A NOTE ON NON-FRAGMENTABLE SUBSPACE OF $\ell_\infty^c(\Gamma)$

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Abstract. In this paper we consider $\ell_\infty^c(\Gamma)$ where Γ is uncountable and introduce subspaces $\{A_{\mathcal{P}}\}_{\mathcal{P}\in\Sigma}$ of $\ell_\infty^c(\Gamma)$ which are fragmented by a metric that generates the discrete topology but $A = \bigcup_{\mathcal{P}\in\Sigma} A_{\mathcal{P}}$ is not countable unions of fragmentable subspaces.

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1. Introduction

A topological space X is fragmentable if there exists a metric $d(.,.)$ on X such that for every $\varepsilon > 0$ and every nonempty set $A \subseteq X$ there exists a nonempty subset $B \subseteq A$ which is relatively open in A and $d\text{-}diam(B) = sup\{d(x,y) : x, y \in B\} < \varepsilon$. In such a case we say that the metric d fragments X. Obviously subspaces of fragmentable space are fragmentable, metric spaces are fragmentable and if τ_1 and τ_2 are two topology on set X such that τ_1 is stronger than τ_2 and (X, τ_2) is fragmentable then (X, τ_1) is fragmentable.

If X is countable union of fragmentable closed subspaces then X is fragmentable [1, Theorem 5.1.10]. This is not true when we replace countable by uncountable. Let Γ be an uncountable set and $Y = \ell_\infty^c(\Gamma)$ be the space of all bounded real-valued functions with countable support defined on Γ. This space by supremum norm is closed subspace of $\ell_\infty^c(\Gamma)$. In next section we introduce a subspace of Y which is uncountable unions of subspaces which (by weak topology) are fragmentable by discrete metric but the space is not even countable unions of fragmentable spaces.

In [6] the following topological game was used to characterize the fragmentability of the space X. Two player $\mathcal{A}$ and $\mathcal{B}$ alternatively select subset of X. The player $\mathcal{A}$ starts the game by choosing some nonempty subset A_1 of X, then the player $\mathcal{B}-$ chooses some nonempty relatively open subset B_1 of A_1. Then again $\mathcal{A}$ selects an arbitrary nonempty subset $A_2 \subseteq B_1$ and $\mathcal{B}$ responds by choosing some nonempty relatively open subset B_2 of A_2. Continuing this alternative selection of sets the two players generate a sequence of sets

$$A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \cdots$$

which we call a play and denote by $p = (A_i, B_i)_{i\geq 1}$. We say that the player $\mathcal{B}$ is winner whenever the set $\bigcap_{i\geq 1} A_i = \bigcap_{i\geq 1} B_i$ contains at most one point, otherwise the player $\mathcal{A}$ is winner. A strategy w for the player $\mathcal{B}$ is a mapping which assigns to each partial play, $A_1 \supseteq B_1 \supseteq A_2 \supseteq B_2 \supseteq \cdots \supseteq A_k$, some nonempty set $B_k = w(A_1, B_1, ..., A_k)$ which is relatively open subset of A_k.

We call the play $p = (A_i, B_i)_{i\geq 1}$, a w-play if, $B_i = w(A_1, B_1, ...s, A_i)$ for every $i \geq 1$. The strategy w is a winning strategy for $\mathcal{B}$ if, the player $\mathcal{B}$ wins every w-play. We denote such a game by G_f.

The following theorem determines the relation between fragmentability and topological game:

Theorem 1.1 [6, Theorem 1.1] *The topological space X is fragmentable if and only if the player $\mathcal{B}$ has a winning strategy for the game G.*

The following theorem is proved in [7, Lemma 3] about spaces which are countable unions of fragmentable spaces:

Theorem 1.2 *If X is countable unions of fragmentable subspaces then there exists a strategy w for the player $\mathcal{B}$ in the game G such that for every w-play $p = (A_i, B_i)_{i\geq 1}$ the set $\bigcap_{i\geq 1} B_i$ contains at most countable point.*

Let τ_1, τ_2 be two (not necessarily distinct) topologies on the set X. We say that (X, τ_1) is fragmented by a metric d which majorizes the topology τ_2 if the topology generated by d is stronger than or equal to the topology τ_2.

Theorem 1.3 [5, Theorem 1.2] *Let τ_1, τ_2 be two (not necessarily distinct) topologies on the set X. The space (X, τ_1) is fragmented by a metric d which majorizes τ_2 if and only if there exists a strategy w for the player $\mathcal{B}$ in the game G in (X, τ_1) such that, for every w-play $p = (A_i, B_i)_{i\geq 1}$ either $\bigcap_{i\geq 1} A_i = \bigcap_{i\geq 1} B_i = \emptyset$ or $\bigcap_{i\geq 1} B_i = \{x\}$ for some $x \in X$, and for every τ_2-open set U that contains x, there exists some integer $k > 0$ with $B_k \subseteq U$.*

Let (X,τ) be a topological space which fragmented by metric d. By use of recent Theorem we can determine that d generates the topology τ or not.

In general, d does not generate the topology τ. For example $(X=\ell_\infty, weak)$ is fragmentable since $(B_X, weak^*)$ is metrizable and $X=\bigcup_{n\in\mathbb{N}} nB_X$ but it is proved in [4, Example 3.2] that this space is not fragmented by a metric which majorizes the weak topology.

If the topology on X is discrete then obviously X is fragmented by each metric on it and then X is fragmented by a metric which generates the discrete topology.

2. Results

Let $x\in Y=\ell^c_\infty(\Gamma)$, $supp(x)=\{\alpha\in\Gamma : x(\alpha)\neq 0\}$, $A=\{x\in\ell^c_\infty(\Gamma) : x(\alpha)=1,\ \alpha\in supp(x)\}$.

Let Σ be the collection of all partitions of Γ such that, for each partition $\mathcal{P}\in\Sigma$, if $I\in\mathcal{P}$, then I is countable.

Let $\mathcal{P}\in\Sigma$, define $A_\mathcal{P}=\{x\in A : supp(x)=I$, for some $I\in\mathcal{P}\}$, obviously $A=\bigcup_{\mathcal{P}\in\Sigma} A_\mathcal{P}$.

Theorem 2.1 *If $\mathcal{P}\in\Sigma$ then $(A_\mathcal{P}, weak)$ is discrete.*

Proof. Let $x_0\in A_\mathcal{P}$. We show that $(\{x_0\}, weak)$ is open in $A_\mathcal{P}$.

If $\alpha\in supp(x_0)$, then $x_0(\alpha)=1$ and $x(\alpha)=0$ for other $x\in A_\mathcal{P}$, that implies $x_0\notin\overline{A_\mathcal{P}\setminus\{x_0\}}$. Therefore, there exists $f\in Y^*$ such that $f(x_0)=1$ and $f(x)=0$ for other $x\in A_\mathcal{P}$. Put $B=\{x\in A_\mathcal{P} : |f(x-x_0|<\frac{1}{2}\}$. B is open in $A_\mathcal{P}$ by weak topology and contains just x_0. ∎

For every $\mathcal{P}\in\Sigma$, the set $A_\mathcal{P}$ by weak topology is closed in A. Theorem 2.1 implies the following theorem:

Theorem 2.2 *If $\mathcal{P}\in\Sigma$ then $(A_\mathcal{P}, weak)$ is fragmented by a metric which generates the discrete topology.*

It is proved in [2, Theorem 3.1] that $(A, weak)$ is not fragmentable. By use of property of Y^*, we show that $(A, weak)$ is not countable unions of fragmentable subspaces.

Lemma 2.3 *Let Γ_1 be an uncountable subset of Γ and $y\in Y^*$, then there exists an uncountable subset $J_y(\Gamma_1)$ of Γ_1 such that $y(x)=0$ for each $x\in A$ where $supp(x)\subseteq J_y(\Gamma_1)$.*

Proof. It is proved in [3] that for $y\in Y^*$, there exists a countable subset I_y of Γ such that $y(x)=0$ for $x\in A$ where $supp(x)\subseteq {I_y}^c$. If $J_y(\Gamma_1)=\Gamma_1\cap I^c_y$, then $y(x)=0$, for $x\in A$ where $supp(x)\subseteq J_y(\Gamma_1)$. ∎

Corollary 2.4 *Let $\Gamma_1 \subseteq \Gamma$ be uncountable and $y_1, y_2, \ldots, y_n \in Y^*$, then there exists an uncountable subset $J_{y_1 y_2 \cdots y_n}(\Gamma_1)$ of Γ_1 such that*

$$y_1(x) = y_2(x) = \cdots = y_n(x) = 0 \text{ for each } x \in A,$$

where

$$supp(x) \subseteq J_{y_1 y_2 \cdots y_n}(\Gamma_1).$$

Proof. We get $J_1 = J_{y_1}(\Gamma_1)$ and $J_2 = J_{y_2}(J_1)$ and continue this process to get

$$J_n = J_{y_n}(J_{n-1}).$$

Put $J_{y_1 y_2 \cdots y_n}(\Gamma_1) = J_n$, then

$$y_1(x) = y_2(x) = \cdots = y_n(x), \text{ for } x \in A,$$

where

$$supp(x) \subseteq J_{y_1 y_2 \cdots y_n}(\Gamma_1).$$

■

Theorem 2.5 $(A, weak)$ *is not countable unions of fragmentable subspaces.*

Proof. By Theorem 1.2 it is enough to show that there exists a play $p = (A_i, B_i)_{i \geq 1}$ in the game G such that $\bigcap_{i \geq 1} B_i$ has uncountable point. Let player $\mathcal{A}$ select $A_1 = A$ and player $\mathcal{B}$ select non-empty and relatively open subset $B_1 \subseteq A_1$. Let $x_1 \in B_1$, then there are $y_{11}, y_{12}, \ldots, y_{1n_1} \in Y^*$ and $\varepsilon_1 > 0$ such that $B'_1 \subseteq B_1$, where $B'_1 = \{x \in A_1 : |y_{11}(x - x_1)| < \varepsilon_1, \ldots, |y_{1n_1}(x - x_1)| < \varepsilon_1\}$. Put $I_1 = supp(x_1)$ and $J_1 = J_{y_{11} y_{12} \cdots y_{1n_1}}({I_1}^c)$. Let

$$A_2 = \{x \in B'_1 : I_1 \subseteq supp(x) \subseteq I_1 \cup J_1\}.$$

Let $B_2 \subseteq A_2$ (non-empty and relatively open) be selected. Let $x_2 \in B_2$, then there are $y_{21}, y_{22}, \ldots, y_{2n_2} \in Y^*$ and $\varepsilon_2 > 0$ such that $B'_2 \subseteq B_2$ where

$$B'_2 = \{x \in A_2 : |y_{21}(x - x_2)| < \varepsilon_2, \ldots, |y_{2n_2}(x - x_2)| < \varepsilon_2\}.$$

Put $I_2 = supp(x_2)$ and $J_2 = J_{y_{21} y_{22} \cdots y_{2n_2}}({I_2}^c \cap J_1)$. Let

$$A_3 = \{x \in B'_2 : I_2 \subseteq supp(x) \subseteq I_2 \cup J_2\}.$$

We get B'_3 similarly. Following this process, in m'th stage we have $I_m = supp(x_m)$ and $J_m = J_{y_{m1} y_{m2} \cdots y_{mn_m}}({I_m}^c \cap J_{m-1})$ and

$$A_m = \{x \in B'_{m-1} : I_m \subseteq supp(x) \subseteq I_m \cup J_m\}.$$

We have

$$I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \subseteq \cdots, J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m \supseteq \cdots.$$

Since I_n and $J_n \setminus J_{n+1}$ are countable for each $n \in \mathbb{N}$, $\left(\bigcup_{n\in\mathbb{N}} I_n\right)$ is countable and J is uncountable where $J = \bigcap_{n\in\mathbb{N}} J_n$. For each $n \in \mathbb{N}$, we have $I_n \cap J_n = \varnothing$, then

$$\left(\bigcup_{n\in\mathbb{N}} I_n\right) \cap \left(\bigcap_{n\in\mathbb{N}} J_n\right) = \varnothing.$$

Let $x \in A$ such that $x(\alpha) = 1$, for every $\alpha \in \bigcup_{n\in\mathbb{N}} I_n$ and for one $\alpha \in J$ and $x(\alpha) = 0$, for other α. We have $x = x_1 + x'_1$ where $x'_1 \in A$ and $supp(x'_1) \subseteq J_1$. Then $y_{1i}(x - x_1) = y_{1i}(x'_1) = 0$, for each $1 \le i \le n_1$, that follows $x \in B'_1$ and $x \in A_2$. Also we have $x = x_2 + x'_2$, where $x'_2 \in A$ and $supp(x'_2) \subseteq J_2$, then $y_{2i}(x - x_2) = y_{2i}(x'_2) = 0$, for each $1 \le i \le n_2$, that follows $x \in B'_2$ and $x \in A_3$. By continuing this process we have $x \in A_n$ for each $n \in \mathbb{N}$, then

$$x \in \bigcap_{n\in\mathbb{N}} A_n.$$

Since J is uncountable, there are uncountable choices for x, then

$$\bigcap_{n\in\mathbb{N}} A_n = \bigcap_{n\in\mathbb{N}} B_n$$

contains uncountable point. ■

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