigcap_{i\in\mathscr{I}} A_i$ is hyper stable in $A_j$. Suppose that $x, y \in A_j$ where $j \neq i_0$ such that $x * y < \bigcap_{i\in\mathscr{I}} A_i$ and $y \in \bigcap_{i\in\mathscr{I}} A_i$. Then $y \in A_i$ for all $i \in \mathscr{I}$. Since $\bigcap_{i\in\mathscr{I}} A_i \subseteq A_i$ for all $i \in \mathscr{I}$, it follows by Lemma 3.3 that $x * y < A_i$ for all $i \in \mathscr{I}$. If $i = i_0$, then $x \in A_{i_0}$ since $A_{i_0}$ hyper KS-ideal of $H$. Note that by assumption, $x \in A_j$ and so $x \in A_i$ for all $i \in \mathscr{I}$. Hence, $x \in \bigcap_{i\in\mathscr{I}} A_i$. Thus, $\bigcap_{i\in\mathscr{I}} A_i$ is a hyper KS-ideal of $A_j$. Moreover, if $A_{i_0}$ is reflexive in $H$, then for all $x \in A_j$, $x * x, xx \subseteq A_{i_0}$ and since $A_j$ is a hyper subKS-semigroup, $x * x, xx \subseteq A_j$ and so $x * x, xx \subseteq A_i$ for all $i \in \mathscr{I}$. Therefore, $x * x, xx \subseteq \bigcap_{i\in\mathscr{I}} A_i$ implying that $\bigcap_{i\in\mathscr{I}} A_i$ is reflexive in $A_j$. ■

**Definition 3.11** Let $\theta$ be an equivalence relation on a hyper KS-semigroup $H$ and $A, B$ be nonempty subsets of $H$.

(i) $A\overline{\theta}B$ means that, for all $a \in A$ there exists $b \in B$ such that $a\theta b$ and for all $b \in B$ there exists $a \in A$ such that $a\theta b$,

(ii) $\theta$ is said to be a *congruence relation* on $H$ if for all $x, y, u, v \in H$, $x\theta y$ and $u\theta v$ imply that $(x * u)\overline{\theta}(y * v)$ and $(xu)\overline{\theta}(yv)$.

From [13], the relation " $\sim_I$ " on H is defined by $x \sim_I y$ if and only if $x * y \subseteq I$ and $y * x \subseteq I$ for all $x, y \in H$. Then, $\sim_I$is an equivalence relation on H. Also, the relation " $\sim_I$ " on $P^*(H)$ is defined by $A \eqsim_I B$ if and only if for all $a \in A$ there exists $b \in B$ such that $a \sim_I b$, and for all $b \in B$ there exists $a \in A$ such that $a \sim_I b$, for all $A, B \in P^*(H)$. Then, $\eqsim_I$ is an equivalence relation on $P^*(H)$, where $I$ is a reflexive hyper KS-ideal of $H$. In [12], it was proved that $x\theta y$ and $u\theta v$ imply $(x * u)\overline{\theta}(y * v)$ for all $x, y, u, v \in H$, (i) $I = I_0$ and (ii) $I_x = I_y$ if and only if $x \sim_I y$.

**Theorem 3.12** *The relation $\sim_I$ is a congruence relation on a hyper KS-semigroup $H$.*

**Proof.** Let $x, y, u, v \in H$ such that $x \sim_I y$ and $u \sim_I v$. Then $x * y, y * x$, $u * v, v * u \subseteq I$. We only need to show that $xu \eqsim_I yv$. Let $a \in xu$ and $b \in yv$. Then

$$a * b \subseteq xu * xv = x(u * v) \subseteq I \text{ and } b * a \subseteq xv * xu = x(v * u) \subseteq I.$$

Thus, $a \sim_I b$ and so $xu \eqsim_I xv$. Also, let $c \in xv$ and $d \in yv$. Then $c*d \subseteq xv*yv = (x * y)v \subseteq I$ and $d * c \subseteq yv * xv = (y * x)v \subseteq I$. Thus, $c \sim_I d$ and so $xv \eqsim_I yv$. Since $\eqsim_I$ is transitive, $xu \eqsim_I yv$.

Therefore, $\sim_I$ is a congruence relation on $H$. ■

In [12], if $(H, *, 0)$ is a hyper BCK-algebra and $I$ is a reflexive hyper BCK-ideal, then $(H/I, \circledast, I_0)$ is a hyper BCK-algebra under the hyperoperation " $\circledast$ " and hyper order " $\ll$ " which is defined as $I_x \circledast I_y = \{I_z : z \in x * y\}$ and $I_x \ll I_y \Leftrightarrow I_0 \in I_x \circledast I_y$ where $I_x = \{y \in H : x \sim_I y\}$ and $H/I = \{I_x : x \in H\}$.

**Theorem 3.13** *Let $I$ be a reflexive hyper KS-ideal of a hyper KS-semigroup $H$. Define the hyperoperation "$\odot$" on $H$ by $I_x \odot I_y = \{I_z : z \in xy\}$ for all $I_x, I_y \in H/I$. Then $(H/I, \circledast, \odot, I_0)$ is a hyper KS-semigroup.*

**Proof.** We only need to show that $(H/I, \odot)$ is a semihypergroup such that $I_0$ is a bilaterally absorbing element and that " $\odot$ " is left and right distributive over " $\circledast$ ". First, we show that the hyperoperation " $\odot$ " on $H/I$ is well-defined. Let

$x, x', y, y' \in H$ such that $I_x = I_{x'}$ and $I_y = I_{y'}$. Then $x \sim_I x'$ and $y \sim_I y'$. Since $\sim_I$ is a congruence relation, $xx' \overline{\approx}_I yy'$, it follows that for all $z \in xx'$, there exists $w \in yy'$ such that $z \sim_I w$ and for all $t \in yy'$, there exists $s \in xx'$ such that $t \sim_I s$. Thus, $I_z = I_w \in I_y \odot I_{y'}$ implying that $z \in yy'$ and also $I_t = I_s \in I_x \odot I_{x'}$ implying that $t \in xx'$. Hence, $I_x \odot I_{x'} \subseteq I_y \odot I_{y'}$ and $I_y \odot I_{y'} \subseteq I_x \odot I_{x'}$. Therefore, $I_x \odot I_{x'} = I_y \odot I_{y'}$ and so "$\odot$" is well-defined. Now, we show that $H/I$ is associative.

Let $I_x, I_y, I_z \in H/I$ and let $I_w \in (I_x \odot I_y) \odot I_z$. Then $I_w \in I_u \odot I_z$ for some $u \in xy$. Thus, $w \in uz \subseteq (xy)z = x(yz)$ and so $I_w \in I_x \odot (I_y \odot I_z)$. Hence, $(I_x \odot I_y) \odot I_z \subseteq I_x \odot (I_y \odot I_z)$. Similarly, let $I_s \in I_x \odot (I_y \odot I_z)$. Then $I_s \in I_x \odot I_t$ for some $t \in yz$. Thus, $s \in xt \subseteq x(yz) = (xy)z$ and so $I_s \in (I_x \odot I_y) \odot I_z$. Hence, $I_x \odot (I_y \odot I_z) \subseteq (I_x \odot I_y) \odot I_z$. Therefore, $(I_x \odot I_y) \odot I_z = I_x \odot (I_y \odot I_z)$ and so $(H/I, \odot)$ is a semihypergroup. Moreover, for all $I_x \in H/I$,

$$I_x \odot I_0 = \{I_z : z \in x0 = \{0\}\} = \{I_0\} = \{I_z : z \in 0x = \{0\}\} = I_0 \odot I_x.$$

Furthermore, let $I_w \in I_x \odot (I_y \circledast I_z)$. Then $I_w \in I_x \odot I_u$ for some $u \in y * z$. By Definition 3.1 (iii), $w \in xu \subseteq x(y * z) = xy * xz$. This implies that $I_w \in (I_x \odot I_y) \circledast (I_x \odot I_z)$ and so $I_x \odot (I_y \circledast I_z) \subseteq (I_x \odot I_y) \circledast (I_x \odot I_z)$. On the other hand, let $I_{w'} \in (I_x \odot I_y) \circledast (I_x \odot I_z)$. Then $I_{w'} \in I_{u'} \circledast I_{v'}$ for some $u' \in xy$ and for some $v' \in xz$. By Definition 3.1 (iii), $w' \in u' * v' \subseteq xy * xz = x(y * z)$. Hence, $I_{w'} \in I_x \odot (I_y \circledast I_z)$ and so $(I_x \odot I_y) \circledast (I_x \odot I_z) \subseteq I_x \odot (I_y \circledast I_z)$. Therefore, $(I_x \odot I_y) \circledast (I_x \odot I_z) = I_x \odot (I_y \circledast I_z)$ and so "$\odot$" is left distributive over "$\circledast$".

Finally, let $I_w \in (I_y \circledast I_z) \odot I_x$. Then there exists $u \in y * z$ such that $I_w \in I_u \odot I_x$. By Definition 3.1 (iii), $w \in ux \subseteq (y * z)x = yx * zx$. Hence, $I_w \in (I_y \odot I_x) \circledast (I_z \odot I_x)$ and so $(I_y \circledast I_z) \odot I_x \subseteq (I_y \odot I_x) \circledast (I_z \odot I_x)$. On the other hand, let $I_{w'} \in (I_y \odot I_x) \circledast (I_z \odot I_x)$. Then $I_{w'} \in I_{u'} \circledast I_{v'}$ for some $u' \in yx$ and for some $v' \in zx$. By Definition 3.1 (iii), $w' \in u' * v' \subseteq yx * zx = (y * z)x$. Hence, $I_{w'} \in (I_y \circledast I_z) \odot I_x$ and so $(I_y \odot I_x) \circledast (I_z \odot I_x) \subseteq (I_y \circledast I_z) \odot I_x$. Therefore, $(I_y \odot I_x) \circledast (I_z \odot I_x) = (I_y \circledast I_z) \odot I_x$ and so "$\odot$" is right distributive over "$\circledast$".

Therefore, $(H/I, \circledast, \odot, I_0)$ is a hyper KS-semigroup. ∎

## 4. Hyper KS-semigroup Homomorphism

In this section, we study the concept of hyper KS-semigroup homomorphism. A hyper KS-semigroup homomorphism is a function between two hyper KS-semigroups which respects the hyperoperations. More precisely:

**Definition 4.1** Let $(H_1, *_1, \cdot_1, 0_1)$ and $(H_2, *_2, \cdot_2, 0_2)$ be hyper KS-semigroups and $f : H_1 \rightarrow H_2$ be a map. Then $f$ is called a *hyper KS-semigroup homomorphism* if $f(x *_1 y) = f(x) *_2 f(y)$ and $f(x \cdot_1 y) = f(x) \cdot_2 f(y)$ for all $x, y \in H_1$.

**Theorem 4.2** *Let $f : G \rightarrow H$ be a hyper KS-semigroup homomorphism. Then $Ker\, f$ is a hyper KS-ideal of $G$.*

**Theorem 4.3** *Let $f : G \rightarrow H$ be a hyper KS-semigroup epimorphism and $I$ be a hyper KS-ideal of $G$ containing $Kerf$ such that $Kerf$ is reflexive. Then $f(I)$ is a reflexive hyper KS-ideal of $H$.*

**Proof.** Let $x \in H$ and $a \in f(I)$. Since $f$ is onto, there exists $y \in G$ such that $x = f(y)$. Also, $a = f(b)$ for some $b \in I$. Since $I$ is hyper stable in $G$, $yb, by \subseteq I$ and so $xa = f(y)f(b) = f(yb) \subseteq f(I)$ and $ax = f(b)f(y) = f(by) \subseteq f(I)$. Thus, $f(I)$ is hyper left and hyper right stable in $H$.

Let $a, b \in H$ such that $a*b < f(I)$ and $b \in f(I)$. Then $b = f(y)$ for some $y \in I$ and $a = f(x)$ for some $x \in G$ since $f$ is onto. Also, since $f$ is a homomorphism, $f(x*y) = f(x)*f(y) = a*b < f(I)$. Let $z \in x*y$. Then $f(z) \in f(x*y) < f(I)$ and so $f(z) < f(w)$ for some $w \in I$. This means that $0 \in f(z)*f(w) = f(z*w)$ since $f$ is a homomorphism. This implies that $0 = f(t)$ for some $t \in z*w$. Since $f(t) = 0$, $t \in Kerf$ and so $(z*w) \cap Kerf \neq \varnothing$. Since $Kerf$ is reflexive, by Lemma 2.5, $z*w \subseteq Kerf \subseteq I$. By Theorem 2.3(a2), $z*w < I$. Since $I$ is a hyper KS-ideal and $w \in I$, it follows that $z \in I$ and so $x*y \subseteq I$. By Theorem 2.3(a2), $x*y < I$. Since $I$ is a hyper KS-ideal and $y \in I$, $x \in I$. Hence, $a = f(x) \in f(I)$ and so $f(I)$ is a hyper KS-ideal of $H$.

Let $y \in H$. Then $f(x) = y$ for some $x \in G$ since $f$ is onto and $x*x, xx \subseteq Kerf \subseteq I$ since $Kerf$ is reflexive. Now, $f$ a homomorphism implies $y*y = f(x)*f(x) = f(x*x) \subseteq f(Kerf) \subseteq f(I)$ and $yy = f(x)f(x) = f(xx) \subseteq f(Kerf) \subseteq f(I)$. Hence, $f(I)$ is reflexive. ■

**Theorem 4.4** *Let $f : G \to H$ be a hyper KS-semigroup homomorphism and $I$ be a hyper KS-ideal of $H$. Then $f^{-1}(I)$ is a hyper KS-ideal of $G$. Moreover, if $I$ is reflexive in $H$, then $f^{-1}(I)$ is reflexive in $G$.*

In [12], if $I$ is a reflexive hyper BCK-ideal of a hyper BCK-algebra $H$, then there exists a canonical surjective homomorphism $\varphi : H \to H/I$ given by $\varphi(x) = I_x$ with kernel $I$.

**Theorem 4.5** *If $I$ is a reflexive hyper KS-ideal of $H$, then the mapping $\varphi : H \to H/I$ given by $\varphi(x) = I_x$ is an epimorphism with kernel $I$.*

The proof follows by showing that $\varphi(x \cdot_1 y) = \varphi(x) \cdot_2 \varphi(y)$ for all $x, y \in H$.

**Theorem 4.6** (First Isomorphism Theorem) *Let $f : G \to H$ be a hyper KS-semigroup homomorphism and $Kerf$ be a reflexive hyper KS-ideal of $G$. Then $G/Kerf \cong Imf$.*

**Proof.** Since $Kerf$ is a reflexive hyper KS-ideal and by Theorem 3.13, $G/Kerf$ is a hyper KS-semigroup. Define a mapping $\varphi : G/Kerf \to Imf$ by $\varphi(J_x) = f(x)$ for all $J_x \in G/Kerf$.

Let $J_x, J_y \in G/Kerf$ such that $J_x = J_y$. Then $x \sim_{Kerf} y$ and so $x*y, y*x \subseteq Kerf$. Now, $f$ homomorphism implies $f(x)*f(y) = f(x*y) \subseteq f(Kerf) = \{0\}$ and $f(y)*f(x) = f(y*x) \subseteq f(Kerf) = \{0\}$. Since $x*y, y*x \neq \varnothing$, it follows that $f(x)*f(y) = \{0\}$ and $f(y)*f(x) = \{0\}$, that is, $0 \in f(x)*f(y)$ and $0 \in f(y)*f(x)$. So, $f(x) < f(y)$ and $f(y) < f(x)$. By Definition 2.2(H4), $f(x) = f(y)$ and so $\varphi(J_x) = \varphi(J_y)$. Hence, $\varphi$ is well-defined.

Let $J_x, J_y \in G/Kerf$ such that $\varphi(J_x) = \varphi(J_y)$. Then $f(x) = f(y)$. By Theorem 2.3(a1), $f(x) < f(y)$ and $f(y) < f(x)$. Thus, by Definition 2.2 and since $f$ is a homomorphism, we have $0 \in f(x) * f(y) = f(x * y)$ and $0 \in f(y) * f(x) = f(y * x)$. This means that there exists $t \in x * y$ and $s \in y * x$ such that $f(t) = 0$ and $f(s) = 0$. This implies that $t, s \in Kerf$ and so $(x * y) \cap Kerf \neq \varnothing$ and $(y * x) \cap Kerf \neq \varnothing$. By Lemma 2.5, $x * y, y * x \subseteq Kerf$. Hence, $x \sim_{Kerf} y$ and so $J_x = J_y$. Therefore, $\varphi$ is one-to-one. Clearly, $\varphi$ is onto. Now, for all $J_x, J_y \in G/Kerf$, $\varphi(J_x \circledast J_y) = \{\varphi(J_t) : J_t \in J_x \circledast J_y\} = \{f(t) : t \in x * y\} = f(x * y) = f(x) * f(y) = \varphi(J_x) * \varphi(J_y)$ and $\varphi(J_x \odot J_y) = \{\varphi(J_t) : J_t \in J_x \odot J_y\} = \{f(t) : t \in xy\} = f(xy) = f(x)f(y) = \varphi(J_x)\varphi(J_y)$. Hence, $\varphi$ is an isomorphism. Therefore, $G/Kerf \cong Imf$. ▪

**Definition 4.7** Let $M$ be a nonempty subset of $H$ and $N$ be a reflexive hyper KS-ideal of $H$. Define the subset $MN$ of $H$ by

$$MN = \bigcup_{m \in M} N_m,$$

where $N_m = \{x \in H : m \sim_N x\}$.

From the preceding definition and since $\sim_N$ is reflexive, for all $m \in M$, $m \sim_N m$. Hence, $m \in N_m \subseteq \bigcup_{m \in M} N_m = MN$. Thus, we $M \subseteq MN$.

**Proposition 4.8** *Let $M$ be a nonempty subset of $H$ and $N$ be a reflexive hyper KS-ideal of $H$. If $M \approx_N N$, then $M \subseteq N$ and $MN = N$.*

**Proof.** The result follows from Definition 4.7 ▪

**Theorem 4.9** *Let $M$ be a hyper subKS-semigroup and $N$ be a reflexive hyper KS-ideal of a hyper KS-semigroup $H$. Then $MN$ is a hyper subKS-semigroup of $H$.*

**Proof.** Let $x, y \in MN$. Then $x \in N_a$ and $y \in N_b$ for some $a, b \in M$. This implies that $x \sim_N a$ and $y \sim_N b$. By Theorem 3.12, we have $x * y \approx_N a * b$ and $xy \approx_N ab$. Thus, for all $w \in x * y$, there exists $z \in a * b \subseteq M$ such that $w \sim_N z$ and for all $u \in xy$, there exists $v \in ab \subseteq M$ such that $u \sim_N v$. This means that $w \in N_z \subseteq MN$ and $u \in N_v \subseteq MN$ and so $w, u \in MN$. Hence, $x * y, xy \subseteq MN$. Therefore, $MN$ is a hyper subKS-semigroup of $H$. ▪

**Theorem 4.10** (Second Isomorphism Theorem) *Let $M$ be a hyper subKS-semigroup and $N$ be a reflexive hyper KS-ideal of $H$. Then $M/(M \cap N) \cong MN/N$.*

**Proof.** Let $M$ be a hyper subKS-semigroup and $N$ be a reflexive hyper KS-ideal of $H$. Define $\pi : H \to H/N$ by $\pi(x) = N_x$ for all $x \in H$. Define $f = \pi_{|M}$. By

Theorem 4.5, $\pi$ is a well-defined homomorphism. It follows that $f$ is also a well-defined homomorphism. By Theorem 4.6, $M/Kerf \cong Imf$. Now, by Theorem 2.3(a2) and Definition 3.4 (iv), we have

$$\begin{aligned} Kerf &= \{x \in M \mid f(x) = N_0\} = \{x \in M \mid N_x = N_0\} \\ &= \{x \in M \mid x \sim_N 0\} = \{x \in M \mid x \in N\} = M \cap N. \end{aligned}$$

Furthermore, we show that $Imf = MN/N$. Let $N_x \in Imf$. Then there exists $m \in M$ such that $f(m) = N_x$. This implies that $\pi(m) = N_m = N_x$ and so $m \sim_N x$. This means that $x \in N_m \subseteq MN$. Hence, $N_x \in MN/N$ and so $Imf \subseteq MN/N$. On the other hand, let $N_y \in MN/N$. Then $y \in MN$, that is, $y \in N_x$ for some $x \in M$. Thus, $y \sim_N x$ and so $N_y = N_x = f(x) \in Imf$. Hence, $MN/N \subseteq Imf$ and so $Imf = MN/N$. Therefore, $M/(M \cap N) \cong MN/N$. ∎

**Lemma 4.11** *Let $I$ and $J$ be hyper KS-ideals of $H$ such that $I$ is reflexive and $I \subseteq J$. Then*

(i) *$I$ is a reflexive hyper KS-ideal of the hyper subKS-semigroup $J$, and*

(ii) *the quotient hyper KS-semigroup $J/I$ is a reflexive hyper KS-ideal of $H/I$.*

**Proof.** The result follows from Definition 3.4 (v) and Proposition 3.7.

**Theorem 4.12** (Third Isomorphism Theorem) *Let $I$ and $J$ be hyper KS-ideals of $H$ such that $I$ is reflexive and $I \subseteq J$. Then $(H/I)/(J/I) \cong H/J$.*

**Proof.** Let $I$ and $J$ be hyper KS-ideals of $H$ such that $I$ is reflexive and $I \subseteq J$. Observe that for all $x \in H$, $x * x, xx \subseteq I \subseteq J$ and so it follows that $J$ is also reflexive. By Lemma 4.11, $J/I$ and $(H/I)/(J/I)$ are defined. Define the function $f : H/I \to H/J$ by $f(I_x) = J_x$. Note that by Theorem 2.7, $f$ is a well-defined homomorphism with respect to the hyperoperation " $*$ ", $Ker\ f = J/I$ and $Im\ f = H/J$. Thus, we only need to show that $f$ is a homomorphism with respect to the hyperoperation " $\cdot$ ". Let $I_x, I_y \in H/I$. Then

$$f(I_x \odot I_y) = \{f(I_t) : I_t \in I_x \odot I_y\} = \{f(I_t) : t \in xy\} = \{J_t : t \in xy\} = J_x \odot J_y.$$

Hence, $(H/I)/(J/I) \cong H/J$. ∎

## 5. Hyper Product of Hyper KS-semigroups

In this section, we develop some more of the abstract theory of hyper KS-semigroups. In particular, we show that the hyper product of any nonempty finite family of hyper KS-semigroups is also a hyper KS-semigroup.

Let $(H_1, *_1, \cdot_1, 0_1)$ and $(H_2, *_2, \cdot_2, 0_2)$ be hyper KS-semigroups with hyper orders denoted by $<_1$ and $<_2$, respectively. Define the structure $H_1 \times H_2$ by $H_1 \times H_2 = \{(a, b) : a \in H_1, b \in H_2\}$. In [2], $(H_1 \times H_2, \circledast, (0_1, 0_2))$ is a hyper

BCK-algebra where $(a_1, b_1) \circledast (a_2, b_2) = (a_1 *_1 a_2, b_1 *_2 b_2)$ and $(a_1, b_1) \ll (a_2, b_2) \Leftrightarrow a_1 <_1 a_2$ and $b_1 <_2 b_2$ for all $(a_1, b_1), (a_2, b_2) \in H_1 \times H_2$. For any sets $A \in P^*(H_1)$ and $B \in P^*(H_2)$, $(A, B) = \{(a, b) : a \in A, b \in B\}$. $H_1$ and $H_2$ shall mean the hyper KS-semigroups $(H_1, *_1, \cdot_1, 0_1)$ and $(H_2, *_2, \cdot_2, 0_2)$ with hyper orders $<_1$ and $<_2$, respectively.

**Definition 5.1** Define the hyperoperation "$\odot$" on $H_1 \times H_2$ by $(a_1, b_1) \odot (a_2, b_2) = (a_1 \cdot_1 a_2, b_1 \cdot_2 b_2)$ for all $(a_1, b_1), (a_2, b_2) \in H_1 \times H_2$. Then $(H_1 \times H_2, \circledast, \odot, (0_1, 0_2))$ is called the *hyper product* of $H_1$ and $H_2$.

The following theorem holds since the hyperoperation $\odot$ is componentwise.

**Theorem 5.2** *The hyper product of two hyper KS-semigroups is also a hyper KS-semigroup.*

Now, we extend the hyper product $H_1 \times H_2$ of $H_1$ and $H_2$ to any finite family of hyper KS-semigroups and obtain the following result.

**Theorem 5.3** *Let $\{H_i | i = 1, 2, \ldots, n\}$ be a nonempty family of hyper KS-semigroups. Then $\left(\prod_{i=1}^{n} H_i, \circledast, \odot, 0\right)$ is a hyper KS-semigroup.*

**Theorem 5.4** *Let $H_1$ and $H_2$ be hyper KS-semigroups.*

(i) *If $I_1$ and $I_2$ are hyper subKS-semigroups of $H_1$ and $H_2$, respectively, then $I_1 \times I_2$ is a hyper subKS-semigroup of $H_1 \times H_2$.*

(ii) *If $I_1$ and $I_2$ are hyper KS-ideals of $H_1$ and $H_2$, respectively, then $I_1 \times I_2$ is a hyper KS-ideal of $H_1 \times H_2$.*

(iii) *If $I_1$ and $I_2$ are reflexive in $H_1$ and $H_2$, respectively, then $I_1 \times I_2$ is reflexive in $H_1 \times H_2$.*

**Proof.** Let $(x, y), (p, q) \in I_1 \times I_2$. Then $x, p \in I_1$ and $y, q \in I_2$.

(i) Suppose that $I_1$ and $I_2$ are hyper subKS-semigroups of $H_1$ and $H_2$, respectively. Then $x * p, xp \subseteq I_1$ and $y * q, yq \subseteq I_2$ implying that $(x, y) \circledast (p, q) = (x * p, y * q) \subseteq I_1 \times I_2$ and $(x, y) \odot (p, q) = (xp, yq) \subseteq I_1 \times I_2$. Thus, $I_1 \times I_2$ is a hyper subKS-semigroup of $H_1 \times H_2$.

(ii) Suppose that $I_1$ and $I_2$ are hyper KS-ideals of $H_1$ and $H_2$, respectively. Let $(r, s) \in H_1 \times H_2$. Then $r \in H_1$ and $s \in H_2$. Since $I_1$ and $I_2$ are hyper stable in $H_1$ and $H_2$, respectively, $rx, xr \subseteq I_1$ and $sy, ys \subseteq I_2$ where $x \in H_1, y \in H_2$. It follows that $(r, s) \odot (x, y) = (rx, sy) \subseteq I_1 \times I_2$ and $(x, y) \odot (r, s) = (xr, ys) \subseteq I_1 \times I_2$ and so $I_1 \times I_2$ is hyper stable in $H_1 \times H_2$. Suppose that $(m, n), (r, s) \in H_1 \times H_2$ such that $(m, n) \circledast (r, s) \ll I_1 \times I_2$ and $(r, s) \in I_1 \times I_2$. Then $(m * r, n * s) \ll I_1 \times I_2$. This means that for all $(a, b) \in (m * r, n * s)$, there exists $(c, d) \in I_1 \times I_2$ such that $(a, b) \ll (c, d)$, that is, $a < c$ and $b < d$. Note that $a \in m * r$ and $c \in I_1$ imply $m * r < I_1$. Since $r \in I_1$, we have $m \in I_1$. Also, $b \in n * s$ and $d \in I_2$ imply $n * s < I_2$. Since $s \in I_2$, it follows that $n \in I_2$. Hence, $(m, n) \in I_1 \times I_2$. Therefore, $I_1 \times I_2$ is a hyper KS-ideal of $H_1 \times H_2$.

(iii) Let $(r, s) \in H_1 \times H_2$. Then $r \in H_1$ and $s \in H_2$. Since $I_1$ and $I_2$ are reflexive in $H_1$ and $H_2$, respectively, it follows that $r * r, rr \subseteq I_1$ and $s * s, ss \subseteq I_2$. Thus, $(r*r, s*s), (rr, ss) \subseteq I_1 \times I_2$. Therefore, $I_1 \times I_2$ is reflexive in $H_1 \times H_2$. ∎

**Theorem 5.5** *Let $\varphi_1 : G_1 \to H_1$ and $\varphi_2 : G_2 \to H_2$ be hyper KS-semigroup homomorphisms. Define $\varphi : G_1 \times G_2 \to H_1 \times H_2$ by $\varphi((a_1, a_2)) = (\varphi_1(a_1), \varphi_2(a_2))$ for all $(a_1, a_2) \in G_1 \times G_2$. Then*

(i) *$\varphi$ is a hyper KS-homomorphism;*

(ii) *$Ker\varphi = Ker\varphi_1 \times Ker\varphi_2$;*

(iii) *$Im\varphi = Im\varphi_1 \times Im\varphi_2$; and*

(iv) *$\varphi$ is a monomorphism (resp. epimorphism) if and only if $\varphi_i$ is a monomorphism (resp. epimorphism) for each $i = 1, 2$.*

**Proof.** Define $\varphi : G_1 \times G_2 \to H_1 \times H_2$ by $\varphi((a_1, a_2)) = (\varphi_1(a_1), \varphi_2(a_2))$ for all $(a_1, a_2) \in G_1 \times G_2$. Since $\varphi_1$ and $\varphi_2$ are well-defined, it also follows that $\varphi$ is well-defined.

Let $(a_1, a_2), (b_1, b_2) \in G_1 \times G_2$. Then

$$\begin{aligned}
\varphi((a_1, a_2) \circledast (b_1, b_2)) &= \varphi((a_1 * b_1, a_2 * b_2)) \\
&= \{\varphi((s, t)) \mid s \in a_1 * b_1, t \in a_2 * b_2\} \\
&= \{(\varphi_1(s), \varphi_2(t)) \mid s \in a_1 * b_1, t \in a_2 * b_2\} \\
&= (\varphi_1(a_1 * b_1), \varphi_2(a_2 * b_2)) \\
&= (\varphi_1(a_1) * \varphi_1(b_1), \varphi_2(a_2) * \varphi_2(b_2)) \\
&= (\varphi_1(a_1), \varphi_2(a_2)) \circledast (\varphi_1(b_1), \varphi_2(b_2)) \\
&= \varphi(a_1, a_2) \circledast \varphi(b_1, b_2).
\end{aligned}$$

and

$$\begin{aligned}
\varphi((a_1, a_2) \odot (b_1, b_2)) &= \varphi((a_1 b_1, a_2 b_2)) \\
&= \{\varphi((s, t)) \mid s \in a_1 b_1, t \in a_2 b_2\} \\
&= \{(\varphi_1(s), \varphi_2(t)) \mid s \in a_1 b_1, t \in a_2 b_2\} \\
&= (\varphi_1(a_1 b_1), \varphi_2(a_2 b_2)) \\
&= (\varphi_1(a_1)\varphi_1(b_1), \varphi_2(a_2)\varphi_2(b_2)) \\
&= (\varphi_1(a_1), \varphi_2(a_2)) \odot (\varphi_1(b_1), \varphi_2(b_2)) \\
&= \varphi(a_1, a_2) \odot \varphi(b_1, b_2).
\end{aligned}$$

Thus, $\varphi$ is a homomorphism. Furthermore,

$$\begin{aligned}
Ker\varphi &= \{(a_1, a_2) \mid \varphi((a_1, a_2)) = (0_1, 0_2) \in H_1 \times H_2\} \\
&= \{(a_1, a_2) | (\varphi_1(a_1), \varphi_2(a_2)) = (0_1, 0_2)\} \\
&= \{(a_1, a_2) \mid \varphi_1(a_1) = 0_1, \varphi_2(a_2) = 0_2\} \\
&= \{(a_1, a_2) \mid a_1 \in Ker\varphi_1, a_2 \in Ker\varphi_2\} \\
&= Ker\varphi_1 \times Ker\varphi_2
\end{aligned}$$

and

$$\begin{aligned} Im\varphi &= \{\varphi((a_1,a_2)) \mid (a_1,a_2) \in G_1 \times G_2\} \\ &= \{(\varphi_1(a_1),\varphi_2(a_2)) \mid a_1 \in G_1, a_2 \in G_2\} \\ &= \{(\varphi_1(a_1),\varphi_2(a_2)) \mid \varphi_1(a_1) \in Im\varphi_1, \varphi_2(a_2) \in Im\varphi_2\} \\ &= Im\varphi_1 \times Im\varphi_2. \end{aligned}$$

Moreover, suppose that $\varphi$ is one-to-one. Let $a_1, b_1 \in G_1$ and $a_2, b_2 \in G_2$ such that $\varphi_1(a_1) = \varphi_1(b_1)$ and $\varphi_2(a_2) = \varphi_2(b_2)$. Then $\varphi((a_1,a_2)) = (\varphi_1(a_1),\varphi_2(a_2)) = (\varphi_1(b_1),\varphi_2(b_2)) = \varphi((b_1,b_2))$. Since $\varphi$ is one-to-one, $(a_1,a_2) = (b_1,b_2)$ and so $a_1 = b_1$ and $a_2 = b_2$. Thus, $\varphi_1$ and $\varphi_2$ are one-to-one. Conversely, suppose that $\varphi_1$ and $\varphi_2$ are one-to-one. Let $(a_1,a_2),(b_1,b_2) \in G_1 \times G_2$ such that $\varphi((a_1,a_2)) = \varphi((b_1,b_2))$. Then $(\varphi_1(a_1),\varphi_2(a_2)) = \varphi((a_1,a_2)) = \varphi((b_1,b_2)) = (\varphi_1(b_1),\varphi_2(b_2))$. This implies that $\varphi_1(a_1) = \varphi_1(b_1)$ and $\varphi_2(a_2) = \varphi_2(b_2)$. Since $\varphi_1$ and $\varphi_2$ are one-to-one, it follows that $a_1 = b_1$ and $a_2 = b_2$ and so $(a_1,a_2) = (b_1,b_2)$. Hence, $\varphi$ is one-to-one.

Finally, assume that $\varphi$ is onto. Let $x_1 \in H_1$ and $x_2 \in H_2$. Then $(x_1,x_2) \in H_1 \times H_2$. Since $\varphi$ is onto, there exists $(a_1,a_2) \in G_1 \times G_2$ such that $(\varphi_1(a_1),\varphi_2(a_2)) = \varphi((a_1,a_2)) = (x_1,x_2)$. It follows that $\varphi_1(a_1) = x_1$ and $\varphi_2(a_2) = x_2$. Hence, $\varphi_1$ and $\varphi_2$ are onto. Conversely, assume that $\varphi_1$ and $\varphi_2$ are onto. Let $(x_1,x_2) \in H_1 \times H_2$. Then $x_1 \in H_1$ and $x_2 \in H_2$. Since $\varphi_1$ and $\varphi_2$ are onto, there exists $a_1 \in G_1$ and $a_2 \in G_2$ such that $\varphi_1(a_1) = x_1$ and $\varphi_2(a_2) = x_2$ and so $\varphi((a_1,a_2)) = (\varphi_1(a_1),\varphi_2(a_2)) = (x_1,x_2)$. Therefore, $\varphi$ is onto. ■

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Accepted: 12.07.2014