ON CHARACTERISTIC SUBGRAPH OF A GRAPH

Zahra Yarahmadi

Department of Mathematics, Faculty of Science
Khorramabad Branch
Islamic Azad University
Khorramabad
I.R. Iran
e-mails: z.yarahmadi@gmail.com, z.yarahmadi@khoiau.ac.ir

Ali Reza Ashrafi

Department of Pure Mathematics
Faculty of Mathematical Sciences
University of Kashan
Kashan 87317-51167
I.R. Iran
e-mail: ashrafi@kashanu.ac.ir

Abstract. A subgraph H of a graph G is called characteristic if $\varphi(H) = H$, for each automorphism $\varphi \in Aut(G)$. In this paper, the main properties of this new concept in algebraic graph theory are presented.

Keywords: characteristic subgraph, Boolean algebra, automorphism group.

AMS Subject Classifications: 05C10, 20E45.

1. Introduction

Throughout this paper graph means simple finite graph and we follow the terminology and notation of [1, 3] for graphs. An trivial graph on n vertices consists of n isolated vertices with no edges. This graph is denoted by $\emptyset_n$. We refer to [2] for general properties of lattices and Boolean algebras.

We assume that G is a graph and u, v are vertices of G. The edge connecting u and v is denoted by uv and the distance $d_G(u,v)$ is defined as the length of a shortest path connecting u and v in G. The eccentricity $\varepsilon(u)$ is the largest distance between u and any other vertex v of G. The maximum eccentricity over all vertices of G is called the diameter of G and denoted by $d(G)$. The minimum eccentricity is said to be radius and denoted by $r(G)$. The center of G is the set of all vertices u such that $\varepsilon(u) = r(G)$. A self-centered graph is one that the center is the same as vertex set. Consider the graph G whose vertices are the n-tuple $(b_1, b_2, ..., b_n)$

with $b_i \in \{0,1\}$ and two vertices are adjacent if the corresponding tuples differ in precisely one place. Such a graph is called a $n-$dimentional hypercube and denote it by Q_n.

Suppose G and H are graphs. H is said to be a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and it is a spanning subgraph of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is called characteristic if for every automorphism $\beta \in Aut(G)$, $\beta(H) = H$. We use the notation $H \leq_{ch} G$ to show that H is a characteristic subgraph of G. It is easily seen that the graph G itself and its trivial spanning subgraph are characteristic in G.

It is easy to see that, if P_n is a path of length n with vertex set $V(P_n) = \{v_0, v_1, ..., v_n\}$, then, for each i, $1 \leq i \leq n/2$, $v_i v_{i+1} \cdots v_{n-i}$ is a characteristic subgraph of P_n.

2. Main results

In this section some basic properties of characteristic subgraphs are investigated. We first introduce an important class of characteristic subgraphs of a given graph G. Define $G[i] = \{x \in V(G) \mid \deg_G(x) = i\}$. Then it is easy to see that for each i, $1 \leq i \leq \Delta(G)$, where Δ is maximum degree of vertices, $\langle G[i] \rangle$ is a characteristic subgraph of G. We start by considering the center of a graph.

Theorem 2.1. $\langle C(G) \rangle \leq_{ch} G$.

Proof. Let $a \in C(G)$, then $\varepsilon(a) = r(G)$ and there exists $b \in V(G)$, such that $\varepsilon(a) = d(a,b) = m$, for a positive integer m. Thus, there exists a path $P : a = a_0, a_1, ..., a_m = b$ connecting a and b. Let $\varphi \in Aut(G)$ be arbitrary. Then, $\varphi(a) = \varphi(a_0), \varphi(a_1), ..., \varphi(a_m) = \varphi(b)$ is a path. Hence $\varepsilon(\varphi(a)) \geq \varepsilon(a)$. Now, suppose $\varepsilon(\varphi(a)) = k$, then there exists $d \in V(G)$, such that $d(\varphi(a), d) = k$ and $\varphi(a) = d_0, d_1, ..., d_k = d$ is a path. Since φ is an automorphism, for each $1 \leq i \leq k$ there exists $a_i \in V(G)$ such that $d_i = \varphi(a_i)$ and $\varphi(a) = \varphi(a_0), \varphi(a_1), ..., \varphi(a_k) = d$ is a path. Hence the following path exists in G, $a = a_0, a_1, ..., a_k$. Therefore, $\varepsilon(a) \geq k = \varepsilon(\varphi(a))$ and then $\varepsilon(a) = \varepsilon(\varphi(a))$. This shows that $\varphi(\langle C(G) \rangle) = \langle C(G) \rangle$, as desired. ∎

Corollary 2.2. *If T is a tree and $\varphi \in Aut(T)$ then φ has at least a fixed vertex or a fixed edge.*

Proof. By a well-known fact in graph theory the center of a tree T is a single vertex or two adjacent vertices of T. So, every automorphism of T fix a vertex or an edge of T. ∎

Theorem 2.3. *The number of characteristic spanning subgraphs of a non-trivial graph is always even.*

Proof. Suppose H is a characteristic spanning subgraph of G. For each characteristic spanning subgraph H, we define a spanning subgraph H' with edge set

$E(H') = E(G) \backslash E(H)$. Since H is a characteristic subgraph, for each $\varphi \in Aut(G)$, $\varphi(H) = H$, it means that $E(\varphi(H)) = E(H)$ and so

$$E(G) \backslash E(\varphi(H)) = E(G) \backslash E(H).$$

By the definition of $E(H')$, we have $E(\varphi(H')) = E(G) \backslash E(\varphi(H))$, and then $E(\varphi(H')) = E(H')$. This implies that $\varphi(H') = H'$ and so H' is a characteristic spanning subgraph of G. Since H and H' are different, the number of characteristic spanning subgraphs of a non-trivial graph is always even. ∎

Remark. Notice that trivial graphs have exactly one characteristic spanning subgraph.

Corollary 2.4. *Suppose G is not regular or self-centered then G has at least four characteristic subgraphs.*

Proof. Our main proof is separated into two different cases as follows:

Case 1: *G is self-centered.* In this case, there are positive integers i, j such that $\langle G[i] \rangle \leq_{ch} G$ and $\langle G[j] \rangle \leq_{ch} G$. Define H_i and H_j to be spanning subgraphs of G such that $E(H_i) = E(\langle G[i] \rangle)$ and $E(H_j) = E(\langle G[j] \rangle)$. Obviously, H_i and H_j are characteristic spanning subgraphs of G. Let $D = \{deg_G(u) | u \in V(G)\}$, if $D = \{i, j\}$, then $H'_i = H_j$ and also $H'_j = H_i$. By considering G and trivial spanning subgraph, we obtain at least four characteristic. Now if $D \supsetneqq \{i, j\}$, by Theorem 2.3, H'_i and H'_j are different characteristic subgraphs of G. By considering G and trivial spanning subgraph, we obtain at least six characteristic subgraphs.

Case 2: *G is not self-centered.* In this case $\langle C(G) \rangle \lneqq G$. Let H be a spanning subgraph of G, with edge set $E(H) = E\big(\langle C(G) \rangle\big)$. By Theorem 2.1, $H \leq_{ch} G$ and by Theorem 2.3 and considering G itself and trivial spanning subgraph of G, we obtain at least four characteristic subgraphs, as desired. ∎

Definition 2.5. A subgraph K of G is minimal characteristic in G if no proper nontrivial subgraph of K is characteristic in G.

It is easy to see that every characteristic subgraph of G contains a minimal characteristic subgraph of G.

Theorem 2.6. *Let G be a graph and H be a minimal characteristic subgraph of G. Then there exists positive integer i such that $H \leq \langle G[i] \rangle$.*

Proof. Suppose H has a vertex of degree i and $K = \{v \in V(H) | \deg_G(v) = i\}$. It is clear that $\langle K \rangle$ is a characteristic subgraph of G. Since $\langle K \rangle \leq H$ and H is a minimal characteristic subgraph, $\langle K \rangle = H$. It means that for each $v \in V(H)$, $\deg_G(v) = i$, proving the result. ∎

Corollary 2.7. *Let G be a graph and for a positive integer i, $G[i]$ is singleton, then $\langle G[i] \rangle$ is a characteristic subgraph of G.*

The converse of Corollary 2.7 does not hold. To do this, we assume that G is a cycle of length n. Then $G[2] = V(G)$ and $\langle G[2] \rangle$ is minimal in G.

Theorem 2.8. *A graph G is vertex transitive graph if and only if except G and its trivial spanning subgraph, it doesn't have any spanning characteristic subgraph.*

Proof. Suppose that G is a vertex transitive graph having a non-trivial characteristic subgraph H, $H \neq G$. Let $v \in V(H)$ and $u \in V(G)/V(H)$. Since G is vertex transitive, there exists $\alpha \in Aut(G)$ such that $\alpha(v) = u$, which is impossible. Conversely, we assume that G is a graph with exactly two spanning characteristic subgraph. If G is not vertex transitive, then there exist vertices u and v such that for each $\alpha \in Aut(G)$, $\alpha(v) \neq u$. Define H to be the orbit of u under the natural action of $Aut(G)$ on $V(G)$. A simple calculation shows that $\langle H \rangle$ is a characteristic subgraph of G, a contradiction. ∎

Suppose $S(G)$ and $CSS(G)$ denote the set of all spanning and characteristic spanning subgraphs of G, respectively. In the following theorem, we prove that $S(G)$ has a Boolean algebra structure.

Theorem 2.9. *Let G be a graph on n vertices, then $S(G)$ is closed under taking intersection and union and $(S(G), \cap, \cup, ')$ is a Boolean algebra in which for each element $H \in S(G)$, H' is a spanning subgraph such that $E(H') = E(G) - E(H)$. Moreover, $|S(G)| = 2^{|E(G)|}$.*

Proof. Let $H, K \in S(G)$, then $H \cap K$ and $H \cup K$ are spanning subgraphs of G. One can see that $(S(G), \cap, \cup)$ is a bounded distributive lattice in which $0_{S(G)} = \emptyset_n$ and $1_{S(G)} = G$. Moreover, $H \cap H' = \emptyset_n = 0_{S(G)}$ and $H \cup H' = G = 1_{S(G)}$ and so $S(G)$ is complemented. Therefore, $(S(G), \cap, \cup, ')$ is a Boolean algebra.

Notice that spanning subgraphs of G with exactly one edge are atoms of this Boolean algebra and conversely, each atom of this Boolean algebra has this form. Since the number of atoms of this Boolean algebra is $|E(G)|$ and every finite Boolean algebra is atomic, by a well-known theorem in Boolean algebras, $|S(G)| = 2^{|E(G)|}$. ∎

Theorem 2.10. *$CSS(G)$ is closed under taking intersection and union and $(CSS(G), \cap, \cup, ')$ is a sub-Boolean algebra of $(S(G), \cap, \cup, ')$.*

Proof. Let $H, K \in CSS(G)$. Obviously, for each $\varphi \in Aut(G)$,

$$\varphi(H \cap K) = \varphi(H) \cap \varphi(K) = H \cap K \text{ and } \varphi(H \cup K) = \varphi(H) \cup \varphi(K) = H \cup K.$$

By a similar argument as the proof of Theorem 2.9, one can show that $(CSS(G), \cap, \cup, ')$ is a Boolean algebra with $0_{CSS(G)} = \emptyset_n$ and $1_{CSS(G)} = G$. Since $CSS(G) \subseteq S(G)$, then $(CSS(G), \cap, \cup, ')$ is sub-Boolean algebra of $(S(G), \cap, \cup, ')$. ∎

Corollary 2.11. *$H \in CSS(G)$ is an atom of Boolean algebra $(CSS(G), \cap, \cup, ')$ if and only if H is a minimal characteristic subgraph of G. In particular, if G is a graph with exactly m minimal characteristic subgraph then $|CSS(G)| = 2^m$.*

Corollary 2.12. *Every characteristic spanning subgraph of G can be represented as the union of some minimal characteristic subgraphs of G.*

Proof. By Theorem 2.9 every element of a Boolean algebra is a supremum of some atoms, as desired. ■

In the end of this paper, we construct a graph $S'(G)$ with $V(S'(G)) = S(G)$ and two spanning subgraph H and K are adjacent if and only if $E(H) \subseteq E(K)$ and $|E(H)| = |E(K)| + 1$. In the following theorem, it is proved that $S'(G)$ is a hypercube.

Theorem 2.13. *$S'(G) \cong Q_m$, where $m = |E(G)|$ and Q_m is $m-$dimentional hypercube.*

Proof. Suppose $E(G) = \{e_1, e_2, ..., e_m\}$. We can represent each spanning subgraph $H \in S(G)$, by an $m-$array $(e'_1, e'_2, ..., e'_m)$ as follows:

$$e'_i = \begin{cases} e_i & e_i \in E(H), \\ 0 & e_i \notin E(H). \end{cases}$$

Define $\psi : Q_m \longrightarrow S(G)$ by $\psi\big((a_1, a_2, ...a_m)\big) = (b_1, b_2, ...b_m)$ such that

$$b_i = \begin{cases} e_i & a_i = 1, \\ 0 & a_i = 0. \end{cases}$$

Suppose that $\alpha, \beta \in V(Q_m)$ such that $\alpha\beta \in E(Q_m)$. Then, by the definition of the hypercube, $\alpha = (t_1, t_2, ..., t_m)$ and $\beta = (s_1, s_2, ..., s_m)$ such that there exists a positive integer $k, 1 \leq k \leq m$, $t_k \neq s_k$ and for all $l \neq k$, $t_l = s_l$. Obviously, $(t_k = 0 \textit{ and } s_k = 1)$ or $(t_k = 1 \textit{ and } s_k = 0)$. Without losing generality, we assume that $t_k = 0 \textit{ and } s_k = 1$. Then, by definition of ψ:

$$\psi(\alpha) = \psi\big((t_1, t_2, ..., t_k = 0, ..., t_m)\big) = (b_1, b_2, ..., b_k = 0, ..., b_m),$$

$$\psi(\beta) = \psi\big((s_1, s_2, ..., s_k = 1, ..., s_m)\big) = (b_1, b_2, ..., b_k = e_k, ..., b_m),$$

and, by adjacency in $S'(G)$, it can easily seen that the subgraph by representing vector $\psi(\alpha) = (b_1, b_2, ..., 0, ..., b_m)$ is adjacent with subgraph by representing vector $\psi(\beta) = (b_1, b_2, ..., e_k, ..., b_m)$. Hence $\psi(\alpha)\psi(\beta) \in E(S(G))$, and so ψ is a homomorphism of graphs. Since ψ is bijective, it is isomorphism and so $S'(G) \cong Q_m$. ■

Acknowledgement. The research of the second author is partially supported by the University of Kashan under grant no 159020/62.

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Accepted: 02.09.2013