REGULAR SUB-SEQUENTIALLY DENSE INJECTIVE IN THE CATEGORY OF S-POSETS

M. Haddadi

Department of Mathematics
Faculty of Mathematics, Statistics and Computer Science
Semnan University, Semnan
Iran
e-mails: haddadi_1360@yahoo.com, m.haddadi@semnan.ac.ir

Gh. Moghaddasi

Department of Mathematics
Hakim Sabzevary University, Sabzevar
Iran
e-mails: r.moghadasi@hsu.ac.ir, g.moghaddasi@gmail.com

Abstract. Sequentially dense monomorphisms were first introduced and studied by Giuli for projection algebras and followed by Ebrahimi, Mahmoudi, Moghaddasi and Shahbaz for S-acts. In this paper we use the notion of sub-Cauchy sequences and introduce the class of regular sub-sequentially dense monomorphisms for S-posets, denoted by $\mathcal{M}_s$. We investigate the properties of the class $\mathcal{M}_s$ and study $\mathcal{M}_s$-injectivity of S-posets. Further, we find $\mathcal{M}_s$-injective hull of S-posets over some kind of semigroups.

1. Introduction and preliminary

In [5], M.M. Ebrahimi and M. Mahmoudi introduced the interesting concept of a Cauchy sequence in an S-set, a set A together with an action of a semigroup S on A, see [11]. Then the concept of the notion of sequentially dense monomorphism and injectivity with respect to this kind of monomorphism have been investigated, see [6].

Introducing the category of S-posets, a poset A together with an action of a posemigroup on A, and algebraic study on this category was initiated by S. Fakhruddin, see [9, 10]. Recently, there are more people interested on studying on this category [3], [2], [13], [8]. In this paper, we are going to study sub-sequentially dense injective objects and the behavior of this type of injectivity in the category of S-posets. To do so, we first introduce the notions of sub-Cauchy sequences and convergent sub-Cauchy sequences. We then, in Section 2, use these concepts to define a categorical closure operator C_s, as introduced by Tholen and

Dikranjan in [14], on the category of S-posets. We also consider the regular and dense monomorphisms with respect to the closure C_s, that is $\mathcal{M}_s$-monomorphisms, and briefly study some categorical properties of this class of monomorphism which is needed subsequently. In Section 3, we study essential monomorphisms and injectivity relative to this class of monomorphisms in the category of S-posets. And finally, in Section 4, we give an example about what we did through the paper and study these concepts in the left zero posemigroups.

Now we briefly recall some concepts and we then give the preliminaries. A semigroup S is said to be *partially ordered* (or simply, a *posemigroup*) if it is also a poset whose partial order is compatible with the binary operation. An *S-poset* is a (possibly empty) poset A together with a monotone map $\lambda : A \times S \to A$, called the action of S on A, such that, for all $a \in A$ and $s, t \in S$, we have $a(st) = (as)t$, where $A \times S$ is considered as a poset with componentwise order and we denote $\lambda(a, s)$ by as. By an S-poset morphism (or morphism), we mean a monotone map between S-posets which is equivariant (or preserves the action).

Definition 1.1. Let S be a posemigroup and A be an S-poset. Then

(i) by a *sub-Cauchy sequence* over an S-poset A we mean a family $(a_s)_{s\in S}$ of ascending elements of A, that is $a_s \leq a_t$ if $s \leq t$, with $a_s t \geq a_{st}$ for all $s, t \in S$.

(ii) by a *sub-Cauchy sequence induced by* $a \in A$ we mean the sub-Cauchy sequence $(a_s = as)_{s\in S}$ and we show it by λ_a.

(iii) by a *limit* of a sub-Cauchy sequence $(a_s)_{s\in S}$ over A in some extension B of A we mean an element $b \in B$ such that $bs = a_s$, for all $s \in S$, and denote it by $lim(a_s)_{s\in S} = b$.

(iv) An S-poset A is said to be *sub-sequentially complete* (or simply *sub-s-complete*) if any sub-Cauchy sequence over A has a limit in A.

Separated S-posets, in which any two distinct points a and b in A can be separated by at least one $s \in S$, by $as \neq bs$, are an important class of s-posets. Here we should note that limits are not necessarily unique, unless A is separated.

In the next definition we generalize the concept of separated S-poset to the sub-separated one.

Definition 1.2.

(i) Let S be a posemigroup. An S-poset A is called *sub-separated* if $a \leq b$ in A whenever $as \leq bs$ for all $s \in S$.

(ii) For an S-poset A, we define $\mathcal{S}(A)$ to be the poset of all sub-Cauchy sequences over A with point-wise order. More explicitly, $\mathcal{S}(A) = \{(a_s)_{s\in S} \mid a_s t \geq a_{st}\}$ and $(a_s)_{s\in S} \leq (a'_s)_{s\in S}$ whenever $a_s \leq a'_s$, for all $s \in S$.

Remark 1.3. Every Sub-separated S-poset is a separated one. Indeed, $as = bs$, for all $s \in S$ implies $as \leq bs$ and $bs \leq as$, for all $s \in S$. Hence $a \leq b$ and $b \leq a$, by sub-separateness, and hence $a = b$.

Theorem 1.4.

(i) *The poset $\mathcal{S}(A)$ is an S-poset.*

(ii) *For every separated S-poset A, the S-poset $\mathcal{S}(A)$ is an extension of A. Also if A is sub-separated then $\mathcal{S}(A)$ is a regular extension of A.*

(iii) *The S-poset $\mathcal{S}(A)$ is sub-separated if $S^2 = S$.*

Proof. (i) To prove, first for a given S-poset A we define an action of S over $\mathcal{S}(A)$ as follows:

$$(a_s)_{s\in S} \cdot t = (a_{ts})_{s\in S},$$

for every $(a_s)_{s\in S} \in \mathcal{S}(A)$.

We note that, $\mathcal{S}(A)$ is closed under the defined action. Indeed, for every sub-Cauchy sequence $(a_s)_{s\in S}$ and $t \in S$, the sequence $(a_s)_{s\in S} \cdot t = (a_{ts})_{s\in S}$ is also a sub-Cauchy sequence, since for all $s \in S$ we have $a_{ts}s' \geq a_{(ts)s'}$.

Now we check the properties of being an S-posets:

(1) For every $t, t' \in S$, and a sub-Cauchy sequence $(a_s)_{s\in S}$, we have $((a_s)_{s\in S} \cdot t) \cdot t' = (a_{ts})_{s\in S} \cdot t'$. Take $(a_{ts})_{s\in S}$ to be $(b_s)_{s\in S}$, so $(a_{ts})_{s\in S} \cdot t' = (b_s)_{s\in S} \cdot t' = (b_{t's})_{s\in S} = (a_{t(t's)})_{s\in S} = (a_{(tt')s)})_{s\in S} = (a_s)_{s\in S} \cdot (tt')$.

(2) For sub-Cauchy sequences $(a_s)_{s\in S}$, $(a'_s)_{s\in S}$ and $t \in S$, if $(a_s)_{s\in S} \leq (a'_s)_{s\in S}$ then $(a_{ts})_{s\in S} \leq (a'_{ts})s \in S$, for every $s, t \in S$. Therefore $(a_s)_{s\in S} \cdot t \leq (a'_s)_{s\in S} \cdot t$.

(3) And, finally, for $t, t' \in S$ and sub-Cauchy sequence $(a)_{s\in S}$, if $t \leq t'$ then $ts \leq t's$, and hence $a_{ts} \leq a_{t's}$, by definition of sub-Cauchy sequence. Therefore, $(a_{ts})_{s\in S} \leq (a_{t's})_{s\in S}$, that is, $(a_s)_{s\in S}t \leq (a_s)_{s\in S}t'$.

(ii) To prove this part we consider the map $f : A \to \mathcal{S}(A)$ defined by $f(a) = \lambda_a$, see Definition 1.1 (ii). The defined f is one-one, since A is separated. Also since A is an S-poset, $as \leq bs$, for every $s \in S$, if $a \leq b$. This means that $\lambda_a \leq \lambda_b$, if $a \leq b$. That is f is order preserving. And finally f is equivariant, because $f(at) = (ats = a_{ts})_{s\in S} = (a_s)_{s\in S} \cdot t$.

Now suppose that A is, moreover, sub-separated. We show that the above S-poset embedding is an order embedding, too. Because if $\lambda_a \leq \lambda_b$ then $as \leq bs$, for every $s \in S$, and hence $a \leq b$, by sub-separateness of A.

(iii) To see that $\mathcal{S}(A)$ is sub-separated, take sub-Cauchy sequences $\gamma = (a_s)_{s\in S}$ and $\gamma' = (a'_s)_{s\in S}$ with $\gamma \cdot t \leq \gamma' \cdot t$ for all $t \in S$. This means that $a_{ts} \leq a'_{ts}$ for all $t, s \in S$. Now since $S^2 = S$, each $r \in S$ is in the form of st for some $t, s \in S$, then we have $a_r = a_{st} \leq a'_{st} = a'_r$ which means $\gamma \leq \gamma'$. ∎

2. Sub-sequential closure operator and dense regular monomorphisms

In this section, we are going to introduce a categorical closure operator C_s, in the sense of Tholen and Dikranjan in [14]. This is a weakening of C^d closure operator in [12], on the category of S-posets. We also consider the class of regular and dense monomorphisms with respect to the closure operator C_s and briefly study some categorical properties of this class of monomorphism which is needed in the sequel.

Definition 2.1. For an S-poset B, and a sub S-poset A of B, by the *s-closure* of A in B we mean $C_s(A) = \{b \in B | \forall s \in S, \exists a \in A; bs \leq a\}$.

We say that A is sub-*s-dense* in B if $C_s(A) = B$.

An S-poset morphism $f : A \to B$ is said to be *sub-sequentially dense* (or, simply, *sub-s-dense*) if $f(A)$ is a sub-s-dense sub S-poset of B.

Note 2.2. We show that the above introduced C_s is a categorical closure operator, in the sense of [14].

(Extensive) To prove $A \subseteq C_s(A)$, let $x \in A$. Then, since $xs \in A$ for all $s \in S$ and $xs \leq xs$ for $xs \in A$, $A \subseteq C_s(A)$.

(Monotonicity) To prove $A_1 \subseteq A_2$ implies $C_s(A_1) \subseteq C_s(A_2)$, let $x \in C_s(A_1)$. Then, for each $s \in S$ there exists $a \in A_1$ such that $xs \leq a$. But, since $A_1 \subseteq A_2$, there exists $a \in A_2$ such that $xs \leq a$. Therefore $C_s(A_1) \subseteq C_s(A_2)$.

(Continuity) We show that $f(C_s(A)) \leq C_s(f(A))$, for all S-poset morphism f from A. Let $y \in f(C_s(A))$. Then, there exists $x \in C_s(A)$ such that $y = f(x)$. Now for each $s \in S$ there exist $a \in A$ such that $xs \leq a$. Therefore, $ys = f(x)s = f(xs) \leq f(a)$, that is, $y \in C_s(A)$.

Theorem 2.3. *The sub-sequential closure operator is idempotent, that is, $C_s(C_s(A)) = C_s(A)$, if $S^2 = S$.*

Proof. By extensive, we have $C_s(A) \subseteq C_s(C_s(A))$. For the converse, let $x \in C_s(C_s(A))$. Then, for each $s \in S$, there exists $a_s \in C_s(A)$ such that $xs \leq a_s$. But, since $a \in C_s(A)$, there exist $a' \in A$ such that $as \leq a'_s$, for every $s \in S$. Now, using $S = S^2$, we have $xs = xs_1s_2 = (xs_1)s_2 \leq a_{s_1}s_2 \leq a'_{s_2}$, hence $x \in C_s(A)$. Therefore, $C_s(C_s(A)) \subseteq C_s(A)$ and hence $C_s(C_s(A)) = C_s(A)$. ∎

Theorem 2.4. *Let B be an S-poset and A be a sub S-poset of B in which every subset has a least element. Then A in B is sub-s-dense if and only if for every sub-Cauchy sequence λ_b, induced by $b \in B$, there exists a sub-Cauchy sequence $(a_s)_{s\in S}$ in A such that $\lambda_b \leq (a_s)_{s\in S}$.*

Proof. ($\Rightarrow$) Let A be sub-*s-dense* in B and λ_b be the sub-Cauchy sequence induced by $b \in B$. Then for each $s \in S$ consider the subset $A_s = \{a \in A \mid bs \leq a\}$ of A. Then A_s has a least element such as a_s. Now we consider $(a_s)_{s\in S}$ which is a sub-Cauchy sequence. Because $b_{st} = bst = b_st \leq a_st \in A_{st}$, hence $a_{st} \leq a_st$.

($\Leftarrow$) The converse is trivial. ∎

Theorem 2.5. *Let A be an S-poset in which every subset has a least element. Then the S-poset morphism $f : A \to \mathcal{S}(A)$ which maps every $a \in A$ to λ_a is sub-s-dense.*

Proof. We show that $C_s(f(A)) = \mathcal{S}(A)$. For each $\gamma = (a_s)_{s\in S} \in \mathcal{S}(A)$, since every subset of A has a least element, the set of upper bounds of $\gamma = (a_s)_{s\in S}$ has a least element, say $a \in A$. Therefore $\gamma \cdot t \leq \lambda_a$, for every $t \in S$, because $\gamma \cdot t = (a_{ts})s \in S$ and $a_{ts} \leq a_t s \leq as$. ∎

In Theorem 1.4, part (ii) it has been shown that every separated S-poset A is embedded in $\mathcal{S}(A)$. Now, by Theorem 2.5, we get the following corollary:

Corollary 2.6. *Every sub-separated S-poset A in which every subset has a least element is regular sub-s-dense sub-S-poset of $\mathcal{S}(A)$.*

Notation 2.7. In the rest of this paper we are going to study injectivity with respect to the class of regular sub-sequentially dense monomorphisms in the category of S-posets. So from now on we denote the class of regular sub-sequentially dense monomorphisms by $\mathcal{M}_s$.

In the following we see that the class of sub-s-dense S-poset monomorphisms has "good" properties with respect to composition.

Theorem 2.8. *Let $\mathcal{M}_s$ be the class of all sub-s-dense S-poset monomorphisms. Then,*

(i) *the class $\mathcal{M}_s$ is composition closed if $S^2 = S$.*

(ii) *the class $\mathcal{M}_s$ is isomorphism closed; that is, contains all isomorphisms and is closed under composition with isomorphisms.*

(iii) *the class $\mathcal{M}_s$ is left cancelable; that is, $gf : A \to B \to C, g : B \to C \in \mathcal{M}_s$ imply $f \in \mathcal{M}_s$.*

Proof. (i) It is easy to see that the composition of regular monomorphisms is a regular monomorphism. Now let $f : A \to B$ and $g : B \to C$ be $\mathcal{M}_s$-monomorphisms between S-poset. Then for every $c \in C$ and $s \in S$, since $S^2 = S$, the inequality $cs = cs_1s_2 \geq g(b_{s_1})s_2 = g(b_{s_1}s_2)$, for some b_{s_1}, is given by the fact that g is sub-s-dense. Now since f is sub-s-dense, $b_{s_1}s_2 \geq f(a_s)$, for some $a_s \in A$. So $cs = cs_1s_2 \geq g(b_{s_1})s_2 = g(b_{s_1}s_2) \geq g(f(a_s))$.

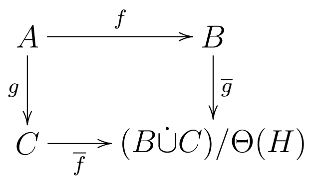
(ii) It is trivial.

(iii) Suppose that $f(a_1) \leq f(a_2)$. Then $gf(a_1) \leq gf(a_2)$ and hence, by regularity of gf, we have $a_1 \leq a_2$. That is f is regular. Also for each $b \in B$ and $s \in S$, there exists $a_s \in A$ such that $g(bs) = g(b)s \geq gf(a_s)$. Now, using regularity of g, we get $bs \geq f(a_s)$. That is $f \in \mathcal{M}_s$. ∎

Theorem 2.9. *In the category of S-posets, pushouts transfer $\mathcal{M}_s$-morphisms.*

Proof. As mentioned in [3], the pushout of S-poset maps $f : A \to B$ and $g : A \to C$ is the quotient of the coproduct $B\dot{\cup}C$ by the congruence $\Theta(H)$

generated by $H = \{(f(a), g(a)) \mid a \in A\}$. Now consider the pushout diagram



in which f is an $\mathcal{M}_s$-morphisms. If $\overline{f}(c_1) = [c_1]_\Theta \leq [c_2]_\Theta = \overline{f}(c_2)$, then there exist $a_1, ..., a_n \in A$ such that

$$c_1 = g(a_1)\Theta(H)f(a_1) \leq f(a_2)\Theta(H)g(a_2) \leq \cdots \leq f(a_n)\Theta(H)g(a_n) = c_2.$$

Now, by regularity of f, we have

$$a_1 \leq a_2, a_3 \leq a_4, ..., a_{n-1} \leq a_n,$$

hence, since g is order preserving, we have

$$c_1 = g(a_1) \leq g(a_2) \leq g(a_3) \leq g(a_4) \leq \cdots \leq g(a_{n-1}) \leq g(a_n) = c_2.$$

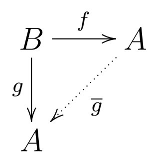
That is, $\overline{f}$ is regular. Also, $\overline{f}$ is sub-sequentially dense, because, for each $[x]_\Theta \in (B\dot{\cup}C)/\Theta(H)$, $[x]_\Theta = [c]_\Theta$ or $[x]_\Theta = [b]_\Theta$, for some $c \in C$ and $b \in B$. In the first case $[x]_\Theta s = [c]_\Theta s = [cs]_\Theta = \overline{f}(cs)$, and in the second case, since f is sub-sequentially dense, there exists $a_s \in A$ with $bs \leq f(a_s)$. Hence $[x]_\Theta s = [b]_\Theta s = [bs]_\Theta \leq [f(a_s)]_\Theta = [g(a_s)]_\Theta = \overline{f}(g(a_s))$. ∎

3. Regular sub-sequential dense injectivity

This section is devoted to the investigation of the behavior of $\mathcal{M}_s$-injectivity in the category of S-posets. First we give some definitions.

Definition 3.1. An S-poset A is called:

(1) *regular sub-sequentially dense injective* or briefly *$\mathcal{M}_s$-injective* if it is injective with respect to $\mathcal{M}_s$-monomorphisms. That is every commutative diagram



with $f \in \mathcal{M}_s$, can be completed by some S-poset morphism $\overline{g}$.

(2) *regular sub-sequentially absolute retract* or briefly *$\mathcal{M}_s$-absolute retract* if it is a retract of each of its regular sub-s-dense extensions.

Theorem 3.2. *An S-poset A is $\mathcal{M}_s$-injective if and only if for every $\mathcal{M}_s$-monomorphism $h : B \to B \sqcup cS$ to a singly generated extension of B, every S-poset morphism $f : B \to A$ can be extended to $g : B \sqcup cS \to A$.*

Proof. One direction is obvious. For the converse, let $h : B \to C$ be a sub-dense regular monomorphism and $f : B \to A$ be an S-poset morphism. Applying Zorn's Lemma on the poset of all sub S-poset D of C with $h(B) \subseteq D$ and such that there exists an S-poset morphism $g : D \to A$ such that $gh = f$, we get a maximal such S-poset, say D. If $D = C$ then the proof is complete, otherwise there exists $c \in C - D$. Now $h : B \to D \sqcup cS$ is an sub-dense regular monomorphism and by hypothesis there is an S-poset morphism $\overline{g}$ which extends g. This contradicts the maximality of D, so $C = D$. ∎

Theorem 3.3. *If a sub-separated S-poset A is sub-complete then it is sub-sequentially absolute retract.*

Proof. Let $f : A \to B$ be an $\mathcal{M}_s$-morphism. Then, by Theorem 2.4, for each $b \in B$, there exists a sub-Cauchy sequence $(a_S)_{s\in S}$ in A such that $\lambda_b \leq (a_s)_{s\in S}$. Since A is sub-complete, $(a_s)_{s\in S}$ has a limit such as $a_b \in A$. Now we define $g : B \to A$ by $g|_A = id_A$ and for $b \in B - A$, $g(b) = a_b$. The defined g is order preserving, because if $b \leq b'$ in B three following cases may occur:

Case 1. Both $b, b' \in B - A$. In this case $bt \leq b't$, for every $t \in S$. Hence $(a_s)_{s\in S} \leq (a'_s)_{s\in S}$, see how we get $(a_s)_{s\in S}$ and $(a'_s)_{s\in S}$ in the proof of Theorem 2.4. Therefore $a_s = a_b s \leq a_{b'} s = a'_s$ and, since A is sub-separated $a_b \leq a_{b'}$, $g(b) \leq g(b')$.

Case 2. Now let $b \in B - A$, $b' \in A$. Then if $b \leq b'$ then, for all $t \in S$, $bt \leq b't$ but, since $b't \in A$ and a_t is a least element in A such that $bt \leq a_t$, we have $a_t \leq b't$, for all $t \in S$. Therefore, $a_b t = a_t \leq b't$ for all $t \in S$. Now $a_b \leq b'$ is a clear result of sub-separateness of A, thus $g(b) \leq g(b')$.

Case 3. If $b, b' \in A$ there is nothing to prove.

Now, we show that g is equivariant. The only thing we have to check is $g(bs) = g(b)s$ when $b \in B - A$, for every $s \in S$. But, two cases may happen:

Case 1. $bs \in A$, for some $s \in S$. Then $g(bs) = bs$. So, by definition of a_s, $a_s = bs$. Since $a_b = \lim(a_s)_{s\in S}$, $a_b s = bs$. That is $g(b)s = g(bs)$.

Case 2. $bs \in B - A$, for some $s \in S$. Then take $g(bs)$ to be $a_{bs} = \lim(a'_{(t)})_{t\in S}$ and $g(b)$ to be $a_b = \lim(a_t)_{t\in S}$. By definition of $(a'_{(t)})_{t\in S}$ and $(a_t)_{t\in S}$, we have $a_{st} = a'_t$, for every $t \in S$. Hence $a_b st = a_{st} = a'_t$, for every $t \in S$. So

$$g(bs) = \lim(a'_{(t)})_{t\in S} = a_b s = g(b)s,$$

and we are done. ∎

Theorem 3.4. *An S-poset A is $\mathcal{M}_s$-injective if and only if it is an $\mathcal{M}$-absolute retract.*

Proof. Since in the category of S-posets, pushouts transfer $\mathcal{M}_s$- morphisms, by Theorem 2.2 in [4] we are done. ∎

Corollary 3.5. *Every sub-separated and sub-complete S-poset A is $\mathcal{M}_s$-injective.*

Proof. By Theorems 3.3 and 3.4 there is nothing to prove. ∎

Now, we study $\mathcal{M}_s$-essential extensions to get the $\mathcal{M}_s$-injective hulls.

Definition 3.6. An $\mathcal{M}_s$-monomorphism $f : A \to B$ is called $\mathcal{M}_s$-*essential* if it is essential with respect to $\mathcal{M}_s$-monomorphisms. That is if $g \circ f : A \to \ B \to C$ is a $\mathcal{M}_s$-monomorphism then g is an $\mathcal{M}_s$-monomorphism.

Also, by a *sub-s-regular injective hull* or, briefly, $\mathcal{M}_s$-*injective hull* of an S-poset A, we mean an $\mathcal{M}_s$-essential extension $f : A \to B$ in which B is an $\mathcal{M}_s$-injective.

Notice that an $\mathcal{M}_s$-injective hull is unique up to isomorphism.

Lemma 3.7. *A regular monomorphism f is $\mathcal{M}_s$-essential if and only if it is essential and s-dense.*

Proof. Let f be sub-s-essential. Then, by definition, f is s-dense. Also it is essential, for if $g : B \to C$ is a morphism such that $g \circ f$ is an S-monomorphism then since $g \circ f : A \to g(B)$ is a sub-dense monomorphism (this is because f is sub-dense and $g : B \to g(B)$) and f is sub-dense essential we get that $g : B \to g(B)$ is a sub-dense monomorphism and hence g is monomorphism. The converse is easy to show. Just notice that if the composition $g \circ f$ of two S-maps is an s-dense monomorphism then g is an s-dense monomorphism, too. ∎

We are going to find an $\mathcal{M}_s$-essential extension for an S-poset A, Banaschewski's condition, and Theorems of well behavior of injectivity for $\mathcal{M}_s$-injectivity, see [1]. To do so, for each S-poset A, we take the poset $\mathcal{C}(A)$ to be the class of all Cauchy sequences; that is $\mathcal{C}(A) = \{(a_s)_{s\in S} \mid a_{st} = a_s t$, for every $s, t \in S$ and $a_s \leq a_t$ if $s \leq t\}$ in which $(a_s)_{s\in S} \leq (a'_s)_{s\in S}$ whenever $a_s \leq a'_s$, for every $s \in S$.

Now, we have the following theorem:

Theorem 3.8. *Given an S-poset A,*

(i) *the poset $\mathcal{C}(A)$ is an S-poset.*

(ii) *if A is sub-separated then the S-poset $\mathcal{C}(A)$ is an $\mathcal{M}_s$-essential extension of A.*

Proof. (i) To make $\mathcal{C}(A)$ an S-poset, we define an action of S on $\mathcal{C}(A)$ as follows:

$$(a_s)_{s\in S} \cdot t = \lambda_{a_t}, \text{ for every } (a_s)_{s\in S} \in \mathcal{C}(A) \text{ and for every } t \in S.$$

Obviously, $\mathcal{C}(A)$ is closed under the defined action. Now note that $\mathcal{C}(A)$ with the defined action is an S-poset, because:

(1) for every $t, t' \in S$, and each Cauchy sequence $(a_s)_{s\in S}$, we have

$$((a_s)_{s\in S} \cdot t) \cdot t' = \lambda_{a_t} \cdot t' = \lambda_{a_t t'} = \lambda_{a_{tt'}} = (a_s)_{s\in S} \cdot (tt').$$

The third equality is given by the definition of the Cauchy sequence $(a_s)_{s\in S}$.

(2) for each pair of the Cauchy sequences $(a_s)_{s\in S}$, $(a'_s)_{s\in S}$ and $t \in S$, if $(a_s)_{s\in S} \leq (a'_s)_{s\in S}$ then $a_t \leq a'_t$, and hence $\lambda_{a_t} \leq \lambda_{a_{t'}}$. That is, $(a_s)_{s\in S} \cdot t \leq (a'_s)_{s\in S} \cdot t$.

(3) and finally for $t, t' \in S$ and a Cauchy sequence $(a)_{s\in S}$, if $t \leq t'$ then $a_t \leq a_{t'}$, by Definition 1.1, therefore $\lambda_a \leq \lambda_{a_{t'}}$.

(ii) To prove this part, we consider the map $f : A \to \mathcal{C}(A)$ defined by $f(a) = \lambda_a$. First we show that f is an S-poset monomorphism. Indeed, f is one-one since A is separated. Also since A is an S-poset, $a \leq b$ implies $as \leq bs$, for every $s \in S$. So, if $a \leq b$ then $\lambda_a \leq \lambda_b$. That is, f is order preserving. And, finally, f is equivariant, because $f(at) = (ats = at_s = a_{ts})_{s\in S} = (a_s)_{s\in S} \cdot t$.

Regularity of f is a clear consequence of the fact that A is sub-separated. Also, f is sub-sequentially dense. Because for every Cauchy sequence γ and every $s \in S$, we have $\gamma \cdot s \leq \lambda_{a_s} \in f(A)$.

Now, we are going to show that f is $\mathcal{M}_s$-essential. Let $gf : A \to \mathcal{C}(A) \to C$ be an $\mathcal{M}_s$-morphism, for some S-poset morphism g, and $g(\gamma) = g(\gamma')$. Then

$$g(\gamma)s = g(\gamma \cdot s) = g(\lambda_{a_s}) = g(\lambda_{a'_s}) = g(\gamma' \cdot s) = g(\gamma')s,$$

that is $gf(a_s) = g(f(a'_s))$. Since gf is one-one, we have $a_s = a'_s$.

Now, suppose that $g(\gamma) \leq g(\gamma')$, then

$$g(\gamma)s = g(\gamma \cdot s) = g(\lambda_{a_s}) = gf(a_s) \leq gf(a'_s) = g(\lambda_{a'_s}) = g(\gamma' \cdot s) = g(\gamma)s,$$

for each $s \in S$. Using the regularity of gf, we have $a_s \leq a'_s$, for every $s \in S$. So $\gamma \leq \gamma'$. Finally, we show that g is sub-sequentially dense. For each $c \in C$, since gf is sub-sequentially dense, for every $s \in S$, there exists $a_s \in A$ such that $cs \leq gf(a_s)$. That is $cs \leq g(\lambda_{(a_s)})$ and we are done. ∎

Theorem 3.9. *Let $S^2 = S$. Then for each S-poset A, the S-poset $\mathcal{C}(A)$ is sub-s-complete.*

Proof. Let $\gamma = (\gamma_s)_{s\in S}$ be a sub-Cauchy sequence in $\mathcal{C}(A)$. That is $\gamma_s = (a^s_t)_{t\in S}$ is a Cauchy sequence, for every $s \in S$. So we have $a^s_t t' = a^s_{tt'}$, for every $s, t, t' \in S$. And hence, for each $t \in S$,

$$\gamma \cdot t = \lambda_{\gamma_t} = (\gamma_t \cdot s)_{s\in S} = (\lambda_{a^t_s})_{s\in S} = (a^t_s t')_{s,t'\in S} = (a^t_{st'})_{s,t\in S} = (a^t_k)_{k\in S} = \gamma_t,$$

in which the sixth equality is given by $S^2 = S$. So γ is a limit for the sub-Cauchy sequence $(\gamma_s)_{s\in S}$. ∎

Theorem 3.10. *Let $S^2 = S$ and A be a sub-separated S-poset. Then $\mathcal{C}(A)$ is the $\mathcal{M}_s$-injective hull of A.*

Proof. By Theorems 3.3 and 3.9, $\mathcal{C}(A)$ is $\mathcal{M}_s$-absolute retract. And since we have Banaschewki's condition for sub-separated S-posets, so $\mathcal{C}(A)$ is injective, see Corollary 3.5. Now, by Theorem 3.8, we are done. ∎

4. Left zero posemigroup

In this short section, we examine the above results for a special case of ordered semigroups. The semigroup we have chosen here is the left zero one. We will see that the $\mathcal{M}_s$-injective hulls in this special and important class of S-posets, have a simpler form.

To do so, we will also use the morphism notation A^S of Cauchy sequences and imitate the above proofs in this notation. We will also see that the fixed elements of an S-poset A in this class play an important role to simplify the $\mathcal{M}_s$-injective hulls.

An element a of an S-poset A is called a *fixed element* if $as = a$ for all $s \in S$. The set of all fixed elements of an S-poset A is denoted by $FixA$, which is in fact a sub S-poset of A.

Note also that, a partial order on a left zero semigroup is automatically compatible with the operation of S. For, if $s \leq t$ and $r \in S$, then we have $sr = s \leq t = tr$ and $rs = r = rt$.

Remark 4.1. Analogously to the proof of Theorem 3.8 part (i), one can easily see that $(FixA)^S$ is an S-poset, except that we should note that the action on $(FixA)^S$ is of the form $f \cdot t : S \to A$ which maps every $s \in S$ to $f(t)s = f(ts) = f(t)$, for every $f \in (FixA)^S$ and every $t \in S$.

Theorem 4.2. *Let S be a left zero posemigroup and A be an S-poset. Then, the S-poset $(FixA)^S$ is sub-complete.*

Proof. Let $\gamma = (f_s)_{s\in S}$ be a sub-Cauchy sequence over $(FixA)^S$. Hence $f_s \cdot r \geq f_{sr} = f_s$ for all $f_s \in (FixA)^S$ and $r \in S$. That is $f_s \cdot r(t) = f_s(rt) = f_s(r) \geq f_s(t)$ for all $r, t \in S$ and this implies that f_s is a constant sequence. So $f_s(t) = f_s(s)$. Now, consider $f : S \to FixA$ which maps each $s \in S$ to $f_s(s)$. Then, for every $s \in S$, we have

$$f \cdot s(t) = f(st) = f(s) = f_s(s) = f_s(t).$$

That is, f is a limit of the sub-Cauchy sequence $(f_s)_{s\in S}$ in $(FixA)^S$. ∎

According to the above theorem and Theorems 3.3 and 3.4, we have the following.

Theorem 4.3. *Let S be a left zero posemigroup and A be a sub-separated S-poset. Then, the S-poset $(FixA)^S$ is sub-sequential regular injective.*

Theorem 4.4. *Let S be a left zero posemigroup and A be a sub-separated S-poset. Then, the S-poset $(FixA)^S$ is the sub-sequential regular injective hull of A.*

Proof. First, we show that $(FixA)^S$ is sub-sequential dense regular essential extension of A. To do so, we define the map $\varphi : A \to (FixA)^S$ by

$$\varphi(a) = \lambda_a : S \to FixA,$$

in which $\lambda_a(s) = as$. We note that $\varphi(A)$ is s-dense in $(FixA)^S$. Because, for each $f \in (FixA)^S$ and $t \in S$ consider $\lambda_{f(t)}$. Then we have

$$f \cdot t(s) = f(ts) = f(t) = f(t)s = \lambda_{f(t)}(s), \tag{4.1}$$

so $(FixA)^S \subseteq C_s(\varphi(A))$. It is clear that φ is a regular monomorphism.

Now, we show that φ is essential. Let $\alpha : (FixA)^S \to B$ be an S-poset morphism such that $\alpha \circ \varphi$ is one-one and $\alpha(f) = \alpha(f')$ for some $f, f' \in (FixA)^S$. Hence, for each $t \in S$ we have $\alpha(f)t = \alpha(f')t$. Thus, by (4.1),

$$\alpha \circ \varphi(f(t)) = \alpha(\lambda_{f(t)}) = \alpha(\lambda_{f'(t)}) = \alpha \circ \varphi(f'(t)),$$

for each $t \in S$, and therefore $f(t) = f'(t)$, for each $t \in S$. Now, Lemma 3.7 ensures that φ is sub s-essential and by the former theorem we are done. ∎

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