A NOTE ON HERMITE-HADAMARD INEQUALITIES FOR PRODUCTS OF CONVEX FUNCTIONS VIA RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS[1]

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Abstract. In this paper, we obtain some new Hermite-Hadamard type inequalities for products of convex functions via Riemann-Liouville integrals. We conclude that Our methods considered here may be a stimulant for further investigations concerning Hermite-Hadamard type inequalities for products of various kinds of convex functions involving Riemann-Liouville fractional integrals.

Keywords: Hermite-Hadamard inequality; convex function; Riemann-Liouville fractional integrals.

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1. Introduction

If $f : I \to R$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

This remarkable result is well known in the literature as the Hermite-Hadamard inequality.

Since then, some refinements of the Hermite-Hadamard inequality on convex functions have been extensively investigated by a number of authors (e.g., [1], [2], [3], [4], [5], [6], [7] and [8]).

In [9], B.G. Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows.

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Theorem 1.1. *Let f and g be real-valued, nonnegative and convex functions on $[a,b]$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b), \tag{1.2}$$

$$2f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b), \tag{1.3}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Some new integral inequalities involving two nonnegative and integrable functions that are related to the Hermite-Hadamard type are also obtained by many authors. In [10], B.G. Pachpatte proposed some Hermite-Hadamard type inequalities involving two log-convex functions. An analogous result for s-convex functions is established by Kirmaci et. al. in [8]. In [12], M.Z. Sarikaya presented some integral inequalities for two h-convex functions. For recent results and generalizations concerning Hermite-Hadamard type inequality for product of two functions see [13] and the references given therein.

It is remarkable that M.Z. Sarikaya et al. [11] proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1.2. *Let $f:[a,b]\to R$ be a positive function with $a<b$ and $f\in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)] \leq \frac{f(a)+f(b)}{2}, \tag{1.4}$$

with $\alpha>0$.

We remark that the symbol $J_{a^+}^\alpha$ and $J_{b^-}^\alpha f$ denote the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha\geq 0$ with $a\geq 0$ which are defined by

$$J_{a^+}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt, \qquad x>a,$$

and

$$J_{b^-}^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)dt, \qquad x<b,$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function defined by

$$\Gamma(\alpha)=\int_0^\infty e^{-t}t^{\alpha-1}dt.$$

In this paper, we obtain some new Hermite-Hadamard type inequalities for products of convex functions via Riemann-Liouville integrals.

2. Main results

Theorem 2.1. *Let f and g be real-valued, nonnegative and convex functions on $[a,b]$. Then*

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b)g(b)+J^\alpha_{b-}f(a)g(a)] &\leq \Big(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\Big)M(a,b)\\ &\quad+\frac{\alpha}{(\alpha+1)(\alpha+2)}N(a,b),\end{aligned}$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$, $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Since f and g are convex on $[a,b]$, then for $t\in[0,1]$ we get

$$f(ta+(1-t)b)\leq tf(a)+(1-t)f(b), \tag{2.1}$$

and

$$g(ta+(1-t)b)\leq tg(a)+(1-t)g(b). \tag{2.2}$$

From (2.1) and (2.2), we get

$$\begin{aligned}f(ta+(1-t)b)g(ta+(1-t)b) &\leq t^2f(a)g(a)+(1-t)^2f(b)g(b)\\ &\quad+t(1-t)[f(a)g(b)+f(b)g(a)].\end{aligned}$$

Similarly, we have

$$\begin{aligned}f((1-t)a+tb)g((1-t)a+tb) &\leq (1-t)^2f(a)g(a)+t^2f(b)g(b)\\ &\quad+t(1-t)[f(a)g(b)+f(b)g(a)].\end{aligned}$$

So

$$\begin{aligned}&f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\\ &\leq(2t^2-2t+1)[f(a)g(a)+f(b)g(b)]+2t(1-t)[f(a)g(b)+f(b)g(a)].\end{aligned}$$

Multiplying both sides of above inequality by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 t^{\alpha-1}f(ta+(1-t)b)g(ta+(1-t)b)dt\\ &\quad+\int_0^1 t^{\alpha-1}f((1-t)a+tb)g((1-t)a+tb)dt\\ &\quad=\int_b^a\Big(\frac{b-u}{b-a}\Big)^{\alpha-1}f(u)g(u)\frac{du}{a-b}+\int_a^b\Big(\frac{v-a}{b-a}\Big)^{\alpha-1}f(v)g(v)\frac{dv}{b-a}\\ &\quad=\frac{\Gamma(\alpha)}{(b-a)^\alpha}[J^\alpha_{a+}f(b)g(b)+J^\alpha_{b-}f(a)g(a)]\end{aligned}$$

$$\leq [f(a)g(a)+f(b)g(b)]\int_0^1 t^{\alpha-1}(2t^2-2t+1)dt$$
$$+2[f(a)g(b)+f(b)g(a)]\int_0^1 t^{\alpha-1}t(1-t)dt$$
$$=\Big(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\Big)[f(a)g(a)+f(b)g(b)]$$
$$+\frac{2}{(\alpha+1)(\alpha+2)}[f(a)g(b)+f(b)g(a)]$$
$$=\Big(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\Big)M(a,b)+\frac{2}{(\alpha+1)(\alpha+2)}N(a,b).$$

So

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)g(b)+J^\alpha_{b^-}f(a)g(a)] \leq \Big(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\Big)M(a,b) +\frac{\alpha}{(\alpha+1)(\alpha+2)}N(a,b),$$

which completes the proof. ∎

Corollary 2.2. *With the notations in Theorem* 2.1*, if we choose* $g:[a,b]\to R$ *as* $g(x)=1$ *for all* $x\in[a,b]$*, we obtain*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)]\leq\frac{f(a)+f(b)}{2},$$

which is the right hand side of (1.4).

Proof. Since $g(x)=1$ for all $x\in[a,b]$, from $M(a,b)=N(a,b)=f(a)+f(b)$, we can get the desire result. ∎

Corollary 2.3. *With the notations in Theorem* 2.1*, if* $\alpha=1$*, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx\leq\frac{1}{3}M(a,b)+\frac{1}{6}N(a,b),$$

which is the right hand side of (1.2).

Theorem 2.4. *Let* f *and* g *be real-valued, nonnegative and convex functions on* $[a,b]$*. Then*

$$2f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)g(b)+J^\alpha_{b^-}f(a)g(a)] + M(a,b)\frac{\alpha}{(\alpha+1)(\alpha+2)} + N(a,b)\Big(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\Big),$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$*,* $N(a,b)=f(a)g(b)+f(a)g(b)$*.*

Proof. We can write

$$\frac{a+b}{2}=\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2},$$

so

$$\begin{aligned}
&f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)\\
&=f\Big(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\Big)g\Big(\frac{ta+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\Big)\\
&\leq\frac{1}{4}\Big[f(ta+(1-t)b)+f((1-t)a+tb)\Big]\Big[g(ta+(1-t)b)+g((1-t)a+tb)\Big]\\
&=\frac{1}{4}\Big[f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\Big]\\
&+\frac{1}{4}\Big[f(ta+(1-t)b)g((1-t)a+tb)+f((1-t)a+tb)g(ta+(1-t)b)\Big]\\
&\leq\frac{1}{4}\Big[f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\Big]\\
&+\frac{1}{4}\Bigg\{\Big[tf(a)+(1-t)f(b)\Big]\Big[(1-t)g(a)+tg(b)\Big]\\
&+\Big[(1-t)f(a)+tf(b)\Big]\Big[tg(a)+(1-t)g(b)\Big]\Bigg\}\\
&=\frac{1}{4}\Big[f(ta+(1-t)b)g(ta+(1-t)b)+f((1-t)a+tb)g((1-t)a+tb)\Big]\\
&+\frac{1}{4}\Bigg\{2t(1-t)\Big[f(a)g(a)+f(b)g(b)\Big]+\Big[(1-t)^2+t^2\Big]\Big[f(a)g(b)+f(b)g(a)\Big]\Bigg\}.
\end{aligned}$$

Multiplying both sides of above inequality by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned}
&f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)\int_0^1 t^{\alpha-1}dt\\
&\leq\frac{1}{4}\Big[\int_0^1 t^{\alpha-1}f(ta+(1-t)b)g(ta+(1-t)b)dt\\
&+\int_0^1 t^{\alpha-1}f((1-t)a+tb)g((1-t)a+tb)dt\Big]\\
&+\frac{1}{4}\Bigg\{\Big[f(a)g(a)+f(b)g(b)\Big]\int_0^1 t^{\alpha-1}2t(1-t)dt\\
&+\Big[f(a)g(b)+f(b)g(a)\Big]\int_0^1 t^{\alpha-1}[(1-t)^2+t^2]dt\Bigg\}.
\end{aligned}$$

That is,

$$\begin{aligned}\frac{1}{\alpha}f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) &\leq \frac{1}{4}\Big[\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}[J_{a^+}^{\alpha}f(b)g(b)+J_{b^-}^{\alpha}f(a)g(a)]\Big]\\ &+ \frac{1}{4}\Big\{M(a,b)\int_0^1 t^{\alpha-1}2t(1-t)dt\\ &+ N(a,b)\int_0^1 t^{\alpha-1}[(1-t)^2+t^2]dt\Big\}.\end{aligned}$$

From

$$\int_0^1 t^{\alpha-1}2t(1-t)dt=\frac{2}{(\alpha+1)(\alpha+2)}$$

and

$$\int_0^1 t^{\alpha-1}[(1-t)^2+t^2)]dt=\Big(\frac{2}{\alpha+2}-\frac{2}{\alpha+1}+\frac{1}{\alpha}\Big),$$

we get

$$\begin{aligned}2f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big) &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha}f(b)g(b)+J_{b^-}^{\alpha}f(a)g(a)]\\ &+ M(a,b)\frac{\alpha}{(\alpha+1)(\alpha+2)}\\ &+ N(a,b)\Big(\frac{\alpha}{\alpha+2}-\frac{\alpha}{\alpha+1}+\frac{1}{2}\Big),\end{aligned}$$

which completes the proof. ∎

Corollary 2.5. *With the notations in Theorem* 2.4*, if* $\alpha=1$*, then*

$$2f\Big(\frac{a+b}{2}\Big)g\Big(\frac{a+b}{2}\Big)\leq\frac{1}{b-a}\int_a^b f(x)g(x)dx+\frac{1}{6}M(a,b)+\frac{1}{3}N(a,b),$$

which is the right hand side of (1.3).

Corollary 2.6. *With the notations in Theorem* 2.4*, if we choose* $g:[a,b]\to R$ *as* $g(x)=1$ *for all* $x\in[a,b]$*, we have*

$$2f\Big(\frac{a+b}{2}\Big)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)]+\frac{f(a)+f(b)}{2}.$$

3. Conclusion

In this paper, we establish some new Hermite-Hadamard type inequalities for products of convex functions via Riemann-Liouville integrals. An interesting topic is whether we can use the methods in this paper to establish the Hermite-Hadamard inequality for other kinds of convex functions.

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