ON HYPER PSEUDO MV-ALGEBRAS

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Abstract. In this paper, we introduce the notion of hyper pseudo MV-algebra as a generalization of pseudo MV-algebra and hyper MV-algebra. Then we investigate some properties of this structure and attempt to construct a hyper pseudo MV-algebra from a l-group. Finally, we proved some theorems about filters, ideals, congruence relations, in hyper pseudo MV-algebras.

Keywords: hyper (Pseudo) MV-algebra, l-group, filter, ideal, congruence.

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1. Introduction

MV-algebras were introduced by C.C. Chang [2] in 1958, as an algebraic model of infinite valued logic. In [10], Mundici showed that any MV-algebra is an interval of an Abelian lattice ordered group with a strong unit. Georgescu and Iorgulescu [4] introduced a new non-commutative algebraic structures, which were called pseudo MV-algebras. It can be obtained by dropping commutative axioms in MV-algebras, which are a generalization of MV-algebras.

The hyper algebraic structure theory was introduced in 1934 [9] by Marty at 8th Congress of Scandinavian Mathematicians. Recently in [6], Sh. Ghorbani et al. applied the hyper structure to MV-algebras and introduced the concept of hyper MV-algebra which is a generalization of MV-algebra and investigated some related results. Since then many researchers have worked on this structure (see [5], [7], [8], [11]).

In this paper, the concept of hyper pseudo MV-algebra was introduced. We show that any pseudo MV-algebra (hyper MV-algebra) is a hyper pseudo MV-algebra and verify some of the properties of this algebra as mentioned in the abstract.

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2. Preliminaries

Definition 2.1. [4] A *Pseudo MV-algebra* is an algebra $(M, +, -, \sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$ such that the following axioms hold for all $x, y, z \in M$,

(A1) $x + (y + z) = (x + y) + z$,
(A2) $x + 0 = 0 + x = x$,
(A3) $x + 1 = 1 + x = 1$,
(A4) $1^- = 1^\sim = 0$,
(A5) $(x^- + y^-)^\sim = (x^\sim + y^\sim)^-$,
(A6) $x + (x^\sim.y) = y + (y^\sim.x) = (x.y^-) + y = (y.x^-) + x$,
(A7) $x.(x^- + y) = (x + y^\sim).y$,
(A8) $(x^-)^\sim = x$.

where $x.y = (x^- + y^-)^\sim$.

Definition 2.2. [1] A *lattice-ordered group (l-group)* is an algebra $(G, \vee, \wedge, +, -, 0)$ such that $(G, \vee, \wedge)$ is a lattice, $(G, +, -, 0)$ is a group and $+$ is an order preserving map. If $(G, \vee, \wedge, +, -, 0)$ is a lattice-ordered group, then we use $[0, u]$ to denote $\{x \in G | 0 \leq x \leq u\}$, for any $0 \leq u$.

Proposition 2.3. *[4] Let $(G, \vee, \wedge, +, -, 0)$ be an l-group and $0 \leq u$, for some $u \in G$. Define $x * y = (x + y) \wedge u$, $x^- = u - x$ and $x^\sim = -x + u$, for any $x, y \in G$. Then $([0, u], *, -, \sim, 0, u)$ is a pseudo MV-algebra.*

Definition 2.4. [6] A *hyper MV-algebra* is a non-empty set M endowed with a binary hyper operation $\oplus$, a unary operation $*$ and a constant 0 satisfying the following conditions: for all $x, y, z \in M$

($hMV1$) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
($hMV2$) $x \oplus y = y \oplus x$,
($hMV3$) $(x^*)^* = x$,
($hMV4$) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$,
($hMV5$) $0^* \in x \oplus 0^*$,
($hMV6$) $0^* \in x \oplus x^*$
($hMV7$) if $x \ll y$ and $y \ll x$, then $x = y$, where $x \ll y$ is defined by $0^* \in x^* \oplus y$.

For every $A, B \subseteq M$, we define $A \ll B$ if and only if there exist $a \in A$ and $b \in B$ such that $a \ll b$ and $A \oplus B = \cup\{a \oplus b | a \in A, b \in B\}$. Also, we define $0^* = 1$ and $A^* = \{a^* | a \in A\}$.

3. Hyper pseudo MV-algebras

Definition 3.1. A *hyper pseudo MV-algebra* is a non-empty set M with a binary hyperoperation $+$, two unary operations $'$, $\sim$ and two constants $0, 1$ satisfying the following conditions: for all $x, y, z \in M$,

($HSMV1$) $x + (y + z) = (x + y) + z$,
($HSMV2$) $1 \in (x + 1) \cap (1 + x)$,

$(HSMV3)$ $1^{\sim} = 1' = 0$,
$(HSMV4)$ $(x' + y')^{\sim} = (x^{\sim} + y^{\sim})'$,
$(HSMV5)$ $x + (x^{\sim} \odot y) = y + (y^{\sim} \odot x) = (x \odot y') + y = (y \odot x') + x$,
$(HSMV6)$ $x \odot (x' + y) = (x + y^{\sim}) \odot y$,
$(HSMV7)$ $(x')^{\sim} = x$,
$(HSMV8)$ $1 \in (x + x^{\sim}) \cap (x' + x)$,
$(HSMV9)$ $1 \in (x' + y) \cap (y' + x)$ implies $x = y$,
$(HSMV10)$ $1 \in x' + y$ if and only if $1 \in y + x^{\sim}$.

where $y \odot x = (x' + y')^{\sim}$, $A' = \{a' | a \in A\}$, $A^{\sim} = \{a^{\sim} | a \in A\}$, $A \odot B = \cup\{a \odot b | a \in A, b \in B\}$ and $A + B = \cup\{a + b | a \in A, b \in B\}$, for any $A, B \subseteq M$. A hyper pseudo MV-algebra M is called *proper* if there exists $a, b \in M$ such that $2 \leq |a + b|$. Let $(M, +, ', \sim, 0, 1)$ be a hyper pseudo MV-algebra, which is not proper. Then for any $x, y \in M$, there exists $a_{x,y} \in M$ such that $x + y = \{a_{x,y}\}$. Define the operation $* : M \to M$ by $x * y = a_{x,y}$. It can be easily obtained that $(M, *, ', \sim, 0, 1)$ is a pseudo MV-algebra and we say that $(M, +, ', \sim, 0, 1)$ is a pseudo MV-algebra.

Example 3.2. Let $(M, +, -, \sim, 0, 1)$ be a pseudo MV-algebra. For any $x, y \in M$, we define $x \oplus y = \{x + y\}$. Then $(M, \oplus, -, \sim, 0, 1)$ is a hyper pseudo MV-algebra.

Example 3.3. Let $(M, \oplus, *, 0, 1)$ be a hyper MV-algebra. Then $(M, \oplus, *, *, 0, 0^*)$ is a hyper pseudo MV-algebra.

Example 3.4. (i) Let $M = \{0, a, b, c, 1\}$. Consider the following tables:

Table 1

+	0	a	b	c	1
0	{0}	{0,a}	{0,b}	{0,c}	M
a	{0,a}	{0,a}	{a,b}	M	M
b	{b,0}	M	{0,b}	{c,b}	M
c	{c,0}	{c,a}	M	{0,c}	M
1	M	M	M	M	M

Table 2

	0	a	b	c	1
′	1	b	c	a	0
∼	1	c	a	b	0

Then we get

Table 3

⊙	0	a	b	c	1
0	M	M	M	M	M
a	M	{1,a}	M	{a,c}	{1,a}
b	M	{a,b}	{1,b}	M	{1,b}
c	M	M	{b,c}	{1,c}	{1,c}
1	M	{a,1}	{b,1}	{c,1}	{1}

It is not difficult to show that $(M, +, ', \sim, 0, 1)$ is a hyper pseudo MV-algebra. Moreover, $+$ is not commutative and so M is not a hyper MV-algebra.

(ii) Let $N = \{0, a, b, c, d, 1\}$ and $M = \{0, a, b, c, 1\}$. Consider the following tables:

Table 4

+	0	a	b	c	d	1
0	{0}	{0,a}	{0,b}	{0,c}	{0,d}	M
a	{0,a}	{0,a}	{a,b}	M	{0,a,d}	M
b	{b,0}	M	{0,b}	{c,b}	{0,b,d}	M
c	{c,0}	{c,a}	M	{0,c}	{0,c,d}	M
d	{d,0}	{d,a}	{d,b}	{d,c}	N	N
1	M	M	M	M	N	M

Table 5

	0	a	b	c	d	1
$'$	1	b	c	a	d	0
$\sim$	1	c	a	b	d	0

Easy calculations show that $(N, +,', \sim, 0, 1)$ is a hyper pseudo MV-algebra.

Remark 3.5.

(i) It is easy, to show that if $(M, +,', \sim, 0, 1)$ is a hyper pseudo MV-algebra such that $x + y = y + x$ and $x' = x^{\sim}$, for any $x, y \in M$, then $(M, +,', 0)$ is a hyper MV-algebra.

(ii) Let $(M, +,', \sim, 0, 1)$ be a hyper pseudo MV-algebra. Define a relation $\leq$ on M by $x \leq y$ if and only if $1 \in x' + y$, for any $x, y \in M$. Then by $(HSMV8)$ and $(HSMV9)$, we conclude that $\leq$ is a reflexive and antisymmetric relation on M.

Proposition 3.6. *For any integer $5 \leq m$, there is at least one proper hyper pseudo MV-algebra of order m.*

Proof. Let $5 \leq m$, $n = m - 2$, $M = \{a_1, a_2, ..., a_n\}$ and $N = M \cup \{0, 1\}$, where $0, 1 \notin M$. Define $a'_i = a_{i+1}$ and $a^{\sim}_j = a_{j-1}$, for any $i, j \in \{1, 2, ..., n\}$, where $a_0 = a_n$ and $a_{n+1} = a_1$. Consider the following hyper operation on M.

$$x + y = \begin{cases} \{0\} & \text{if } x = y = 0, \\ \{0, y\} & \text{if } x = 0,\ y = a_i \text{ or } y = 0,\ x = a_i, \\ N & \text{if} x = 1 \text{ or } y = 1, \\ \{0, a_i\} & \text{if } x = y = a_i, \\ N & \text{if } x = a_i,\ y = a_j,\ j = i - 1, \\ \{a_i, a_j\} & \text{otherwise.} \end{cases}$$

It is easy to show that $(N, +,', \sim, 0, 1)$ is a proper hyper pseudo MV-algebra of order m. ∎

From now on, in this paper, $(M, +,', \sim, 0, 1)$ or simply M will denote a hyper pseudo MV-algebra, unless otherwise stated. If $A, B \subseteq M$, $A \ll B$ $(A \leq B)$ means that $a \leq b$, for some $a \in A$ and $b \in B$ (for any $a \in A$, there exists $b \in B$ such that $a \leq b$). Moreover, if $A = \{a\}$, then we write $a \ll B$ instead of $A \ll B$.

Proposition 3.7. *Suppose that* $(G, \vee, \wedge, +, -, 0)$ *is a* l*-group and* $0 \le u$, *for some* $u \in G$. *Let* $\downarrow A = \{x \in [0,u] | x \le a, \text{ for all } a \in A\}$ *and* $\uparrow A = \{x \in [0,u] | a \le x, \text{ for all } a \in A\}$, *for any* $A \subseteq G$. *Define* $x \oplus y = \downarrow \{x + y, u\} = \downarrow \{x * y\}$, $x^- = u - x$ *and* $x^\sim = -x + u$, *for any* $x, y \in G$, *where* $*$ *is the operation which is defined in Proposition 2.3. Then* $([0,u], \oplus, ', \sim, 0, u)$ *is a hyper pseudo* MV*-algebra.*

Proof. Let $x, y, z \in [0,u]$. By Proposition 2.3, the operations $'$ and $\sim$ are one to one and onto maps. Moreover, $x \le y \Leftrightarrow y' \le x' \Leftrightarrow y^\sim \le x^\sim$, for all $x, y \in [0,u]$.

(1) Let $x \in [0,u]$. Then $0 \le x$ and so $u = 0 + u \le x + u$. Hence $u \in \downarrow \{x+u, u\} = x \oplus u$. By the similar way, $u \in u \oplus x$. Therefore, $u \in (x \oplus u) \cap (u \oplus x)$.

(2) By Proposition 2.3, $u' = u^\sim = 0$ and $(x')^\sim = x$, for all $x \in [0,u]$.

(3) Let $a \in [0,u]$. Then $0 \le a \le u$ and so $0 = a - a \le u - a = u'$, $-a \le 0$. Hence $u - a \le u + 0 = u$. Therefore, $a' \in [0,u]$. By the similar way, we can show that $a^\sim \in [0,u]$. On the other hand, let $b' \in [0,u]$, for some $b \in G$. Then $0 \le u - b \le u$, so $b = 0 + b \le (u-b) + b = u$ and $-b = -u + (u-b) \le -u + u = 0$. Hence $0 \le b$. Therefore, $b \in [0,u]$ if and only if $b' \in [0,u]$, for any $b \in G$. Similarly, we can show that $b \in [0,u]$ if and only if $b^\sim \in [0,u]$. Hence

$$
\begin{aligned}
(x' \oplus y')^\sim &= \{a^\sim \in [0,u] | a \le x' + y', a \in [0,u]\} \\
&= \{a^\sim | a \in [0,u], (x' + y')^\sim \le a^\sim\} \\
&= \{a^\sim | a \in [0,u], (x^\sim + y^\sim)' \le a^\sim\}, \text{ by Proposition 2.3} \\
&= \{a^\sim | (a^\sim)' \in [0,u], (x^\sim + y^\sim)' \le a^\sim\} \\
&= \{t | t' \in [0,u], (x^\sim + y^\sim)' \le t\} \\
&= \{t | t' \in [0,u], t^\sim \le (x^\sim + y^\sim)\} \\
&= \{t | t' \in [0,u], t^\sim \in (x^\sim \oplus y^\sim)\} \\
&= \{t | t' \in [0,u], t \in (x^\sim \oplus y^\sim)'\} \\
&= \{t | t \in [0,u], t \in (x^\sim \oplus y^\sim)'\} \\
&= (x^\sim \oplus y^\sim)'.
\end{aligned}
$$

(4) By Proposition 2.3, $x + x^\sim = u = x' + x$. Hence $u \in \downarrow \{x + x^\sim, u\}$ and $u \in \downarrow \{x' + x, u\}$. Therefore, $u \in (x \oplus x^\sim) \cap (x' \oplus x)$.

(5) Let $u \in (x' \oplus y) \cap (y' \oplus x)$. Then by $u \in x' \oplus y$, we get $u \le x' + y$ and so $u + (-y) \le (x' + y) + (-y) = x'$. Similarly, $u \in y' \oplus x$ implies $x' \le y'$. Hence $x' = y'$ and so $x = y$.

(6) Since $([0,u], +, ', \sim, 0, u)$ is a pseudo MV-algebra, then we have

$$u \in x' \oplus y \Leftrightarrow u \le x' + y \Leftrightarrow y' \le x' \Leftrightarrow y^\sim \le x^\sim \Leftrightarrow -y + u \le x^\sim \Leftrightarrow u \le y + x^\sim$$

(7) We want to show that $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

$$x \oplus (y \oplus z) = \cup\{x \oplus a | a \in y \oplus z\} = \{t \in [0,u] | t \le x * a, a \in y \oplus z\}$$

For any $a \in y \oplus z$, we have $a \le y * z$, $a \in [0,u]$ and so $x + a \le x + (y * z)$. Hence $x * a \le x * (y * z)$, it follows that $x \oplus (y \oplus z) \subseteq \downarrow \{x * (y * z)\}$. Moreover, $y * z \le y * z$

and so $\downarrow \{x*(y*z)\} \subseteq x\oplus(y\oplus z)$. Therefore, $x\oplus(y\oplus z) = \downarrow \{x*(y*z), u\}$. By the similar way, $(x\oplus y)\oplus z = \downarrow \{(x*y)*z, u\}$. Now, by Proposition 2.3, we get $x\oplus(y\oplus z) = (x\oplus y)\oplus z$.

(8) Let $a \odot b = (b^{\sim} \oplus a^{\sim})'$, for all $a, b \in [0, u]$. Then

$$\begin{aligned} x\odot y = (y^{\sim}\oplus x^{\sim})' &= \{a' | a\in[0,u], a\in y^{\sim}\oplus x^{\sim}\} \\ &= \{a'\in[0,u] | a \le y^{\sim} * x^{\sim}\} \\ &= \{a'\in[0,u] | (y^{\sim}*x^{\sim})' \le a'\} \\ &= \{t\in[0,u] | (y^{\sim}*x^{\sim})' \le t\}, \text{ since } ' \text{ is one to one and onto} \\ &= \uparrow \{x.y\}. \end{aligned}$$

Now, we prove that, $(HSMV5)$ holds.

$$x\oplus(x^{\sim}\odot y) = \cup\{x\oplus a | a\in x^{\sim}\odot y\} = \{t\in[0,u] | t\le x*a, a\in\uparrow\{x^{\sim}.y\}\}$$

By Proposition 2.3, $x^{\sim}.y \in [0, u]$ and so $u \in \uparrow \{x^{\sim}.y\}$. Hence $x\oplus(x^{\sim}\odot y) = [0, u]$. By the similar way, we can show that $y\oplus(y^{\sim}\odot x) = (x\odot y')\oplus y = (y\odot x')\oplus x = [0, u]$. Therefore, $(HSMV5)$ holds.

(9) By definition of $\oplus$, it is obvious that $0 \in x'\oplus y$. Hence by Proposition 2.3, we obtain

$$[0, u] = \uparrow \{x*0\} = x\odot 0 \subseteq x\odot x\odot(x'\oplus y) \subseteq [0, u]$$

By the similar way, we obtain $(x\oplus y^{\sim})\odot y = [0, u]$. Therefore, $(HSMV6)$ holds.

From (1)-(9), it follows that $([0, u], \oplus, ', \sim, 0, u)$ is a hyper pseudo MV-algebra. ∎

Lemma 3.8. *The following properties hold: for any $x, y, z \in M$.*

(i) $(x^{\sim})' = x$, $0^{\sim} = 0' = 1$ *and* $1'' = 1 = (1^{\sim})^{\sim}$.
Moreover, $x = y \Leftrightarrow x' = y' \Leftrightarrow x^{\sim} = y^{\sim}$,

(ii) $0 \in (x\odot 0)\cap(0\odot x)$,

(iii) $x\odot y = (y^{\sim} + x^{\sim})'$,

(iv) $0 \in (x\odot x')\cap(x^{\sim}\odot x)$,

(v) $x \ll x+y$ *and* $y \le x+y$,

(vi) $(x\odot y)\odot z = x\odot(y\odot z)$,

(vii) $x\le y \Leftrightarrow 0\in y^{\sim}\odot x \Leftrightarrow 0\in x\odot y' \Leftrightarrow 1\in y+x^{\sim}$,

(viii) $x\odot y \ll z \Leftrightarrow x \ll z+y^{\sim} \Leftrightarrow y \ll x'+z$,

(ix) $(x+y)' = y'\odot x'$, $(x+y)^{\sim} = y^{\sim}\odot x^{\sim}$ *and* $x+y = (y'\odot x')^{\sim} = (y^{\sim}\odot x^{\sim})'$,

(x) $(x\odot y)' = y'+x'$ *and* $(x\odot y)^{\sim} = y^{\sim}+x^{\sim}$,

(xi) $(x^{\sim}\odot y)+y^{\sim} = (y^{\sim}\odot x)+x^{\sim}$,

(xii) $x \odot (x' + y) = y \odot (y' + x)$,

(xiii) $x \leq 1$ *and* $0 \leq x$.

Proof. (i) Since $1 \in x + x^{\sim}$, then by $(HSMV7)$, $1 \in (x + ((x^{\sim})')^{\sim})$. On the other hand, $1 \in ((x^{\sim})' + x^{\sim})$ and so $1 \in (x + u^{\sim}) \cap (u + x^{\sim})$, where $u = (x^{\sim})'$. Hence by $(HSMV9)$, $x = (x^{\sim})'$.
The proof of the other parts follows from $(HSMV7)$ and the first part of (i).

(ii) By $(HSMV2)$, $1 \in 1 + x'$ and so $1^{\sim} \in (1 + x')^{\sim}$. Hence by $(HSMV3)$, $0 \in x \odot 0$. By the similar way, we can show that $0 \in (0 \odot x)$.

(iii) The proof is Straightforward by $(HSMV4)$.

(iv) By $(HSMV8)$, $1 \in (x + x^{\sim}) \cap (x' + x)$, and so by $(HSMV7)$ and (iii), $0 \in (x + x^{\sim})' = ((x')^{\sim} + x^{\sim})' = x \odot x'$. On the other hand by (i), $0 \in (x' + x)^{\sim} = (x' + (x^{\sim})')^{\sim} = x^{\sim} \odot x$.

(v) By $(HSMV2)$ and $(HSMV8)$, we get $1 \in 1 + y \subseteq (x' + x) + y = x' + (x + y)$ and so $1 \in x' + c$, for some $c \in x + y$. Hence $x \leq c$, so $x \ll x + y$. Moreover, $1 \in x + 1 \subseteq x + (y + y^{\sim}) = (x + y) + y^{\sim}$. Hence $1 \in d + y^{\sim}$, for some $d \in x + y$. Therefore, $y \leq d$ and so $y \ll x + y$.

(iv)
$$\begin{aligned}
(x \odot y) \odot z &= (y' + x')^{\sim} \odot z \\
&= \cup\{a^{\sim} \odot z | a \in y' + x'\} \\
&= \cup\{(z' + (a^{\sim})')^{\sim} | a \in y' + x'\} \\
&= \cup\{(z' + a)^{\sim} | a \in y' + x'\}, \text{ by (i)} \\
&= (z' + (y' + x'))^{\sim} \\
&= ((z' + y') + x')^{\sim} \\
&= (((z' + y')^{\sim})' + x')^{\sim}, \text{ by (i)} \\
&= (y \odot z)' + x')^{\sim} \\
&= \cup\{(a' + x')^{\sim} | a \in y \odot z\} \\
&= \cup\{x \odot a | a \in y \odot z\} \\
&= x \odot (y \odot z).
\end{aligned}$$

(vii) Let $x, y \in M$. Then

$$x \leq y \Leftrightarrow 1 \in x' + y \Leftrightarrow 0 \in (x' + y)^{\sim} \Leftrightarrow 0 \in y^{\sim} \odot x \tag{3.1}$$

Moreover, by $(HSMV10)$, we have

$$1 \in x' + y \Leftrightarrow 1 \in y + x^{\sim} \Leftrightarrow 0 \in ((y')^{\sim} + x^{\sim})' \Leftrightarrow 0 \in x \odot y' \tag{3.2}$$

By using of (3.1) and (3.2), the proof is completed.

(viii)
$$\begin{aligned}
x \odot y \ll z &\Leftrightarrow 1 \in (x \odot y)' + z \Leftrightarrow 1 \in (y' + x') + z \\
&\Leftrightarrow 1 \in y' + (x' + z) \Leftrightarrow y \ll x' + z \\
x \odot y \ll z &\Leftrightarrow 1 \in z + (x \odot y)^{\sim} \Leftrightarrow 1 \in z + (y^{\sim} + x^{\sim}) \\
&\Leftrightarrow 1 \in (z + y^{\sim}) + x^{\sim} \Leftrightarrow x \ll z + y^{\sim}
\end{aligned}$$

(ix) By $(HSMV7)$, $(x + y)' = ((x')^{\sim} + (y')^{\sim})' = y' \odot x'$. Similarly, by using of (i), we can show that $(x + y)^{\sim} = y^{\sim} \odot x^{\sim}$.

(x) By $(HSMV7)$ and (iii), we get $(x \odot y)^\sim = ((y^\sim + x^\sim)')^\sim = y^\sim + x^\sim$. The proof of the other part is similar.

(xi) By $(HSMV5)$, we obtain $(x^\sim \odot (y^\sim)') + y^\sim = (y^\sim \odot (x^\sim)') + x^\sim$. Therefore,

$$(x^\sim \odot y) + y^\sim = (y^\sim \odot x) + x^\sim$$

(xii)
$$\begin{aligned} x \odot (x' + y) &= ((x' + y)^\sim + (x')^\sim)' \\ &= (y^\sim \odot x) + x^\sim \\ &= ((x^\sim \odot y) + y^\sim)', \text{ by (xi)} \\ &= y \odot (x^\sim \odot y)' \\ &= y \odot (y' + x) \end{aligned}$$

(xiii) Let $x \in M$. By $(HSMV2)$ and (i), $0 \leq x$. Moreover, $1 \in 1 + x^\sim$ and so $x \leq 1$. ■

Theorem 3.9. *Let $(M, +, ', \sim, 0, 1)$ be a hyper pseudo MV-algebra. Then*

(i) *$(M, \odot, \sim, ' 1, 0)$ is a hyper pseudo MV-algebra, too. It is called the dual hyper pseudo MV-algebra of $(M, +, ', \sim, 0, 1)$.*

(ii) *If $x + y = y + x$, for any $x, y \in M$, then M is a hyper MV-algebra.*

Proof. (i) It follows from Lemma 3.8.

(ii) Suppose that $x + y = y + x$, for any $x, y \in M$. Let $x \in M$. Then by $(HSMV8)$ and Lemma 3.8(i), we get $1 \in x' + x = x' + (x^\sim)' = (x^\sim)' + x'$ and so Lemma 3.8(vii) implies $x^\sim \leq x'$. Moreover, by $1 \in x + x^\sim = (x')^\sim + x^\sim = x^\sim + (x')^\sim$, we conclude that $x^\sim \leq x'$. Hence, $x' = x^\sim$, for any $x \in M$. Now, it is clear that $(M, +, ', 0)$ is a hyper MV-algebra. ■

Proposition 3.10. *Let $x, y \in M$. Then the following properties hold:*

(i) $x \leq y \Leftrightarrow y' \leq x' \Leftrightarrow y^\sim \leq x^\sim$,

(ii) *$x \leq y$ implies $x + a \ll y + a$ and $a + x \ll a + y$, for all $a \in M$,*

(iii) *$x \leq y$ implies $a \odot x \ll a \odot y$ and $x \odot a \ll y \odot a$, for all $a \in M$,*

(iv) $x \odot y \ll x, y$,

(v) *$0 + 0 = \{0\}$ and $1 \odot 1 = \{1\}$,*

(vi) *$y \in x + 0$ ($y \in 0 + x$) implies $y \leq x$. Moreover, if $y \in x \odot 1$ ($y \in 1 \odot x$), then $x \leq y$,*

(vii) $x \in (0 + x) \cap (x + 0)$,

(viii) $x \in (1 \odot x) \cap (x \odot 1)$,

(ix) *$x + 0 = y + 0$ (0+x=0+y) implies $x = y$.*

Proof. (i)

$$y' \leq x' \Leftrightarrow 1 \in x' + (y')^{\sim} = x' + y \Leftrightarrow x \leq y \Leftrightarrow 1 \in y + x^{\sim} = (y^{\sim})' + x^{\sim} \Leftrightarrow y^{\sim} \leq x^{\sim}$$

(ii) Let $x \leq y$ and $a \in M$. Then we have

$$\begin{aligned}(y+a)+(x+a)^{\sim} &= (y+a)+(a^{\sim}\odot x^{\sim})\\ &= y+(a+(a^{\sim}\odot x^{\sim}))\\ &= y+(x^{\sim}+(((x^{\sim})^{\sim}\odot a)),\ \text{by } (HSMV5)\\ &= (y+x^{\sim})+((x^{\sim})^{\sim}\odot a)\\ &\supseteq 1+((x^{\sim})^{\sim}\odot a),\ \text{since } x\leq y.\end{aligned}$$

Hence by $(HSMV2)$, $1 \in (y+a)+(x+a)^{\sim}$ and so $1 \in u+v^{\sim}$, for some $u \in y+a$ and $v \in x+a$. Thus, by Lemma 3.8(vii), $v \leq u$. It follows that, $x+a \ll y+a$. On the other hand,

$$\begin{aligned}(a+x)'+(a+y) &= (x'\odot a')+(a+y),\ \text{by Lemma 3.8(ix)}\\ &= ((x'\odot a')+a)+y\\ &= ((a\odot x'')+x')+y\\ &= (a\odot x'')+(x'+y)\\ &\supseteq (a\odot x'')+1,\ \text{since } x\leq y\end{aligned}$$

Hence by $(HSMV2)$, $1 \in (a+x)'+(a+y)$ and so there exists $u \in a+x$ and $v \in a+y$ such that $1 \in u'+v$, whence $u \leq v$. Therefore, $a+y \ll a+y$.

(iii) Let $x \leq y$ and $a \in M$. Then by (i), $y' \leq x'$ and so by (ii), $a'+y' \ll a'+x'$. Hence $u \leq v$, for some $u \in a'+x'$ and $v \in a'+v'$, whence by (i), $v^{\sim} \leq u^{\sim}$. Therefore, $x \odot a = (a'+x')^{\sim} \ll (a'+y')^{\sim} = y \odot a$. By the similar way, we can show that $a \odot x \ll a \odot y$.

(iv) By Lemma 3.8(v), we know that $x', y' \leq x'+y'$ and so by (i), $(x'+y')^{\sim} \ll (x')^{\sim}, (y')^{\sim}$. Hence $y \odot x \ll x, y$.

(v) Let $b \in 0+0$. Then by $(HSMV8)$, we get $1 \in b+b^{\sim} \subseteq (0+0)+b^{\sim} = 0+(0+b^{\sim})$ and so $1 \in 0+x$, for some $x \in 0+b^{\sim} = 1'+b^{\sim}$. Hence by $(HSMV2,3,10)$, $x = 1$, and so $1 \in 0+b^{\sim}$. Therefore, $b \leq 0$ and so by Lemma 3.8(xiii) and $(HSMV8)$, $b = 0$. The proof of the other part follows from Theorem 3.9.

(vi) Let $x \in M$ and $y \in 0+x$. Then by (v), $0+x = (0+0)+x = 0+(0+x) \supseteq 0+y$ and so

$$1 \in 0+1 \subseteq 0+(y+y^{\sim}) = (0+y)+y^{\sim} \subseteq (0+x)+y^{\sim} = 0+(x+y^{\sim})$$

Hence there exists $u \in x+y^{\sim}$ such that $1 \in 0+u = 1'+u$. Thus $u = 1$ and so $1 \in x+y^{\sim}$. Therefore, $y \leq x$. The proof of other part is similar. Now, let $y \in x \odot 1$. Then $y \in (0+x')^{\sim}$ and so $y' \in 0+x'$. It follows that $y' \leq x'$ and so by (i), $x \leq y$. The proof of the other part is similar.

(vii) Let $x \in M$. Then $1 \in 0+1 \subseteq 0+(x+x^{\sim}) = (0+x)+x^{\sim}$ and so $1 \in u+x^{\sim}$, for some $u \in 0+x$. Hence $x \leq u$. Now, by (vi), we get $x = u$.

Therefore, $x \in x + 0$. On the other hand, $1 \in 1 + 0 \subseteq (x' + x) + 0 = x' + (x + 0)$. Thus $1 \in x' + u$, for some $u \in x + 0$ and so $x \leq u$. Hence by (vi), we get $x = u$ and so $x \in x + 0$.

(viii) Let $x \in M$. By (vii), $x' \in 0 + x'$, and so $x = (x')^{\sim} \in (1' + x')^{\sim} = x \odot 1$. By the similar way, we can show that $x \in 1 \odot x$.

(ix) Let $x + 0 = y + 0$, for some $x, y \in M$. Since $1 \in x' + x$, then we get

$$1 \in (x' + x) + 0 = x' + (x + 0) = x' + (y + 0) = (x' + y) + 0.$$

Hence there exists $u \in x' + 1$ such that $1 \in u + 0 = u + 1^{\sim}$. By $(HSMV2)$ and $(HSMV9)$, we get $u = 1$ and so $1 \in x' + y$. By the similar way, we can show that $1 \in y + x'$. Therefore, $(HSMV9)$ implies $x = y$. The proof of the other part is similar. ∎

4. Some hyper operations in hyper pseudo MV-algebra

In this section we define the concept of hyper operations, $\vee$, $\wedge$, $/$ and $\backslash$ on a hyper pseudo MV-algebras which we will use for definitions of ideals and filters in hyper pseudo MV-algebras. Then we obtain some of theirs properties and relation between $\leq$, $+$, $\odot$ and these hyper operations.

Proposition 4.1. *Consider the hyper operations $/$ and $\backslash$ on M, were defined by $a/b = a + b^{\sim}$ and $a\backslash b = a' + b$. Then, for any $x, y, z \in M$, we have:*

(i) $z \ll z/y$ *and* $z \ll x\backslash z$,

(ii) $x \leq y$ *implies* $x/z \ll y/z$ *and* $z\backslash x \ll z\backslash y$ *(*$z/y \ll z/x$ *and* $y\backslash z \ll x\backslash z$*)*,

(iii) $z \vee y^{\sim} \ll z/y$ *and* $y' \vee z \ll y\backslash z$,

(iv) $y\backslash z, z/y \subseteq \{u | u \odot y \ll z\}$ *and* $\{u | u \odot y \ll z\} \leq z/y, y\backslash z$,

(v) $x\backslash(y/z) = (x\backslash y)/z$,

(vi) $y\backslash(x\backslash z) = (x \odot y)\backslash z$,

(vii) $(z/x)/y = z/(x \odot y)$,

(viii) $(x + y)/z = x + (y/z)$,

(ix) $x/(z\backslash x) = z/(x\backslash z)$.

Proof. (i) It follows from Lemma 3.8(v).

(ii) Let $x \leq y$. Then by Proposition 3.10(ii), we have $x + z^{\sim} \ll y + z^{\sim}$ and $z' + x \ll z' + y$. By the similar way, we can prove the other part.

(iii) By Proposition 3.10(iv), $z^{\sim} \odot y^{\sim} \ll y^{\sim}$ and so there exists $a \in z^{\sim} \odot y^{\sim}$ such that $a \leq y^{\sim}$. Now, by 3.10(ii), we get $z \vee y^{\sim} = z + (z^{\sim} \odot y^{\sim}) \supseteq z + a \ll z + y^{\sim} = z/y$. By the similar way, we can prove the other part.

(iv) Let $a \in z/y$. Then $a \ll z/y = z + y^{\sim}$. Hence by Lemma 3.8(viii), $a \odot y \ll z$ and so $a \in \{u|u \odot y \ll z\}$. Now, let $u \in \{u|u \odot y \ll z\}$. Then by Lemma 3.8(viii), $u \ll z/y$ and so $u \leq b$, for some $b \in z/y$. Therefore, $\{u|u \odot y \ll z\} \leq z/y$. The proof of the other part is similar.

(v) $x\backslash(y/z) = x\backslash(y+z^{\sim}) = x'+(y+z^{\sim}) = (x'+y)+z^{\sim} = (x'+y)/z = (x\backslash y)/z$.

(vi) By Lemma 3.8(x), we get

$$y\backslash(x\backslash z) = y\backslash(x' + z) = y' + (x' + z) = (y' + x') + z = (x \odot y)' + z = (x \odot y)\backslash z.$$

(vii) $(z/x)/y = (z + x^{\sim})/y = (z + x^{\sim}) + y^{\sim} = z + (x^{\sim} + y^{\sim}) = z + (y \odot x)^{\sim} = z/(y \odot x)$.

(viii) $(x + y)/z = (x + y) + z^{\sim} = x + (y + z^{\sim}) = x + (y/z)$.

(ix) $x/(z\backslash x) = x/(z' + x) = x + (z' + x)^{\sim} = x + (z' + (x^{\sim})')^{\sim} = x + (x^{\sim} \odot z)$. Now, by $(HSMV5)$, we obtain $x/(z\backslash x) = z+(z^{\sim}\odot x) = z+(x'+z)^{\sim} = z/(x'+z) = z/(x\backslash z)$. ■

Proposition 4.2. *Let $\vee$ and $\wedge$ be two hyper operations on M, were defined by $x \vee y = x + (x^{\sim} \odot y)$ and $x \wedge y = x \odot (x' + y)$. Then the following hold: for all $x, y, z \in M$,*

(i) $x \in (x \wedge x) \cap (x \vee x)$,

(ii) $x \vee y = y \vee x$ *and* $x \wedge y = y \wedge x$,

(iii) $x \in (x \wedge (x \vee y)) \cap (x \vee (x \wedge y))$,

(iv) $x \in x \vee 0$, $0 \in x \wedge 0$, $x \in x \wedge 1$ *and* $x \in x \vee 1$,

(v) $x \leq y$ *implies* $x \in x \wedge y$ *and* $y \in x \vee y$,

(vi) $x \odot y \ll x \wedge y \ll x, y$,

(vii) $x, y \ll x \vee y$,

(viii) $x \leq y$ *implies* $x \wedge z \ll y \wedge z$ *and* $x \vee z \ll y \vee z$,

(ix) $x \wedge y = (x' \vee y')^{\sim} = (x^{\sim} \vee y^{\sim})'$,

(x) $x \vee y = (x' \wedge y')^{\sim} = (x^{\sim} \wedge y^{\sim})'$,

(xi) $0 \vee 1 = 0 + 1$ *and* $0 \wedge 1 = 1 \odot 0$,

(xii) $\vee$ *is associative if and only if* $\wedge$ *is associative.*

Proof. (i) By Lemma 3.8(iv), $0 \in x^{\sim} \odot x$ and so by Proposition 3.10(vii), $x \in x + 0 \subseteq x + (x^{\sim} + x) = x \vee x$. On the other hand, by Proposition 3.10(viii) and $(HSMV8)$, we have $x \in x \odot 1 \subseteq x \odot (x' + x) = x \wedge x$.

(ii) It follows from $(HSMV7, 8)$.

(iii)

$$\begin{aligned} x \wedge (x \vee y) = x \wedge (x + (x^{\sim} \odot y)) &= x \odot (x' + (x + (x^{\sim} \odot y))) \\ &= x \odot ((x' + x) + (x^{\sim} \odot y)) \\ &\supseteq x \odot 1, \text{ by } (HSMV8) \text{ and } (HSMV2) \\ &\ni x, \text{ by Proposition 3.10(viii)} \end{aligned}$$

By the similar way, we can show that $x \in (x \vee (x \wedge y))$.

(iv) By Lemma 3.8(iv) and 3.10(vii), we get

$$\begin{aligned} &0 \in x \odot x' \subseteq x \odot (x' + 0) = x \wedge 0, \quad \text{and} \\ &x \in x + 0 \subseteq x + (x^{\sim} \odot 0) = x \vee 0. \end{aligned}$$

Moreover, by Proposition 3.10(viii) and $(HSMV8)$, we obtain

$$\begin{aligned} &1 \in x + x^{\sim} \subseteq x + (x^{\sim} \odot 1) = x \vee 1, \quad \text{and} \\ &x \in x \odot 1 \subseteq x \odot (x' + 1) = x \wedge 1. \end{aligned}$$

(v) Let $x \leq y$. Then by Lemma 3.8(vii), $1 \in x' + y$ and $0 \in y^{\sim} \odot x$. Hence by Proposition 3.10 (vii) and (viii), we have

$$\begin{gathered} y \in y + 0 \subseteq y + (y^{\sim} \odot x) = (x \odot y') + y = x \vee y, \\ x \in x \odot 1 \subseteq x \odot (x' + y) = x \wedge y. \end{gathered}$$

(vi) Since by Lemma 3.8(v), $y \leq x' + y$, then Proposition 3.10(iii) implies

$$x \odot y \ll x \odot (x' + y) = x \wedge y.$$

Moreover, by Lemma 3.8(iv) and $(HSMV6)$, $x \wedge y \ll x, y$.

(vii) It is straight consequent of Lemma 3.8(v) and $(HSMV5)$.

(viii) Let $x \leq y$ and $z \in M$. Then by Proposition 3.10(ii), $x + z^{\sim} \ll y + z^{\sim}$ and so by Proposition 3.10(iii), $x \wedge z = (x + z^{\sim}) \odot z \ll (y + z^{\sim}) \odot z = y \wedge z$. Similarly, we can show that $x \vee z \ll y \vee z$.

(ix) We will show that $x \wedge y = (x' \vee y')^{\sim}$. The proof of other part is similar.

$$(x' \vee y')^{\sim} = (x' + ((x')^{\sim} \odot y'))^{\sim} = (x' + (x \odot y'))^{\sim} = (x \odot y')^{\sim} \odot x = (y + x^{\sim}) \odot x = x \wedge y$$

(x) By (ix) and Lemma 3.8(i), we get $(x^{\sim} \wedge y^{\sim})' = (((x^{\sim})' \vee (y^{\sim})')^{\sim})' = x \vee y$. By the similar way, we can show that $x \vee y = (x' \wedge y')^{\sim}$.

(xi) By Proposition 3.10(v), we have $0 + 0 = \{0\}$ and $1 \odot 1 = \{1\}$. Therefore, $0 \vee 1 = 0 + (0^{\sim} \odot 1) = 0 + (1 \odot 1) = 0 + 1$ and $0 \wedge 1 = 1 \odot (1' + 0) = 1 \odot (0 + 0) = 1 \odot 0$.

(xii) Let $\vee$ be a associative hyper operation A be a non-empty subset of M and $x, y, z \in M$. Then by Lemma 3.8(i), we get $\{a' \in M | a' \in A\} = A$ and so

$$\begin{aligned} x\wedge(y\wedge z) &= \cup\{x\wedge a|a\in y\wedge z\} \\ &= \cup\{x\wedge a|a'\in y'\vee z'\},\text{ by (iv)} \\ &= \cup\{(x'\vee a')^{\sim}|a'\in y'\vee z'\},\text{ by (iv)} \\ &= \cup\{(x'\vee a')^{\sim}|a'\in y'\vee z'\},\text{ by (iv)} \\ &= \cup\{(x'\vee a)^{\sim}|a\in y'\vee z'\},\text{ by (iv)} \\ &= (x'\vee(y'\vee z'))^{\sim} \\ &= ((x'\vee y')\vee z')^{\sim} \end{aligned}$$

By the similar way, we can show that $(x\wedge y)\wedge z = ((x'\vee y')\vee z')^{\sim}$. Hence $\wedge$ is associative. The proof of the converse is similar. ■

Remark 4.3. Let $x, y \in M$.

(i) If $x+y=\{0\}$, then $x=y=0$ (by Lemma 3.8(v)).

(ii) If $x\vee y=\{0\}$, then $x=y=0$ (by Proposition 4.2(vii)).

(iii) If $x\wedge y=\{1\}$ $(x\odot y=\{1\})$, then $x=y=1$ (by Proposition 4.2(vi)).

Proposition 4.4. *Let $x, y\in M$. Then*

(i) $x\ll x\odot y$ *implies* $1\in x'\vee y$,

(ii) $y\ll x\odot y$ *implies* $1\in x\vee y^{\sim}$,

(iii) $x+y\ll y$ *implies* $0\in x^{\sim}\wedge y$,

(iv) $x+y\ll x$ *implies* $0\in x\wedge y'$.

Proof. (i) By Lemma 3.8(i), we get

$$1\in x'\vee y\Leftrightarrow 1\in x'+((x^{\sim})'\odot y)\Leftrightarrow 1\in x'+u,(\exists u\in x\odot y)\Leftrightarrow x\ll x\odot y$$

(ii) By Proposition 3.5(vii), we obtain

$$1\in x\vee y^{\sim}\Leftrightarrow 1\in(x\odot(y^{\sim})')+y^{\sim}\Leftrightarrow 1\in v+y^{\sim}(\exists v\in x\odot y)\Leftrightarrow y\ll x\odot y.$$

(iii) Proposition 3.5(vii) implies

$$0\in x^{\sim}\wedge y\Leftrightarrow 0\in x^{\sim}\odot((x^{\sim})'+y)\Leftrightarrow 0\in x^{\sim}\odot u(\exists u\in x+y)\Leftrightarrow x\ll x+y.$$

(iv) $0\in x\wedge y'\Leftrightarrow 0\in(x+(y')^{\sim})\odot y'\Leftrightarrow 0\in v\odot y'(\exists v\in x+y)\Leftrightarrow y\ll x+y.$ ■

5. Filters on hyper pseudo MV-algebras

In this section, we attempt to verify filter and ideal theories in hyper pseudo MV-algebras. Then the relation between them was obtained. We show that I is an ideal of M if and only if I is a filter of dual pseudo MV-algebra of M.

Definition 5.1. Let I be a non-empty subset of M. Then I is called

- a *weak ideal* of M if

  $(x+y)\cap I \neq \emptyset$, for all $x, y \in I$,

  $y \in I$ and $x \leq y$ imply $x \in I$, for all $x \in M$.

- an *ideal* of M if

  $(y/x)' \ll I$ and $y \in I$ imply $x \in I$, for all $x \in M$,

  $(x+y) \subseteq I$, for all $x, y \in I$.

Clearly, any ideal of M is a weak ideal. An ideal (a weak ideal) I of M is called proper, if $I \subsetneqq M$. It is easy to show that if I is an ideal of M, then I is proper if and only if $x \in I$ implies $x', x^{\sim} \in M' - I$, for all $x \in M$. Moreover, if I is an ideal, $x \leq y$ and $y \in I$, then $0 \in x \odot y'$ and so $(y/x)' \ll I$. Hence $x \in I$.

Example 5.2. Consider the hyper pseudo MV-algebra in Example 3.4(i). Then $\{0\}$, $\{0, a\}$, $\{0, b\}$ and $\{0, c\}$ are ideals of M. Moreover, $\{0, a, b\}$, $\{0, b, c\}$ and $\{0, a, c\}$ are weak ideals of M. But, they are not ideal.

Definition 5.3. Let F be a non-empty subset of M. Then F is called

- a *weak filter* of M if

  $x \in F$ and $x \leq y$ imply $y \in F$, for any $y \in M$

  $(x \odot y) \cap F \neq \emptyset$, for all $x, y \in F$,

- a *filter* of M if

  $F \ll x\backslash y$ and $x \in F$ imply $y \in F$, for any $y \in M$,

  $(x \odot y) \subseteq F$, for all $x, y \in F$.

Clearly, any filter of M is a weak filter.

Proposition 5.4. *Let F be a non-empty subset of M such that $(x \odot y) \subseteq F$, for all $x, y \in F$. Then*

(i) *F is a filter if and only if $x \in F$ and $x \leq y$ imply $y \in F$, for any $x, y \in M$.*

(ii) *F is a filter if and only if $F \ll y/x$ and $x \in F$ imply $y \in F$, for any $x, y \in M$.*

Proof. (i) Let F be a filter of M, $x \in F$ and $x \leq y$. Then by $1 \in x\backslash y$, we have $F \ll x\backslash y$ and so by assumption $y \in F$. Conversely, let $x \in F$ and $F \ll x\backslash y$, for some $y \in M$. Then there exists $a \in F$ such that $a \ll x\backslash y$, so $1 \in a\backslash(x\backslash y)$. Hence by Proposition 4.1(vi), $1 \in (x \odot a)\backslash y$. Since $x, y \in F$, then by assumption $x \odot a \subseteq F$ and so there exists $u \in F$ such that $1 \in u\backslash y$. Now, by assumption, we get $y \in F$. Therefore, F is a filter of L.

(ii) Suppose that F be a filter of M, $F \ll y/x$ and $x \in F$, for some $y \in M$. Then there exists $a \in F$ such that $a \ll y + x^{\sim}$ and so $1 \in (y/x)/a$. By Proposition 4.1(vii), we get $1 \in y/(x \odot a)$. Since F is a filter, then $x \odot a \subseteq F$ and so there exists $b \in F$ such that $1 \in y/b$. Hence $b \leq y$ and $b \in F$, whence by (i), $y \in F$. The proof of the converse is straightforward by $(HSMV10)$ and (i). ∎

Example 5.5. Let M be a hyper pseudo MV-algebra in Example 3.4(i). Then

(i) $\{1, a\}$, $\{1, b\}$ and $\{1, c\}$ are three filters of M.

(ii) $\{1, a, b\}$ and $\{1, a, b\}$ are weak filters of M. But, they are not filters.

Proposition 5.6. *Let F be a filter, I be an ideal of M and $x, y \in F$. Then $(x \wedge y) \cap I \neq \emptyset$, $(x \vee y) \cap F \neq \emptyset$, $(x \vee y) \cap I \neq \emptyset$ and $(x \wedge y) \cap F \neq \emptyset$.*

Proof. From Lemma 3.8(v) and (vii), it follows that $(x \wedge y) \cap I \neq \emptyset$ and $(x \vee y) \cap F \neq \emptyset$. By Proposition 4.2(vi), there exists $a \in x^{\sim} \odot y$ such that $a \leq y$ and so $x + a \ll x + y \subseteq I$. Hence $(x \vee y) \cap I = x + (x^{\sim} \odot y) \cap I \neq \emptyset$. On the other hand, by Proposition 4.2(vi), there exits $b \in (x' + y)$ such that $y \leq a$ and so $F \supseteq x \odot y \ll x \odot a$. Hence $(x \wedge y) \cap F = x \odot (x' + y) \cap F \neq \emptyset$. ∎

By Proposition 3.10(v), in any hyper pseudo MV-algebra, $\{1\}$ is a filter and $\{0\}$ is an ideal.

Theorem 5.7. *Let I be a non-empty subset of M. Then the following are equivalent:*

(i) *I is an ideal of M,*

(ii) *$I^{\sim}$ is a filter of M,*

(iii) *I' is a filter of M.*

Proof. (i) $\Rightarrow$ (ii) Suppose that I is an ideal of M and $a, b \in I^{\sim}$. Then there exist $x, y \in I$ such that $a = x^{\sim}$ and $b = y^{\sim}$. By Lemma 3.8(ix), we get $a \odot b = x^{\sim} \odot y^{\sim} = (y + x)^{\sim}$. Since I is an ideal of M and $x, y \in I$, then $(y + x)^{\sim} \subseteq I^{\sim}$. Now, let $I^{\sim} \ll a \backslash b$, for some $a \in I^{\sim}$ and $b \in M$. Then there exist $x, y \in I$ such that $a = x^{\sim}$ and $y^{\sim} \ll x^{\sim} \backslash b$. By Proposition 3.10(i), we conclude that $(x^{\sim} \backslash b)' \ll y$, so $= (x/b')' = (x + (b')^{\sim})' = (x + b)' \ll y$. Since $x \in I$ and I is an ideal of M, then we get $b' \in I$. Therefore, $b \in I^{\sim}$ and so $I^{\sim}$ is a filter of M.

(ii) $\Rightarrow$ (i) Let $I^{\sim}$ be a filter of M and $x, y \in I$. Then $x^{\sim}, y^{\sim} \in I^{\sim}$ and $(x + y)^{\sim} = (y^{\sim} \odot x^{\sim}) \subseteq I^{\sim}$. Hence $x + y \subseteq I$. Now, let $(y/x)' \ll I$, for some $y \in I$ and $x \in M$. Then By Proposition 3.10(i), $I^{\sim} \ll y/x = y + x^{\sim} = (y^{\sim})' + x^{\sim} = y^{\sim} \backslash x^{\sim}$ and $y^{\sim} \in I^{\sim}$. Since $I^{\sim}$ is a filter, then we get $x^{\sim} \in I^{\sim}$ and so $x \in I$. Therefore, I is an ideal of M.

(i) $\Rightarrow$ (iii) Assume that I is an ideal of M and $x, y \in I'$. Then $x = a'$ and $y = b'$, for some $a, b \in I$. By Lemma 3.8(ix), $x \odot y = a' \odot b' = (b + a)' \subseteq I'$. Now, let $I' \ll (x \backslash y)$, $x \in I$ and $y \in M$. Then $(x^{\sim}/y^{\sim})' = y^{\sim} \odot (x^{\sim})' = y^{\sim} \odot x = (x' + y)^{\sim} \ll I$. By $x \in I$, we get $y^{\sim} \in I$ and so $y \in I'$. Therefore, I' is a filter of M.

(iii) $\Rightarrow$ (i) It is easy to show that $x + y \subseteq I$, for all $x, y \in I$. Let $(y/x)' \ll I$, for some $y \in I$ and $x \in M$. Then by Proposition 3.10(i), $I' \ll (y/x)'' = (x \odot y')' = (y')' + x' = y' \backslash x'$. Since I' is a filter of M and $y' \in I'$, then we get $x' \in I'$ and so $x = (x')^{\sim} \in I$. Therefore, I is an ideal of M. ∎

Corollary 5.8. *Let I be a non-empty subset of M such that $x + y \subseteq I$, for any $x, y \in I$. Then I is an ideal if and only if $y \in I$ and $x \leq y$ imply $x \in I$, for any $x \in M$.*

Proof. Suppose that $y \in I$ and $x \leq y$ imply $x \in I$, for any $x \in M$. We show that $I^{\sim}$ is a filter. Let $x, y \in I^{\sim}$. Then $x', y' \in I$ and so $y' + x' \subseteq I$. It follows that $x \odot y \subseteq I^{\sim}$. Now, let $x \leq y$ and $x \in I^{\sim}$. Then $y' \leq x'$ and $x' \in I$, so by assumption, $y' \in I$. Hence $y \in I^{\sim}$, whence by Proposition 5.4, $I^{\sim}$ is a filter of M. Therefore, by Theorem 5.7, I is an ideal of M. The proof of the converse is clear. ∎

Definition 5.9. Let $(M', +, ', \sim, 0, 1)$ be a hyper pseudo MV-algebra and $f : M \to M'$ be a map such that $f(0) = 0$ and $f(x') = f(x)'$, for all $x \in M$. Then f is called

(i) a *weak homomorphism* if $f(x + y) \subseteq f(x) + f(y)$, for all $x, y \in M$,

(ii) a *homomorphism* if $f(x + y) = f(x) + f(y)$, for all $x, y \in M$.

For any weak homomorphism $f : M \to M'$, we use $ker(f)$ to denote $f^{-1}(0) = \{x \in M | f(x) = 0\}$. A homomorphism $f : M \to M'$ is called *isomorphism* if f is a one to one and onto map. We use $M \cong M'$ to denote, there exists a isomorphism from M to M'.

Example 5.10. Let M be a hyper pseudo MV-algebra in Example 3.4(i).

(i) Let $M' = \{0, a, b, c, d, 1\}$. Consider the following tables:

Table 6

+	0	a	b	c	d	1
0	{0}	{0,a}	{0,b}	{0,c}	{0,d}	M'
a	{0,a}	{0,a}	{a,b}	M'	{a,d}	M'
b	{b,0}	M'	{0,b}	{c,b}	{b,d}	M'
c	{c,0}	{c,a}	M'	{0,c}	{c,d}	M'
d	{d,0}	{d,a}	{d,b}	{d,c}	M'	M'
1	M'	M'	M'	M'	M'	M'

Table 7

	0	a	b	c	d	1
$'$	1	b	c	a	d	0
$\sim$	1	c	a	b	d	0

Easy calculations show that $(M', +, ', \sim, 0, 1)$ is a hyper pseudo MV-algebra. Let $f : M \to M'$ be a map was defined by $f(x) = x$, for all $x \in M$. Then f is a weak homomorphism.

(ii) Define $f : M \to M$ by $f(0) = 0$, $f(1) = 1$, $f(a) = b$, $f(b) = c$ and $f(c) = a$. It is not difficult to check that $f : M \to M$ is a homomorphism.

Theorem 5.11. *Let $f : M \to M'$ be a weak homomorphism and I be an ideal of M'. Then*

(i) *$f(x^{\sim}) = f(x)^{\sim}$. Moreover, $x \leq y$ implies $f(x) \leq f(y)$, for any $x, y \in M$,*

(ii) *$ker(f)$ is an ideal of M,*

(iii) *$f^{-1}(I)$ is an ideal of M,*

(iv) *If F is a filter of M', then $f^{-1}(F)$ is a filter of M,*

(iv) *If f is a homomorphism, then f is one to one if and only if $ker(f) = \{0\}$,*

(v) *F is a filter of M if and only if F is an ideal of dual pseudo MV-algebra of M,*

(vi) *If f is onto and I is an ideal of M containing $ker(f)$, then $f(I)$ is an ideal of M'.*

Proof. (i) Let $x \in M$. Then by $f(x) = f((x^{\sim})') = f(x^{\sim})'$, we get $f(x)^{\sim} = (f(x^{\sim})')^{\sim} = f(x^{\sim})$. Now, let $x \leq y$. then $1 \in x' + y$ and so $1 = f(0)' = f(1) \in f(x' + y) = f(x)' + f(y)$. Hence $f(x) \leq f(y)$.

(ii) Let $x, y \in ker(f)$. Then $f(x) = f(y) = 0$ and so by Proposition 3.10(v), $f(x + y) \subseteq f(x) + f(y) = 0 + 0 = 0$. Hence $x + y \subseteq ker(f)$. Now, let $(y/x)' \ll ker(f)$ and $y \in ker(f)$. Then $x \odot y' \leq a$, for some $a \in ker(f)$. By Lemma 3.8(viii), we get $x \leq a + y$. Since $a, y \in ker(f)$, then $a + y \subseteq ker(f)$ and so $f(x) = 0$. Therefore, $ker(f)$ is an ideal of M.

(iii) Assume that I is an ideal of M'. If $x \odot y' \ll f^{-1}(I)$ and $y \in f^{-1}(I)$, for some $x, y \in M$, then by (i), $f(x) \odot f(y)' \ll I$ and $f(y) \in I$. Since I is an ideal of M', then we have $f(x) \in I$. Hence $x \in f^{-1}(I)$. Now, let $a, b \in f^{-1}(I)$. Then $f(x), f(y) \in I$. Hence $f(x + y) \subseteq f(x) + f(y) \subseteq I$ and so $x + y \subseteq f^{-1}(I)$. Therefore, $f^{-1}(I)$ is an ideal of M.

(iv) Let F be a filter of M'. Then by Theorem 5.7, F' is a filter of M, so by (iii), $f^{-1}(F')$ is an ideal of M. It suffices to show that $f^{-1}(F') = f^{-1}(F)'$. Let $x \in M$. Hence

$$x \in f^{-1}(F') \Leftrightarrow f(x) \in F' \Leftrightarrow f(x^{\sim}) = f(x)^{\sim} \in F \Leftrightarrow x^{\sim} \in f^{-1}(F) \Leftrightarrow x \in f^{-1}(F)'$$

(iv) Clearly, if f is one to one, then $ker(f) = \{0\}$. Suppose that $ker(f) = \{0\}$ and $f(x) = f(y)$, for some $x, y \in M$. Then $1 \in (f(x)' + f(y)) \cap (f(y)' + f(x))$. Since f is a homomorphism, then by (i), we get $0 \in f((x' + y)^{\sim}) \cap f((y' + x)^{\sim})$. Hence by assumption, $0 \in (x' + y)^{\sim} = y^{\sim} \odot x$ and $0 \in (y' + x)^{\sim} = x^{\sim} \odot y$. Now, by Lemma 3.8(vii), we conclude that $x = y$. Therefore, f is a one to one map.

(v) Straightforward.

(vi) Let I be an ideal of M containing $ker(f)$ and $x, y \in f(I)$. Then there exist $a, b \in I$ such that $x = f(a)$ and $y = f(b)$. Hence $x + y = f(a) + f(b) = f(a + b) \subseteq f(I)$. Now, let $(y/x)' \ll f(I)$ and $y \in f(I)$, for some $x, y \in M'$. Since f is onto, then there are $a \in M$ and $b \in I$ such that $x = f(a)$ and $y = f(b)$. By $f((b/a)') = (f(b)/f(a))' \ll f(I)$, we get, there are $u \in I$ and $c \in (b/a)'$ such that $f(c) \leq f(u)$ and so $0 \in (f(u)/f(c))' = f((u/c)')$. Hence $(u/c)' \ll ker(f) \subseteq I$. Since $u \in I$ and I is an ideal of M, then $c \in I$ and so $(b/a)' \ll I$. Therefore, $a \in I$ (since $b \in I$ and I is an ideal) and so $x \in f(I)$. That is, $f(I)$ is an ideal of M'. ∎

Corollary 5.12. *If M' be a hyper pseudo MV-algebra and $f : M \to M'$ be a homomorphism, then $f^{-1}(1)$ is a filter of M.*

Proof. Clearly, $f : (M, \odot, \sim, ', 1, 0) \to (M', \odot, ', 1, 0)$ is a homomorphism, where $(M, \odot, \sim, ', 1, 0)$ and $(M', \odot, ', 1, 0)$ are dual hyper pseudo MV-algebra of M and M', respectively. Hence by Theorem 5.11(ii), $f^{-1}(1) = ker(f)$ is an ideal of $(M, \odot, \sim, ', 1, 0)$. Therefore, by Theorem 5.11(v), $f^{-1}(1)$ is a filter of M. ∎

6. Congruence relation on hyper pseudo MV-algebras

Definition 6.1. Let θ be an equivalence relation on M and A, B be two subsets of M. Then

- $A\theta B$ means that $a\theta b$, for some $a \in A$ and $b \in B$,
- $A\overline{\theta}B$ means that for any $a \in A$, there exists $b \in B$ such that $a\theta b$ and for any $b \in B$ there exists $a \in A$ such that $b\theta a$. Moreover, if $A = \{a\}$, then we write $a\theta B$, for $A\theta B$,
- $A\overline{\overline{\theta}}B$ means that $a\theta b$, for any $a \in A$ and $b \in B$.

It can be easily obtained that, $A\overline{\theta}B$ and $B\overline{\theta}C$ imply $A\overline{\theta}C$, for any subset A, B, C of M.

Definition 6.2. Let θ be an equivalence relation on M. Then θ is called the *congruence relation*, if it satisfies the following conditions:

(C1) $x\theta y$ and $a\theta b$ imply $(x + a)\overline{\theta}(y + b)$,for any $x, y, a, b \in M$,

(C2) $x\theta y$ implies $x'\theta y'$ and $x^{\sim}\theta y^{\sim}$, for any $x, y \in M$,

(C3) $x\theta y$ if and only if $1\theta(x' + y)$ and $1\theta(y' + x)$, for any $x, y \in M$,

(C4) $1\theta(x' + y)$ if and only if $1\theta(y + x^{\sim})$, for any $x, y \in M$.

We use $Con(M)$ to denote the set of all congruence relations on M.

Note 6.3. Let θ be an equivalence relation on M.

(i) If θ is an equivalence relation on M satisfies on (C1), (C2), then $x\theta y$ implies that $1 \in x' + x\overline{\theta}x' + y$ and $1 \in y' + y\overline{\theta}y' + x$.

(ii) If $[1] = \{x \in M | x\theta 1\} = \{1\}$, then by (C3) and $(HSMV9)$ we get $\theta = \{(x, x) | x \in M\}$.

Example 6.4. Let $M = \{0, a, b, c, d, e, f, 1\}$. Consider the following tables:

Table 8

+	0	a	b	c	d	e	f	1
0	{0}	{0,a}	{0,b}	{0,c}	{0,d}	{0,e}	{0,f}	M
a	{0,a}	{0,a}	{0,a,b}	{0,a,c}	{0,a,d}	{0,a,e}	M	M
b	{b,0}	{0,a,b}	{0,b}	{0,b,c}	{0,b,d}	M	M	M
c	{c,0}	{0,c,a}	M	{0,c}	{0,c,d}	{0,c,e}	{0,c,f}	M
d	{d,0}	M	M	{0,d,c}	{d,0}	{0,d,e}	{0,d,f}	M
e	{e,0}	{0,a,e}	{0,b,e}	{0,c,e}	M	{0,e}	{0,e,f}	M
f	{f,0}	{0,a,f}	{0,b,f}	M	M	{0,f,e}	{0,f}	M
1	M	M	M	M	M	M	M	M

Table 9

	0	a	b	c	d	e	f	1
′	1	d	c	f	e	b	a	0
∼	1	f	e	b	a	d	c	0

Then $(M, +,', \sim, 0, 1)$ is a hyper pseudo MV-algebra.

Let $\theta = \{(x, y)|x, y \in \{0, a, c, e\}\} \cup \{(x, y)|x, y \in \{1, b, d, f\}\}$. Easy calculations show that θ is a congruence relation on M.

Proposition 6.5. *Let θ be a congruence relation on M and $[x] = \{a \in M|x\theta a\}$, for any $x \in M$. Then*

(i) *θ is a congruence relation on dual hyper pseudo MV-algebra of $(M, +,', \sim, 0, 1)$,*

(ii) *$[1]$ is a filter of M,*

(iii) *$[0]$ is an ideal of M.*

(iv) *$[x]$ is convex, for any $x \in M$, that is $a \leq b \leq c$ and $a, b \in [x]$ imply $b \in [x]$, for any $a, b, c \in M$,*

(v) *If $x \in M$ and $a, b \in [x]$, then $a \wedge b \cap [x] \neq \emptyset$ and $a \vee b \cap [x] \neq \emptyset$.*

Proof. (i) Let M^* be the dual hyper pseudo MV-algebra of M. Clearly, (C2) holds. Let $x\theta y$ and $a\theta b$. Then by (C2), $x'\theta y'$ and $a'\theta b'$ and so by (C1), we have $a' + x'\overline{\theta}b' + y'$. Hence by (C2), $x \odot a\overline{\theta}y \odot b$, whence (C2) hold in M^*. Now, let $0\theta(x^\sim \odot y)$ and $0\theta(y^\sim \odot x)$, then $0\theta(y' + x)^\sim$ and so $0\theta a^\sim$, for some $a \in y' + x$. Hence $1\theta a$, so $1\theta y' + x$. By the similar way, we can show that $1\theta x' + y$, which implies that $x\theta y$. By Note 6.3(i), (C3), hold. Let $x, y \in M$. Then by (C4), we get

$$0\theta(x^\sim \odot y) \Leftrightarrow 0\theta(y' + x)^\sim \Leftrightarrow 1\theta(y' + x) \Leftrightarrow 1\theta(y' + x) \Leftrightarrow 0\theta(x' + y)^\sim = (y^\sim \odot x)$$

Therefore, θ is a congruence relation on M^*.

(ii) Let $x, y \in [1]$. Then by Proposition 3.10(v), we get $(x \odot y)\overline{\theta}(1 \odot 1) = 1$. Hence $x \odot y \subseteq [1]$. Now, let $a \in [1]$ and $a \leq b$, for some $b \in M$. Then $1 \in a' + b$. Moreover, $(b' + a)\overline{\theta}(b' + 1)$ and $1 \in b' + 1$, so $(b' + a)\theta 1$. It follows from (C3), $a\theta b$. Therefore, by Proposition 5.4, $[1]$ is a filter of M.

(iii) By (i) $[0]$ is a filter of M^*. Since $[0] = [1]'$, then by Theorem 5.7, $[0]$ is an ideal of M.

(iv) Let $b, x \in M$ and $a, c \in [x]$ such that $a \leq b \leq c$. Then $1 \in a' + b$, $1 \in b' + c$. Since $a\theta c$, then $(a' + b)\overline{\theta}(c' + b)$, so by $1 \in a' + b$, we get $1\theta(c' + b)$. By (C3), we conclude that $a\theta c$. Therefore, $[x]$ is convex.

(v) It is easy to show that, $a\theta b$ and $x\theta y$, imply $(a\vee x)\overline{\theta}(b\vee y)$ and $(a\wedge x)\overline{\theta}(b\wedge y)$, for any $x, y, a, b \in M$. Now, let $x \in M$ and $a, b \in [x]$. Then $(a \vee b)\overline{\theta}(x \vee x)$. Hence by Proposition 4.2(i), $(a \vee b)\theta x$. By the similar way, we can show that $(a \wedge b)\theta x$. ■

Theorem 6.6. *Let θ be a congruence relation on M and M/θ be the set of all congruence classes of M with respect to θ. Then $(M/\theta, \oplus, -, *, [0], [1])$ is a hype pseudo MV-algebra, where $[x] \oplus [y] = [x \oplus y]$, $[x]^- = [x']$ and $[x]^* = [x^\sim]$, for any $x, y \in X$.*

Proof. Since θ is a congruence relation on M, then clearly, $\oplus$ is a hyper operation on M/θ and $-$, $*$ are well define. Moreover, by definition of $\oplus$, $-$ and $*$, it can be easily obtained that $(HSMV1)$-$(HSMV8)$ hold in M/θ. Let $x, y \in M$ and $[1] \in ([x]^- \oplus [y]) \cap ([y]^- \oplus [y])$, then $[1] \in [x' + y] \cap [y' + x]$ and so $1\theta(x' + y)$, $1\theta(y' + x)$. Hence by (C3), $x\theta y$. That is $[x] = [y]$. Moreover, by (C4), we have

$$[1] \in [x]^- \oplus [y] \Leftrightarrow [1] \in [x' + y] \Leftrightarrow 1\theta x' + y \Leftrightarrow 1\theta y + x^\sim \Leftrightarrow [1] \in [y] \oplus [x]^*$$

Therefore, $(M/\theta, \oplus, -, *, [0], [1])$ is a hype pseudo MV-algebra. ∎

Definition 6.7. A filter F of M is called *normal filter* or simply N-filter if the relation θ_F by the following definition

$$\theta_F = \{(x, y) \in M \times M | (x' + y) \cap F \neq \emptyset, \quad (y' + x) \cap F \neq \emptyset\}$$

is a congruence relation on M. If F is a normal filter, the we use M/F to denote the set of all equivalence relation of M with respect to θ. We use $NF(M)$ to denote the set of all N-filters of M.

Proposition 6.8. *Let F be a normal filter of M. Then $x \in F$ if and only if $x'' \in F$ if and only if $(x^\sim)^\sim \in F$, for any $x \in M$.*

Proof. Let $x \in M$ and F be a normal filter of M and $*$, $-$ be the hyper operations in Theorem 6.6. Then there exists a congruence relation θ on M such that $F = [1]_\theta$. If $x \in F$, then $[x] = [1]$ and so $[x''] = ([x]^-)^- = ([1]^-)^- = [1]$. Hence $x'' \in F$. Conversely, let $x'' \in F$. Then $[x''] = [1]$ and so $[x]^-)^- = [1]$. Hence $[x] = [1]$, whence $x \in F$. By the similar way, we can show that $x \in F$ if and only if $x'' \in F$. ∎

In the next theorem, we try to find relation between N-filters and congruence relations on M.

Theorem 6.9.

(i) *F is a N-filter of M if and only if there exists a congruence relation θ on M such that $F = [1]$.*

(ii) *There is a bijection map between the set of all N-filters of M and the set of all congruence relations on M.*

Proof. (i) Suppose that F is a N-filter of M. Then the relation

$$\theta_F = \{(x, y) \in M \times M | (x' + y) \cap F \neq \emptyset, \quad (y' + x) \cap F \neq \emptyset\}$$

is a congruence relation on M. We show that $F = [1]$. Let $x \in F$. Then by Proposition 3.10(vii), and $(HSMV2)$ we have $1 \in x' + 1$ and $x \leq 0 + x = 1' + x$ and so by (C3), $x\theta_F 1$. Now, let $x \in [1]$, then by (C3), $1' + x \cap F \neq \emptyset$. Since by Proposition 3.10(vi), $1' + x \leq x$, then by Proposition 5.4, $x \in F$. Therefore, $F = [1]$. Conversely, let there exists a congruence relation ϕ on M such that $F = [1]_\phi$ and

$$\theta = \{(x, y) \in M \times M | (x' + y) \cap F \neq \emptyset, \quad (y' + x) \cap F \neq \emptyset\}.$$

We show that $\theta = \phi$. Let $x\theta y$, for some $x, y \in M$. Then $(x' + y) \cap F \neq \emptyset$ and $(y'+x) \cap F \neq \emptyset$ and so $1\phi(x'+y)$ and $1\phi(y'+x)$. Since ϕ is a congruence relation, then by (C3), $x\phi y$. Now, let $x\phi y$, then by (C3), $1\phi(x' + y)$ and $1\phi(y' + x)$ and so $(x' + y) \cap F \neq \emptyset$ and $(y' + x) \cap F \neq \emptyset$. Hence $x\theta y$. Therefore, θ is a congruence relation on M and so F is a N-filter.

(ii) Define the map $f : Con(M) \rightarrow NF(M)$ by $f(\theta) = [1]_\theta$, for any $\theta \in Con(M)$. It follows from (i), f is an onto map. Moreover, by the proof of (i), we can show that f is a bijection map. ∎

Note 6.10. Let F be a normal filter of M. Then there exists a congruence relation θ on M such that $F = [1]$. In the proof of Theorem 6.9, we shown that $\theta_F = \theta$.

Theorem 6.11. *Let M' be a hyper pseudo MV-algebra and $f : M \rightarrow M'$ be a homomorphism. Then $\theta = \{(x, y) \in M \times M | f(x) = f(y)\}$ is a congruence relation on M.*

Proof. Clearly, θ is an equivalence relation on M. Since f is a homomorphism, then (C1) and (C2) hold. Now, let $1\theta(x' + y)$ and $1\theta(y' + x)$, for some $x, y \in M$. Then $1 = f(1) \in f(x' + y) = f(x)' + f(y)$. By the similar way, $1 \in f(y)' + f(x)$. Since M' is a hyper pseudo MV-algebra, then $f(x) = f(y)$ and so $(x, y) \in \theta$. Hence by Note 6.3(i), (C3) holds. Since M' is a hyper pseudo MV-algebra, then

$$\begin{aligned}(x' + y)\theta 1 \quad &\Leftrightarrow \quad 1 = f(1) \in f(x)' + f(y) \Leftrightarrow 1 \in f(y) + f(x)^{\sim} \\ &\Leftrightarrow \quad f(1) \in f(y + x^{\sim}) \Leftrightarrow (y + x^{\sim})\theta 1.\end{aligned}$$

Therefore, θ is a congruence relation on M. ∎

Corollary 6.12. *Let M' be a hyper pseudo MV-algebra and $f : M \rightarrow M'$ be a homomorphism. Then $f^{-1}(1)$ is a N-filter of M.*

Proof. Since $\{1\}$ is a filter of M', then by Theorem 5.11(iv), $f^{-1}(1)$ is a filter of L. Hence by Theorem 6.9 and 6.11 and $f^{-1}(1) = [1]$, we conclude that $f^{-1}(1)$ is a N-filter of M. ∎

Theorem 6.13. *Let G be a N-filter of M. Then there is a one to one corresponding between the set of all filters of M/G, that is $F(M/G)$ and the set of all filters of M containing F, that is $F(M, G)$.*

Proof. Let $f : F(M/G) \rightarrow F(M, G)$ be a map was defined by $f(H) = \{x \in M \mid x \in [a]$, for some $[a] \in H\}$. Let H be a filter of $M/[F]$. Then $[1] \in H$ and so by Theorem 6.9, $F = [1] \subseteq f(H)$. If $x, y \in F(H)$, then $[x], [y] \in H$ and so $[y \odot x] = ([x]^{-} \oplus [y]^{-})^{*} \subseteq H$. Hence $y \odot x \subseteq f(H)$. Now, let $x \in f(H)$ and $x \leq y$, for some $y \in M$. Then $1 \in x' + y$, so $[1] \in [x]^{-} \oplus [y]$. Hence $[x] \leq [y]$ in hyper pseudo MV-algebra M/F. Since H is a filter of $M/[F]$, then $[y] \in H$ and so $y \in f(H)$. Therefore, $f(H)$ is a filter of M containing F. Clearly, f is one to one. Now, we show that f is onto. Let G be a filter of M containing F. Then we define $H = \{[x] \in M/[F] | x \in G\}$. Clearly, $[1] \in H$. Let $[x], [y] \in H$. Then there exist $a, b \in G$ such that $[x] = [a]$ and $[y] = [b]$. Since G is a filter, then

$(a'+b')^{\sim} \subseteq G$ and so $([x]^{-} \oplus [y]^{-})^{*} = ([a]^{-} \oplus [b]^{-})^{*} = [(a'+b')^{\sim}] \subseteq H$. Let $[x] \in H$ and $[x] \leq [y]$, for some $[y] \in M/F$. Then there exists $a \in G$ such that $[x] = [a]$ and $[1] \in [x]^{-} + [y] = [x' + y]$. Hence $1\theta(x' + y)\overline{\theta}(a' + y)$, where θ is a congruence relation in Definition 6.7. Since $[1] \subseteq G$ and $1\theta a' + y$, then we get $a' + y \cap G \neq \emptyset$ and so by $a \in G$, we conclude that $y \in G$. Hence $[y] \in H$ and so H is a filter of $M/[F]$. It is easy to show that $f(H) = G$. Therefore, f is one to one and onto. ∎

Theorem 6.14. *Let M' be a hyper pseudo MV-algebra, $f : M \to M'$ be a homomorphism and $F = f^{-1}(1)$. Then M/F is a hyper pseudo MV-algebra and $g : M/F \to M'$, was defined by $g([x]) = f(x)$, for any $[x] \in M/F$ is a one to one homomorphism.*

Proof. Since f is a homomorphism, then by Corollary 6.12, F is a N-filter of M and so M/F is a hyper pseudo MV-algebra. If θ be a congruence relation in Theorem 6.11, then by Note 6.10, $\theta = \theta_F$. Clearly, g is a homomorphism. We will show that g is one to one. Let $g([x]) = g([y])$, for some $x, y \in M$. Then by definition of g, we get $f(x) = f(y)$ and so $[x] = [y]$. Hence g is one to one. ∎

From now on in this section, for convenience, we use the same notations for the hyper operation and operations of M and M/θ, for any congruence relation θ on M.

Theorem 6.15. *Let F and G be two normal filters of M such that $F \subseteq G$. Then G/F is a normal filter of M/F and $\dfrac{M/F}{G/F} \cong M/G$.*

Proof. Since F and G are normal filters of M, then there exist two congruence relations θ and φ on M such that $F = [1]_{\varphi}$ and $G = [1]_{\theta}$. Consider the following relation on M/F.

$$\Theta = \{([x]_F, [y]_F) \in M/F \times M/F | (x, y) \in \theta\}$$

Clearly, Θ is an equivalence relation on M/F. We show that it is a congruence relation. Let $([x]_F, [y]_F) \in \Theta$. Then $(x, y) \in \theta$ and so $(x', y') \in \theta$. It follows that $([x']_F, [y']_F) = ([x]'_F, [y]'_F) \in \Theta$. By the similar way, we can show that $([x]^{\sim}_F, [y]^{\sim}_F) \in \Theta$. Let $([a]_F, [b]_F) \in \Theta$. Then $(a, b) \in \theta$. Since θ is a congruence relation on M, then we get $(x + a)\overline{\theta}(y + b)$. By definition of Θ, we conclude that $([x]_F + [a]_F)\overline{\Theta}([y]_F + [a]_F)$. Now, let $a, x \in M$ and $[1]_F\Theta([x]'_F + [a]_F)$ and $[1]_F\Theta([a]'_F + [x]_F)$. Then $[1]_F\Theta[x' + a]_F$, so there is $u \in x' + a$ such that $1\theta u$ and so $1\theta(x' + a)$. By the similar way, $1\theta(a' + x)$. Hence by (C3), $x\theta a$ and so $[x]_F = [a]_F$. Finally, we show that (C4) hold. Let $x, y \in M$ such that $[1]_F\Theta([x]'_F + [y]_F)$, then similar to the above arguments $1\theta(x' + y)$. Since θ is a congruence relation, then by (C4) $1\theta(y + x^{\sim})$ and so $[1]_F\Theta([y]_F + [x]^{\sim}_F)$. Hence Θ is a congruence relation on M/F. By the proof of Theorem 6.13, G/F is a filter of M/F. Suppose that $[x]_F \in M/F$. Then, by

$$[x]_F \in [[1]_F]_{\Theta} \Leftrightarrow [x]_F\Theta[1]_F \Leftrightarrow x\theta 1 \Leftrightarrow x \in [1]_{\theta} = [1]_F \Leftrightarrow x \in G \Leftrightarrow [x]_F \in G/F,$$

we obtain $[[1]_F] = G/F$, and so, by Theorem 6.13(i), G/F is a normal filter of M/F. Define $f : M/F \to M/G$, by $f([1]_F) = [1]_G$, for any $x \in M$. Clearly, f is an onto homomorphism. Moreover, $f^{-1}([1]_G) = \{[x]_F | x \in G\} = G/F$. Therefore, by Theorem 6.14, we have $\dfrac{M/F}{G/F} \cong M/G$. ∎

Definition 6.16. A congruence relation θ on M is called *strong congruence relation* if we replace (C1) with $(C'1)$ in Definition 6.2.

$(C'1)$ $x\theta y$ and $a\theta b$ imply $(x+a)\overline{\overline{\theta}}(y+b)$, for any $x, y, a, b \in M$.

Theorem 6.17. *If θ is a strong congruence relation on M, then M/θ is a pseudo MV-algebra.*

Proof. Let θ be a strong congruence relation on M. It suffices to show that $[x]+[y]$ has only one element, for any $x, y \in M$. Let $x, y \in M$ and $[a], [b] \in [x]+[y]$. Then by definition of $+$, there exist $u, v \in x + y$ such that $a\theta u$ and $b\theta v$. Since θ is a strong congruence relation, then $(x+y)\overline{\overline{\theta}}(x+y)$ and so $u\theta v$. Hence $a\theta b$, so $[a] = [b]$. Therefore, M/θ is a pseudo MV-algebra. ∎

7. Conclusions

In this paper, we defined the concept of hyper pseudo MV-algebras and investigate some properties of this structure. Then we introduced the concept of ideal and filter in hyper pseudo MV-algebras and obtained the relation between them. Finally, we used congruence relation in hyper pseudo MV-algebra and construct quotient hyper MV-algebras. For future research, we can work on category of hyper pseudo MV-algebras, relation between this category and category of hyper MV-algebras, relation between hyper pseudo MV-algebras and hyper pseudo K-algebras and fundamental relations on these structures.

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