CYCLIC HYPERGROUPS WHICH ARE INDUCED BY THE CHARACTER OF SOME FINITE GROUPS

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Abstract. Let G be a finite group and $\hat{G}$ be the set of all irreducible characters of G. In this paper, the hypergroups obtained from the character table $\hat{G}$ are considered. Moreover, we show that $\hat{S}_n$ for $n \geq 3$ and $\hat{A}_n$ for $n \geq 4$ are single-power cyclic hypergroups and $\hat{D_{2n}}$ is cyclic with finite period.

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1. Introduction

Hypergroups have been studied by many researchers in various fields for a long time; for examples, see [2], [12] and [4]. Cyclic hypergroups already considered at the beginning of the theory's history by [13] have been later on studied in depth by Vougiouklis [11] and afterwards by Leoreanu [8]. The hypergroup H will be called cyclic with finite period with respect to $h \in H$, if there exists a positive integer $s \in \mathbb{Z}^+$, such that

$$H = h^1 \cup h^2 \cup ... \cup h^s,$$

where $h^t = \underbrace{h.h...h}_{t\ times}$.

The minimum such a s will be called period of the generator h. If there exists $h \in H$ and $s \in \mathbb{Z}^+$, the minimum one, such that $H = h^s$, then H will be called single -power cyclic and h is a generator with single- power period s. Quasicanonical hypergroups were introduced by P. Corsini and later were studied by P. Bonansinga and Ch. Massouros. They satisfy all the conditions of canonical hypergroups, except the commutativity. Later, S. D. Comer in [1] introduced this class of hypergroups independently, using the name of polygroups. A polygroup is a system $\mathcal{P} =< P, ., e, ^{-1} >$, where $e \in P$, $^{-1}$ is a unitary operation on P, maps $P \times P$ into the non-empty subset of P, and the following axioms hold for all $x, y, z \in P$:

1. $(x.y).z = x.(y.z)$;
2. $e.x = x.e = x$;
3. $x \in y.z$ implies $y \in x.z^{-1}$ and $z \in y^{-1}.x$.

Roth in [10] showed that for a finite group G, there exists a polygroup system $\left\langle \hat{G}, *, \chi_1, ^{-1} \right\rangle$ where $\hat{G}$ is the set of all irreducible characters [6] of G. Later on, $\hat{G}$ have been studied in various fields. McMullen in [9] proved that $\mathbb{C}\hat{G}$ is semisimple and Comer in [1] showed that a natural hypergroup is associated with every character algebra and also showed certain edge coloring of graphs give raise to hypergroups with special properties. Symmetry groups have been widely applied in chemistry [3] and crystallography [7]. Many of these applications, have involved coset decomposition, decompositions into conjugacy classes and group characters. Let S_n be symmetric group on n letters and A_n be alternating group on n letters and D_{2n} be dihedral group. In this paper, we will show that the hypergroups which obtained from character tables of S_n for $n \geq 3$ and A_n for $n \geq 4$ are single-power cyclic hypergroup. In continue, we will show the hypergroup $\hat{D_{2n}}$ is cyclic hypergroup with finite period.

2. Preliminaries

In this section, we mention some fundamental notions and facts of character of finite groups and character hypergroups, referring to Issacs's book [6] and Roth's paper [10].

Let G be a finite group and F be a field. Also, let V be a finite dimensional vector space on F. A representation of G over V is a homomorphism

$$T : G \longrightarrow GL(V), T(xy) = T(x)T(y); \quad \forall x, y \in G.$$

A representation T of G is called irreducible, if V is an irreducible $FG-mod$. Let the dimension of V over F be n. Then $GL(V) \cong GL(n, F)$ where $GL(n, F)$ is the set of all square invertible matrixes. Let T be a representation of G. Then the character χ of G afforded by T is the function given by $\chi(g) = trT(g)$ and χ

is an irreducible character if the representation T is irreducible. For a character χ, the kernel of χ is defined by

$$\ker\chi = \{g \in G : \chi(g) = \chi(e)\}.$$

If $\ker\chi = \{e\}$, then χ is called a faithful character. We assume that the field F is equal to complex number. If χ and ψ are any two complex characters of G, then (χ, ψ) denotes the usual inner product:

$$(\chi, \psi) = \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

Let $Irr(G) = \{\chi_1, \chi_2, ..., \chi_k\}$, where χ_i for $1 \leq i \leq k$ are irreducible characters of G. Since we need some well known results relate to character theory we bring them in follow:

Theorem 2.1. [6](Orthogonality Relations) *Let χ_i, χ_j and χ be complex characters of G and $g, h \in G$. Then*

$$\begin{aligned} \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\chi_j(g^{-1}) &= \delta_{ij}, \\ \frac{1}{|G|} \sum_{\chi \in Irr(G)} \chi(g)\chi(h) &= 0. \end{aligned}$$

Theorem 2.2. [6] *The character table of D_{2n} for even integer $n = 2m$, $\epsilon = e^{\frac{2\pi i}{n}}$ and $1 \leq j \leq m-1$ is as follow:*

Table I

D_{2n}	1	a^m	$a^r(1 \leq r \leq m-1)$	b	ab
$\lvert c(g_i)\rvert$	$2n$	$2n$	n	4	4
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^m$	$(-1)^r$	1	-1
χ_4	1	$(-1)^m$	$(-1)^r$	-1	1
ψ_j	2	$2(-1)^j$	$\epsilon^{jr} + \epsilon^{-jr}$	0	0

Theorem 2.3. [6] *The character table of D_{2n} for odd integer n, $\epsilon = e^{\frac{2\pi i}{n}}$ and $1 \leq j \leq \frac{n-1}{2}$ is as follow:*

Table II

D_{2n}	1	$a^r(1 \leq r \leq \frac{n-1}{2})$	b
$\lvert c(g_i)\rvert$	$2n$	n	2
χ_1	1	1	1
χ_2	1	1	-1
ψ_j	2	$\epsilon^{jr} + \epsilon^{-jr}$	0

Suppose that the group G acts on a set Ω and $g \in G$. Then we define the set of fixed points of g by

$$fix(g) = \{\alpha \in \Omega | \alpha^g = \alpha\}.$$

Theorem 2.4. [6] *In symmetric group S_n, $\chi(g) = |fix(g)| - 1$ is a faithful irreducible character.*

Theorem 2.5. [6] *In alternating group A_n, $(\chi(g)) \downarrow_{A_n}$ is a faithful irreducible character.*

Theorem 2.6. (Cauchy-Frobenius Lemma) [5] *Let G be a finite group acting on a finite set Ω. Then G has m orbits on Ω where*

$$m|G| = \sum_{g \in G} |fix(g)|.$$

Let G be a finite group with $\hat{G} = \{\chi_1, \chi_2, ..., \chi_k\}$. Roth in [10], introduced the character polygroup $< \hat{G}, *, \chi_1, ^{-1} >$ where the product $\chi_i * \chi_j$ is the set of those irreducible components which appear in the element wise product $\chi_i \chi_j$. Further, $\bar{\chi}$, the complex conjugate of χ, is the inverse of χ. If $\theta \in \chi * \psi$, then $(\theta, \chi\psi) > 0$, hence $(\theta\bar{\chi}, \psi) > 0$ and $\psi \in \theta\bar{\chi}$.

Lemma 2.7. [10] *Let G be a finite abelian group. Then $\hat{G}$ is isomorphic to G.*

A key theorem in the study of the character hypergroup $\hat{G}$ is the classical theorem of Burnside:

Theorem 2.8. (Burnside) [6] *Let χ be a faithful character of G and suppose $\chi(g)$ takes on exactly m different values for $g \in G$. Then every $\psi \in Irr(G)$ is a constituent of one of the characters $(\chi)^j$ for $0 \leq j < m$.*

3. Main results

In this section, we will obtain the main results related to the character table of symmetric group S_n, A_n and D_{2n}. In fact, we give Theorems 3.4, 3.5, 3.8 and 3.11 as the main results. For an irreducible character χ_i, we let $\chi_i^t = \underbrace{\chi_i * \chi_i * ... * \chi_i}_{t \ times}$, where the hyper operation $*$ is as above.

Lemma 3.1. *In the symmetric group S_n for $n \geq 3$, χ takes on exactly n different values for any $g \in S_n$.*

Proof. We know that S_n has conjugacy classes of the forme $1^{n-i}i$ where $0 \leq i \leq n$ and $i \neq 1$. Moreover, in each of classes we have $\chi(g) = n - i - 1$. ∎

Lemma 3.2. *In the alternating group A_n for $n \geq 4$, $\chi \downarrow_{A_n}$ takes on exactly $n - 1$ different values for $g \in A_n$.*

Proof. A_n has conjugacy classes of the forme $1^{n-i}i$ where i is an odd integer such that $0 \leq i \leq n$ and $i \neq 1$ and also has conjugacy classes of the forme $1^{n-i^2}i^2$ and $1^{n-ij}ij$ for even integer i, j. ∎

Let Ω be a finite set and for any positive integer t, $\Omega^t = \underbrace{\Omega \times \Omega \times ... \times \Omega}_{t\ times}$. Then we give a corollary of Cauchy-Frobenius Lemma as follow:

Corollary 3.3. *Let Ω be a finite set and G be a finite group acting on Ω^t. Then G has m orbits on Ω^t where*

$$\frac{1}{G}\sum_{g\in G}|fix(g)|^t = m.$$

Proof. Consider the set

$$\mathcal{F} = \{(\omega_1, \omega_2, ..., \omega_t, g) \in \Omega \times \Omega \times ... \times \Omega \times G \mid (\omega_1, \omega_2, ..., \omega_t)^g = (\omega_1, \omega_2, ..., \omega_t)\}.$$

we shall count the number of $\mathcal{F}$ in two ways. First, suppose that the orbits of G are $\Omega_1', \Omega_2', ..., \Omega_m'$. Then, using the orbit-stabilizer property, we have

$$|\mathcal{F}| = \sum_{i=1}^{m}\sum_{(\omega_1,\omega_2,...,\omega_t)\in\Omega_i'}\frac{|G|}{|\Omega_i'|} = \sum_{i=1}^{m}|G| = m|G|.$$

Second,

$$|\mathcal{F}| = \sum_{g\in G}|fix(g)|.$$

The result follows. ∎

Theorem 3.4. *For $n \geq 3$, $\hat{S_n}$ is a single-power cyclic polygroup with respect to generator $\chi(g) =\mid fix(g) \mid -1$. In fact $(\hat{S_n}) = \chi^{n-1}$.*

Proof. By the Burnside theorem 2.8, we have $\hat{S_n} = \chi^0 \cup \chi^1 \cup ... \cup \chi^{n-1}$. We must prove that $\chi \in \chi^2$. We know that for $\Omega = \{1, 2, ..., n\}$, S_n acts on Ω, Ω^2 and Ω^3. The action on Ω^2 has two orbits $A_1 = \{(i,i) \mid i \in \Omega\}$ and $A_2 = \{(i,j) \mid i,j \in \Omega, i \neq j\}$. Similarly the action on Ω^3 has five orbits.

Now, put $F(g) = |fix(g)|$. Then, by the Cauchy-Frobenius Lemma 2.6 and the previous lemma:

$$\frac{1}{|G|}\sum_{g\in G}F(g) = 1, \qquad \frac{1}{|G|}\sum_{g\in G}F(g)^2 = 2, \qquad \frac{1}{|G|}\sum_{g\in G}F(g)^3 = 5.$$

Hence,

$$(\chi, \chi^2) = \frac{1}{|G|}\sum_{g\in G}\chi(g)^3 = \frac{1}{G}\sum_{g\in G}(F(g)-1)^3 = 1.$$

Therefore, $\chi \in \chi^2$. Consequently,

$$\chi^2 \subseteq \chi^3 \subseteq ... \subseteq \chi^{n-1}.$$

Hence,

$$\hat{S_n} = \chi^0 \cup \chi^1 \cup ... \cup \chi^{n-1} = \chi^{n-1}. \qquad \blacksquare$$

Since the alternating group A_n is an important simple group, we would like to give some results as above on it.

Theorem 3.5. *For $n \geq 4$, $\hat{A_n}$ is a single-power cyclic polygroup with respect to generator $\chi \downarrow_{A_n}$. In fact, $\hat{A_n} = (\chi \downarrow_{A_n})^{n-2}$.*

Proof. By the Burnside theorem 2.8, we have

$$\hat{A_n} = (\chi \downarrow_{A_n})^0 \cup (\chi \downarrow_{A_n})^1 \cup ... \cup (\chi \downarrow_{A_n})^{n-2}.$$

Since $(\chi, \chi^2) \neq 0$, we have $(\chi \downarrow_{A_n}, (\chi \downarrow_{A_n})^2) \neq 0$. Hence, $\chi \downarrow_{A_n} \in (\chi \downarrow_{A_n})^2$. Therefore,

$$(\chi \downarrow_{A_n})^2 \subseteq (\chi \downarrow_{A_n})^3 \subseteq ... \subseteq (\chi \downarrow_{A_n})^{n-2}. \qquad \blacksquare$$

Since dihedral groups are famous between of all non-abelian groups, we show that $\hat{D_{2n}}$ has a cyclic hypergroup structure.

Lemma 3.6. *Consider the dihedral group D_{2n} for even integer $n = 2m$. Let m be an even integer. Then:*

Case (a). *For an even integer j, $\psi_j \in \psi_1^m$. Also the multiplicities of ψ_j in ψ_1^m is $\begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix}$.*

Case (b). *For an odd integer j, $\psi_j \in \psi_1^{m-1}$. Also the multiplicities of ψ_j in ψ_1^{m-1} is $\begin{pmatrix} m-1 \\ \frac{m-j-1}{2} \end{pmatrix}$.*

Proof. Case (a). Let j be an even integer. Then, we have

$$(\psi_1^m, \psi_j) = \frac{2^{m+1} + 2^{m+1}}{2n} + \frac{(\epsilon + \epsilon^{-1})^m (\epsilon^j + \epsilon^{-j}) + ...}{n}$$

$$= \frac{2^{m+1} + \sum_{k=0}^{m} \epsilon^{-m+2k+j} + \sum_{k=0}^{m} \epsilon^{-m+2k-j} + \sum_{k=0}^{m} \epsilon^{-2m+4k+2j} + ...}{n}$$

$$\begin{array}{llllll}
= \frac{1}{n}\Big(2^{m+1} & +\epsilon^{-m+j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-m+2+j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{m-2+j} & +\epsilon^{m+j} \\
 & +\epsilon^{-2m+2j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-2m+4+2j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{2m-4+2j} & +\epsilon^{2m+2j} \\
\vdots & & & & & \\
 & +\epsilon^{-m^2+m(1+j)-j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-m^2+m(3+j)-2-j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{m^2+m(j-3)+2-j} & +\epsilon^{m^2+m(j-1)-j} \\
 & +\epsilon^{-m-j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-m+2-j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{m-2-j} & +\epsilon^{m-j} \\
 & +\epsilon^{-2m-2j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-2m+4-2j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{2m-4-2j} & +\epsilon^{2m-2j} \\
\vdots & & & & & \\
 & +\epsilon^{-m^2+m(1-j)+j} & +\begin{pmatrix} m \\ 1 \end{pmatrix}\epsilon^{-m^2+m(3-j)-2+j} & +... & +\begin{pmatrix} m \\ m-1 \end{pmatrix}\epsilon^{m^2-m(3+j)+2+j} & +\epsilon^{m^2-m(1+j)+j}\Big).
\end{array}$$

In this equality, we consider the column A_k, A_k', B_k and B_k' as follows: for all $0 \leq k \leq \frac{m-j}{2}$, A_k is equal to the column

$$\begin{array}{l}\epsilon^{-m+2k+j}\\ +\epsilon^{-2m+4k+2j}\\ \vdots\\ +\epsilon^{-m^2+2mk+m-2k+mj-j},\end{array}$$

B_k is equal to the column

$$\begin{array}{l}\epsilon^{-m+2k-j}\\ +\epsilon^{-2m+4k-2j}\\ \vdots\\ +\epsilon^{-m^2+2mk+m-2k-mj+j}.\end{array}$$

and for all $\frac{m-j}{2} < k \leq m$, A_k' is equal to the column

$$\begin{array}{l}\epsilon^{-m+2k+j}\\ +\epsilon^{-2m+4k+2j}\\ \vdots\\ +\epsilon^{-m^2+2mk+m-2k+mj-j},\end{array}$$

and B_k' is equal to column

$$\begin{array}{l}\epsilon^{-m+2k-j}\\ +\epsilon^{-2m+4k-2j}\\ \vdots\\ +\epsilon^{-m^2+2mk+m-2k-mj+j}.\end{array}$$

Now by using the orthogonality relations 2.1 for A_k, A_k', B_k and B_k' and by some manipulations we get that

$$A_k + B_k' = -2\begin{pmatrix} m \\ k \end{pmatrix}, A_k' + B_k = -2\begin{pmatrix} m \\ k \end{pmatrix}, k \neq \frac{m-j}{2}.$$

And, for $k = \frac{m-j}{2}$, we have for each component of A_k and B_k is equal to one.

Hence,

$$(\psi_1^m, \psi_j) = \frac{1}{2m}(2^{m+1} + 2(m-1)\begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix} - 2\sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix} + 2\begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix}).$$

but

$$-2\sum_{k=0}^{m}\begin{pmatrix} m \\ k \end{pmatrix} = -2(1+1)^m = -2. - 2^m = -2^{m+1}.$$

So

$$(\psi_1^m, \psi_j) = \frac{1}{2m}.2m\begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix} = \begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix}.$$

and hence the proof of Case (a) is completed.

Case (b). The proof is similar to Case (a). ∎

Lemma 3.7. *Consider the dihedral group D_{2n} for even integer $n = 2m$. Let m be an odd integer. Then:*

Case (a). *For an even integer j, $\psi_j \in \psi_1^{m-1}$ and the multiplicities of ψ_j in ψ_1^{m-1} is $\begin{pmatrix} m-1 \\ \frac{m-j-1}{2} \end{pmatrix}$.*

Case (b). *For an odd integer j, $\psi_j \in \psi_1^{m}$ and the multiplicities of ψ_j in ψ_1^{m} is $\begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix}$.*

Proof. The proof is similar to Lemma 3.6. ■

Theorem 3.8. *Consider the dihedral group D_{2n} and $n = 2m$. Then $\hat{D_{2n}}$ is a cyclic hypergroup with generator ψ_1. In fact, $\hat{D_{2n}} = \psi_1^{m-1} \cup \psi_1^m$.*

Proof. First let m be an even integer. By Lemma 3.6 it is enough to show that for $1 \leq i \leq 4$, χ_i are in ψ_1^m. Since m is an even integer and by the character value number of χ_1, χ_2 in the character table of D_{2n}, we have:

$$\chi_1, \chi_2 \in \psi_1^m.$$

Now, for χ_3 we have:

$$(\psi_1^m, \chi_3) = \frac{2^m + 2^m}{2n} + \frac{-(\epsilon + \epsilon^{-1})^m + (\epsilon^2 + \epsilon^{-2})^m - \dots}{n}$$

$$= \frac{2^m - \sum_{k=0}^{m} \epsilon^{-m+2k} + \sum_{k=0}^{m} \epsilon^{-2m+4k} - \dots}{n}$$

$$= \frac{1}{n}\Bigg(2^m \quad -\epsilon^{-m} \quad - \begin{pmatrix} m \\ 1 \end{pmatrix} \epsilon^{-m+2} \quad -\dots \quad - \begin{pmatrix} m \\ m-1 \end{pmatrix} \epsilon^{m-2} \quad -\epsilon^{m}$$
$$+\epsilon^{-2m} \quad + \begin{pmatrix} m \\ 1 \end{pmatrix} \epsilon^{-2m+4} \quad +\dots \quad + \begin{pmatrix} m \\ m-1 \end{pmatrix} \epsilon^{2m-4} \quad +\epsilon^{2m}$$
$$\vdots$$
$$+\epsilon^{-m^2+m} \quad + \begin{pmatrix} m \\ 1 \end{pmatrix} \epsilon^{-m^2+3m-2} \quad +\dots \quad + \begin{pmatrix} m \\ m-1 \end{pmatrix} \epsilon^{m^2-3m+2} \quad +\epsilon^{m^2-m} \Bigg).$$

In this equality, we consider the column A_k and B_k as follows: for all $0 \leq k \leq \frac{m}{2}$, A_k is equal to the column

$$\begin{array}{l} \epsilon^{-m+2k} \\ +\epsilon^{-2m+4k} \\ \vdots \\ +\epsilon^{-m^2+2mk+m-2k}. \end{array}$$

and, for all $\frac{m}{2} < k \leq m$, B_k its equal to column

$$\begin{aligned}&\epsilon^{m-2k}\\&+\epsilon^{2m-4k}\\&\vdots\\&+\epsilon^{m^2-2mk-m+2k}.\end{aligned}$$

Now, by using the orthogonality relations for A_k and B_k and some manipulations, we get that

$$A_k + B_k = -2\begin{pmatrix} m \\ k \end{pmatrix}.$$

for $k = \dfrac{m}{2}$, we have that each component of A_k is equal to one. Hence,

$$(\psi_1^m, \chi_3) = \frac{1}{2m}\Big(2^m + (m-1)\begin{pmatrix} m \\ \frac{m}{2} \end{pmatrix} - 2\sum_{k=1}^{\frac{m}{2}-1}\begin{pmatrix} m \\ k \end{pmatrix} + 2m - 2\Big).$$

But

$$-2\sum_{k=1}^{\frac{m}{2}-1}\begin{pmatrix} m \\ k \end{pmatrix} = -2^m + \begin{pmatrix} m \\ \frac{m}{2} \end{pmatrix} + 2.$$

So

$$(\psi_1^m, \chi_3) = \frac{2m + m\begin{pmatrix} m \\ \frac{m}{2} \end{pmatrix}}{2m} \neq 0.$$

Therefore, $\chi_3 \in \psi_1^m$, and similarly $\chi_4 \in \psi_1^m$. Hence the proof is completed.

Now, let m be an odd integer. The proof in this case is similar to the above. ∎

Lemma 3.9. *Consider the dihedral group D_{2n} for odd integer n. Put $m = \frac{n-1}{2}$ and let m be an even integer. Then:*

Case (a). *For an even integer j, $\psi_j \in \psi_1^m$ and the multiplicities of ψ_j in ψ_1^m is $\begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix}$.*

Case (b). *For an odd integer j, $\psi_j \in \psi_1^{m-1}$ and the multiplicities of ψ_j in ψ_1^{m-1} is $\begin{pmatrix} m-1 \\ \frac{m-j-1}{2} \end{pmatrix}$.*

Proof. The proof is similar to Lemma 3.6. ∎

Lemma 3.10. *Consider the dihedral group D_{2n} for odd integer n. Put $m = \frac{n-1}{2}$ and let m be an odd integer. Then:*

Case (a). *For an even integer j, $\psi_j \in \psi_1^{m-1}$ and the multiplicities of ψ_j in ψ_1^{m-1} is $\begin{pmatrix} m-1 \\ \frac{m-j-1}{2} \end{pmatrix}$.*

Case (b). *For an odd integer j, $\psi_j \in \psi_1^m$ and the multiplicities of ψ_j in ψ_1^m is $\begin{pmatrix} m \\ \frac{m-j}{2} \end{pmatrix}$.*

Proof. The proof is similar to Lemma 3.6. ∎

Theorem 3.11. *Consider the dihedral group D_{2n} for odd integer $n > 3$. Then $\hat{D_{2n}}$ is a cyclic with finite period hypergroup with generator ψ_1 where $m = \frac{n-1}{2}$. In fact, $\hat{D_{2n}} = \psi_1^{m-1} \cup \psi_1^m$.*

Proof. By Lemmas 3.9 and 3.10, it is enough to show that χ_1 and χ_2 are in ${\psi_1}^{m-1} \cup {\psi_1}^m$. By the character value number of χ_1 and χ_2 in the character table of D_{2n}, we have $\chi_1, \chi_2 \in {\psi_1}^m$ if m is an even integer and $\chi_1, \chi_2 \in {\psi_1}^{m-1}$ if m is an odd integer. This completes the proof. ∎

4. Conclusion

In this paper, a relation between character theory and polygroup theory has obtained. In fact, we could give a structure of hypergroup by character tables and using a special hyperaction on them. Now there is a question, can we extended this idea to an arbitrary finite group in which its character tables is known?

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