A NEW CHARACTERIZATION OF SIMPLETIC GROUP $S_8(2)$

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Abstract. It is a well-known fact that characters of a finite group can give important information of the group's structure. Also it was proved by chen [1] that a finite simple group can be uniquely determined by its character table. In this paper, the authors attempt to investigate how to characterize a finite almost simple group by using less information of its character table, and successfully characterize the simpletic group $S_8(2)$ by its order and at most two irreducible character degrees of its character table.

Keywords: finite group, character degree, simple group, order.

2000 AMS Subject Classification: 20C15.

1. Introduction

In this paper, G represents a finite group. We use $Irr(G)$ to denote the set of all irreducible complex characters of G, and $cd(G) = \{\nu(1)|\nu \in Irr(G)\}$ the set of all irreducible character degrees of G. Moreover, $cd^*(G)$ denotes the Multi-set of degrees of irreducible characters, i.e., each element of the set $cd^*(G)$ can occur many times upon the number of characters of the same degree. In particular, $|cd^*(G)| = |Irr(G)|$. $Syl_p(G)$ denotes the set of all Sylow p-subgroups of G, where $p \in \pi(G)$, and G_p denotes a Sylow p-subgroup of G. $H \cdot M$ denotes the

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non-split extension of H by M and $H : M$ the split extension of H by M. For any finite group G, $L_i(G)$ denotes the ith largest irreducible character degree of G. Particularly, $L_1(G)$ and $L_2(G)$ are the largest and the second largest irreducible character degree of G, respectively. All further unexplained symbols and notations are standard and can be found, for instance, in [2].

A finite group is called a K_n-group if $|\pi(G)| = n$, where n is a positive integer. K_3-simple groups were classified many years ago, which are A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ and $U_4(2)$ (cf. [3]). Classifying finite groups by the properties of their characters is an interesting and difficult topic in group theory. It is a well-known fact that characters of a finite group can give important information about the group's structure. For example, in [1], Chen proved that a finite simple group can be uniquely determined by its character table. In 2000, Huppert [4] put forward the following conjecture.

Huppert's Conjecture. *Let H be any nonabelian simple group, and G a group such that $cd(G) = cd(H)$, then $G \cong H \times A$, where A is an abelian group.*

Huppert conjectured that each finite non-abelian simple group G is characterized by $cd(G)$, the set of degrees of its complex irreducible characters. In [4, 5, 6], he confirmed that the conjecture holds for simple groups such as $L_2(q)$ and $S_z(q)$. Moreover, he also proved this conjecture holds for 19 out of 26 sporadic simple groups, and a few others (cf. [4], [5], [6]). In [7], Daneshkhah showed that the conjecture holds for another three sporadic simple groups Co_1, Co_2 and Co_3. Xu, et al. attempt to characterize the finite simple groups by less information of its characters, and for the first time successfully characterize the simple K_3-groups (cf. [8]) and sporadic simple groups by their orders and one or both of its largest and second largest irreducible character degrees (cf. [9], [10], [11]). For convenience, we summarize some results of these articles which will be used later in the following propositions:

Proposition. (cf. [10]) *Let G be a finite group and M a non-Abelian simple group, then the following assertions hold:*

(i) *If M is one of the simple groups: A_5, $L_2(7)$, $L_2(17)$, $L_3(3)$, $U_4(2)$, M_{11}, M_{23}, J_1, J_3 and J_4, then $G \cong M$ if and only if $|G| = |M|$ and $L_1(G) = L_1(M)$;*

(ii) *If M is isomorphic to one of the simple groups: simple K_3-groups, Mathieu simple groups and Janko simple groups, then $G \cong M$ if and only if $|G|=|M|$, $L_1(G) = L_1(M)$ and $L_2(G) = L_2(M)$;*

(iii) *If $M = A_6$, then G is isomorphic to one of the groups: $G \cong A_6$, $G \cong S_3 \times A_5$ and $G \cong Z_3 \rtimes S_5$, if and only if $|G| = |M|$ and $L_1(G) = L_1(M)$;*

(iv) *If $M = M_{22}$, then $G \cong M_{22}$ or $H \times M_{11}$, where H is a Frobenius group with elementary kernel of order 8 and a cyclic complement of order 7, if and only if $|G| = |M_{22}|$ and $L_1(G) = L_1(M_{22})$.*

In the following, we continue this investigation, and show that the simpletic simple group $S_8(2)$ by its order and at most two irreducible character degrees of its character tables.

Our main results are the following:

Theorem. *Let G be a finite group and $|G| = |S_8(2)|$, then $G \cong S_8(2)$ if and only if $L_1(G) = L_1(S_8(2))$ and $L_2(G) = L_2(S_8(2))$.*

2. Preliminaries

In this section, we consider some results which will be applied for our further investigations.

Lemma 2.1. *Let G be a finite solvable group of order $q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$, where $q_1, q_2, \cdots, q_s$ are distinct primes, and if $(kq_s + 1) \nmid q_i^{\alpha_i}$ for each $i \leq s-1$ and $k > 0$, then the Sylow q_s-subgroup is normal in G.*

Proof. Let N be a minimal normal subgroup of G. Since G is solvable, then we have $|N| = q^m$. If $q = q_s$, by induction on G/N, it is easy to see the Sylow q_s-subgroup is normal in G. Now, assume that $q = q_i$ for some $i < s$. By induction of the factor group G/N, we obtain that the Sylow q_s-subgroup Q/N of G/N is normal in G/N. Thus $Q \trianglelefteq G$. Let P be a Sylow q_s-subgroup of Q. Then $Q = NP$. By Sylow's Theorem, $|Q : N_Q(P)| = q_i^l (l \leq m \leq \alpha_i)$ and $q_s \big| q_i^l - 1$. But this means that $(kq_s + 1) \big| q^{\alpha_i}$, and then $k = 0$ by assumption. Hence $P \trianglelefteq Q$. Since $Q \trianglelefteq G$, we have $P \trianglelefteq G$, as required.

Lemma 2.2. *Let G be a non-solvable group, then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \big| |Out(K/H)|$.*

Proof. Let G be a non-solvable group and M/N a minimal normal subgroup of G/N. Then M/N is a direct product of isomorphic non-abelian simple groups. Hence, $C_{G/N}(M/N) \bigcap M/N = Z(M/N) = 1$, and so

$$M/N \cong \frac{M/N \times C_{G/N}(M/N)}{C_{G/N}(M/N)} \leq \frac{G/N}{C_{G/N}(M/N)}.$$

Thus $\dfrac{G/N}{C_{G/N}(M/N) \times M/N}$ is a subgroup of $Out(M/N)$.

Let $K/N = C_{G/N}(M/N) \times M/N$ and $H/N = C_{G/N}(M/N)$, then $G/K \leq Out(M/N)$ and $K/H \cong M/N$, and the normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, as desired.

Lemma 2.3. (cf. [2]) *Let S be a finite non-abelian simple group with $\pi(S) \subseteq \{2, 3, 5, 7, 17\}$, then S is isomorphic to one of the following simple groups listed in Table 1. In particular, if $|\pi(Out(S))| \neq 1$, then $\pi(Out(S)) \subseteq \{2, 3\}$.*

Table 1. Finite nonabelian simple groups with $\pi(S) \subseteq \{2, 3, 5, 7, 17\}$

G	$\|G\|$	$Out(G)$	G	$\|G\|$	$Out(G)$
A_5	$2^2 \cdot 3 \cdot 5$	2	$U_3(5)$	$2^4 \cdot 3^2 \cdot 5^3 \cdot 7$	6
$L_2(7)$	$2^3 \cdot 3 \cdot 7$	2	$S_4(7)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 7^4$	2
A_6	$2^3 \cdot 3^2 \cdot 5$	4	A_9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	2
$L_2(8)$	$2^3 \cdot 3^2 \cdot 7$	3	$O_8^+(2)$	$2^{12} \cdot 3^5 \cdot 5^2 \cdot 7$	6
$L_2(17)$	$2^4 \cdot 3^2 \cdot 17$	2	$S_4(4)$	$2^8 \cdot 3^2 \cdot 5^2 \cdot 17$	4
$L_2(16)$	$2^4 \cdot 3 \cdot 5 \cdot 17$	4	$S_6(2)$	$2^9 \cdot 3^4 \cdot 5 \cdot 7$	1
A_7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	2	$L_2(49)$	$2^4 \cdot 3 \cdot 5^2 \cdot 7^2$	4
$U_3(3)$	$2^5 \cdot 3^3 \cdot 7$	2	J_2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$	2
He	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$	2	A_{10}	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	2
$O_8^-(2)$	$2^{12} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$	2	$L_2(31)$	$2^5 \cdot 3 \cdot 5 \cdot 31$	2
$L_4(4)$	$2^{12} \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 17$	4	$U_4(3)$	$2^7 \cdot 3^6 \cdot 5 \cdot 7$	8
A_8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	2	$U_4(2)$	$2^6 \cdot 3^4 \cdot 5$	2
$L_3(4)$	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	12	$S_8(2)$	$2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$	1

3. Proof of Theorem

In this section, we will prove the following theorem:

Theorem. *Let G be a finite group and $|G| = |S_8(2)|$, then $G \cong S_8(2)$ if and only if $L_1(G) = L_1(S_8(2))$ and $L_2(G) = L_2(S_8(2))$.*

Proof. By hypothesis, we have that $|G| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$, $L_1(G) = 2 \cdot 3^4 \cdot 5^2 \cdot 17$ and $L_2(G) = 2^{16}$. Let $\chi, \beta \in Irr(G)$ such that $\chi(1) = L_1(G) = 2 \cdot 3^4 \cdot 5^2 \cdot 17$ and $\beta(1) = L_2(G) = 2^{16}$.

We first prove that G is nonsolvable. Otherwise, G is a solvable group. Let T be a Hall-subgroup of G with order $|T| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 17$. Considering the permutation representation of G on the right cosets of T with kernel T_G, we have that $G/T_G \lesssim S_7$. Since orders of solvable subgroups of S_7 divided by 7 are 7, 14 and 21, then $|T_G|$ can be equal to one of $2^{16} \cdot 3^5 \cdot 5^2 \cdot 17$, $2^{15} \cdot 3^5 \cdot 5^2 \cdot 17$ and $2^{16} \cdot 3^4 \cdot 5^2 \cdot 17$.

If $|T_G| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 17$, let $\Delta \in Irr(T_G)$ such that $[\beta_{T_G}, \Delta] \neq 0$, then we have $\beta(1)/\Delta(1) \big| |G : T_G| = 7$, and so $\Delta(1) = 2^{16}$. Let $\Omega \in Irr(T_G)$ such that $[\chi_{T_G}, \Omega] \neq 0$, then $\chi(1)/\Omega(1) \big| |G : T_G| = 7$. Thus $\Omega(1) = 2 \cdot 3^4 \cdot 5^2 \cdot 17$. Let Q be a minimal normal subgroup of T_G, then we assert that $|Q| = 3$.

If $|Q| = 2$, then $\Delta(1) \big| |T_G : Q| = 2^{15} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$, a contradiction.

Similarly, we can prove that $|Q| \neq 2^i$, where $2 \leq i \leq 16$.

If $|Q| = 3^2$, then $\Omega(1) \big| |T_G : Q| = 2^{16} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 17$, a contradiction.

By the similar arguments as above, we can prove that $|Q| \neq 3^m$, 5, 5^2 or 17, where $3 \leq m \leq 5$. Hence, we have that $|Q| = 3$, and so the assertion holds. Let $\lambda \in Irr(Q)$ such that $[\Delta_Q, \lambda] \neq 0$. Set $e = [\Delta_Q, \lambda]$, $t = |T_G : I_{T_G}(\lambda)|$, we have

$\Delta(1) = et\lambda(1)$. Since Q is abelian and $Aut(Q) \cong Z_2$, we have that $\lambda(1) = 1$, and hence $t \leq 2$. Thus $e \geq 2^{15}$. But $[\Delta_Q, \Delta_Q] = e^2 t \geq 2^{31} > |T_G : Q| = 2^{16} \cdot 3^4 \cdot 5^2 \cdot 17$, a contradiction to [13, 2.29].

If $|T_G| = 2^{15} \cdot 3^5 \cdot 5^2 \cdot 17$, let $\theta \in Irr(T_G)$ such that $[\chi_{T_G}, \theta] \neq 0$, then $\chi(1)/\theta(1) \big| |G : T_G| = 14$, and so $\theta(1) = 2 \cdot 3^4 \cdot 5^2 \cdot 17$. Hence $\theta(1)^2 > |T_G|$, a contradiction.

If $|T_G| = 2^{16} \cdot 3^4 \cdot 5^2 \cdot 17$, let $\vartheta \in Irr(T_G)$ such that $[\beta_{T_G}, \vartheta] \neq 0$, then $\beta(1)/\vartheta(1) \big| |G : T_G| = 21$. Hence $\vartheta(1) = 2^{16}$. But $\vartheta(1)^2 > |T_G|$, a contradiction.

Therefore, G is nonsolvable. By Lemma 2.2, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \big| |Out(K/H)|$. Since $|G| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$, then by Table 1 we have that K/H can be isomorphic to one of the groups: A_5, A_6, $U_4(2)$, $L_2(7)$, $L_2(8)$, $U_3(3)$, A_7, $L_3(4)$, A_8, A_9, J_2, A_{10}, $S_6(2)$, $O_8^+(2)$, $L_2(17)$, $L_2(16)$, $S_4(4)$, $O_8^-(2)$, $L_4(4)$, $A_5 \times A_5$, $A_6 \times A_6$ and $S_8(2)$.

Case 1. $K/H \cong A_5$

By Lemma 2.2, we have $|G : K| = 1$ or 2. We will only discuss the case $|G : K| = 1$ since by the same reason we can prove that $|G : K| \neq 2$.

If $|G : K| = 1$, then $|H| = 2^{14} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$. If H is solvable, then H has a Hall-subgroup D with index 7. Considering the permutation representation of H on the right cosets of D with kernel D_H, we have that $H/D_H \lesssim S_7$. Since orders of solvable subgroups of S_7 divided by 7 are 7, 14 and 21. Hence $|D_H| = 2^{14} \cdot 3^4 \cdot 5 \cdot 17$, $2^{13} \cdot 3^4 \cdot 5 \cdot 17$ and $2^{14} \cdot 3^3 \cdot 5 \cdot 17$.

If $|D_H| = 2^{13} \cdot 3^4 \cdot 5 \cdot 17$, let $\alpha \in Irr(D_H)$ such that $[\beta_{D_H}, \alpha] \neq 0$, then $\beta(1)/\alpha(1) \big| |G : D_H| = 2^{13} \cdot 3 \cdot 5 \cdot 7$, and so $\alpha(1) = 2^{13}$. But $\alpha(1)^2 > |D_H|$, a contradiction.

Similarly, we can prove that $|D_H| \neq 2^{14} \cdot 3^4 \cdot 5 \cdot 17$ or $2^{14} \cdot 3^3 \cdot 5 \cdot 17$.

Therefore, H is nonsolvable. By Lemma 2.2, H has a normal series $1 \trianglelefteq N \trianglelefteq M \trianglelefteq H$ such that M/N is a direct product of isomorphic non-abelian simple groups and $|H/M| \big| |Out(M/N)|$. Since $|H| = 2^{14} \cdot 3^4 \cdot 5 \cdot 7 \cdot 17$, then by Table 1 M/N can be isomorphic to one of A_5, A_6, $U_4(2)$, $L_2(7)$, $L_2(8)$, $U_3(3)$, A_7, $L_3(4)$, A_8, A_9, $S_6(2)$, $L_2(17)$, $L_2(16)$ and $O_8^-(2)$.

Subcase 1.1. $M/N \cong A_5$

By Table 1, we have $|H/M| \big| |Out(A_5)| = 2$. Hence $|H : M| = 1$ or 2.

If $|H : M| = 1$, then $|N| = 2^{12} \cdot 3^3 \cdot 7 \cdot 17$. Let $\Delta \in Irr(N)$ such that $[\beta_N, \Delta] \neq 0$, we have that $\beta(1)/\Delta(1) \big| |G : N| = 2^4 \cdot 3^2 \cdot 5^2$. Therefore, $\Delta(1) = 2^{12}$. But $\Delta(1)^2 > |N|$, a contradiction.

If $|H : M| = 2$, then $|N| = 2^{11} \cdot 3^3 \cdot 7 \cdot 17$. By the same reason as above, N is nonsolvable. By Lemma 2.2, N has a normal series: $1 \trianglelefteq B \trianglelefteq A \trianglelefteq N$ such that A/B is a direct product of isomorphic non-abelian simple groups and $|N/A| \big| |Out(A/B)|$. Since $|N| = 2^{12} \cdot 3^3 \cdot 7 \cdot 17$, then we have that $A/B \cong L_2(7)$, $L_2(8)$, $U_3(3)$ or $L_2(17)$.

If $A/B \cong L_2(7)$, then $|N : A| = 1$ or 2. Assume that $|B| = 2^v \cdot 3^2 \cdot 17$, where $8 \leq v \leq 9$. Let $\Omega \in Irr(B)$ such that $[\beta_B, \Omega] \neq 0$, we have $\beta(1)/\Omega(1) \big| |G : B| = 2^{16-v} \cdot 3^3 \cdot 5^2 \cdot 7$. Hence $\Omega(1) = 2^v$. But $\Omega(1)^2 > |B|$, a contradiction.

Similarly, we can prove that $A/B \ncong L_2(8)$, $U_3(3)$ or $L_2(17)$.

By the same reason as Subcase 1.1, M/N cannot be isomorphic to A_5, A_6 or $U_4(2)$.

Subcase 1.2. $M/N \cong L_2(7)$

In this case, $|H/M|\big||Out(L_2(7))| = 2$. Hence $|H : M| = 1$ or 2. Since the method of proof for $|H : M| = 2$ is the same as $|H : M| = 1$, we only discuss the case $|H : M| = 1$.

If $|H : M| = 1$, we have $|N| = 2^{10} \cdot 3^3 \cdot 5 \cdot 17$. If N is solvable, then N has a Hall-subgroup R with index 5. Considering the permutation representation of N on the right cosets of R with kernel R_N, we have that $N/R_N \lesssim S_5$. Since orders of solvable subgroups of S_5 divided by 5 are 5, 10 and 20, then $|R_N|$ can be equal to one of $2^{10} \cdot 3^3 \cdot 17$, $2^9 \cdot 3^3 \cdot 17$ and $2^8 \cdot 3^3 \cdot 17$.

If $|R_N| = 2^{10} \cdot 3^3 \cdot 17$, let $\Lambda \in Irr(R_N)$ such that $[\beta_{R_N}, \Lambda] \neq 0$, one has that $\beta(1)/\Lambda(1)\big||G : R_N| = 2^6 \cdot 3^2 \cdot 5^2 \cdot 7$, and then $2^{10}\big|\Lambda(1)$. But $\Lambda(1)^2 > |R_N|$, a contradiction. Similarly, we can prove that $|R_N| \neq 2^9 \cdot 3^3 \cdot 17$.

If $|R_N| = 2^8 \cdot 3^3 \cdot 17$, let U be a minimal normal subgroup of G and $U \leq R_N$. We assert that $|U| = 3$.

If $|U| = 2$, then $\beta(1)\big||G : U| = 2^{15} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$, a contradiction. Similarly, we have $|U| \neq 2^i$, where $2 \leq i \leq 10$. If $|U| = 3^2$, then $\chi(1)\big||G : U| = 2^{16} \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 17$, a contradiction. Also, we can prove that $|U| \neq 3^3$ or 17. Therefore, $|U| = 3$, and so the assertion holds. In this case, $|R_N/U| = 2^8 \cdot 3^2 \cdot 17$. Since R_N is solvable, then there exists a Hall-subgroup V of R_N with index 9. Let $V_{R_N} = \bigcap_{x \in R_N} V^x \leq V$. Then $R_N/V_{R_N} \lesssim S_9$. Since orders of solvable subgroups of S_9 divided by 9 are 9, 18, 36, 72, 144, 288, 576 and 1152, then we have that the order of V_{R_N} may be one of $2^8 \cdot 3 \cdot 17$, $2^7 \cdot 3 \cdot 17$, $2^6 \cdot 3 \cdot 17$, $2^5 \cdot 3 \cdot 17$, $2^4 \cdot 3 \cdot 17$, $2^3 \cdot 3 \cdot 17$, $2^2 \cdot 3 \cdot 17$ and $2 \cdot 3 \cdot 17$.

If $|V_{R_N}| = 2^8 \cdot 3 \cdot 17$, let $\theta \in Irr(V_{R_N})$ such that $[\beta_{V_{R_N}}, \theta] \neq 0$, then $\beta(1)/\theta(1)\big||G : V_{R_N}| = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, and so $\theta(1) = 2^8$. But $\theta(1)^2 > |V_{R_N}|$, a contradiction. Similarly, we have that $|V_{R_N}| \neq 2^7 \cdot 3 \cdot 17$, $2^6 \cdot 3 \cdot 17$, $2^2 \cdot 3 \cdot 17$ or $2 \cdot 3 \cdot 17$.

If $|V_{R_N}| = 2^5 \cdot 3 \cdot 17$, Considering the permutation representation of V_{R_N} on the right cosets of P with kernel $P_{V_{R_N}}$, where $|V_{R_N} : P| = 3$, we have that $V_{R_N}/P_{V_{R_N}} \lesssim S_3$. Hence $|P_{V_{R_N}}| = 2^5 \cdot 17$ or $2^4 \cdot 17$.

If $|P_{V_{R_N}}| = 2^5 \cdot 17$, let $\mu \in Irr(P_{V_{R_N}})$ such that $[\beta_{P_{V_{R_N}}}, \mu] \neq 0$, then $\beta(1)/\mu(1)\big||G : P_{V_{R_N}}| = 2^{11} \cdot 3^4 \cdot 5^2 \cdot 7$, and so $\mu(1) = 2^5$. Thus $\mu(1)^2 > |P_{V_{R_N}}|$, which is a contradiction. If $|P_{V_{R_N}}| = 2^4 \cdot 17$, let $\digamma \in Irr(P_{V_{R_N}})$ such that $[\chi_{P_{V_{R_N}}}, \digamma] \neq 0$, then $\chi(1)/\digamma(1)\big||G : P_{V_{R_N}}| = 2^{12} \cdot 3^4 \cdot 5^2 \cdot 7$, and hence $\digamma(1) = 17$. In the case that $\digamma(1)^2 > |P_{V_{R_N}}|$, a contradiction.

Therefore, N is nonsolvable, and so the assertion is true. By Lemma 2.2, N has a normal series $1 \trianglelefteq B \trianglelefteq A \trianglelefteq N$ such that A/B is a direct product of isomorphic non-abelian simple groups and $|N/A|\big||Out(A/B)|$. By Table 1, we have $A/B \cong A_5$, A_6, $L_2(17)$ or $L_2(16)$.

If $A/B \cong A_5$, then $|N : A| = 1$ or 2. If $|N : A| = 1$, we have $|B| = 2^8 \cdot 3^2 \cdot 17$.

Let $\zeta \in Irr(B)$ such that $[\beta_B, \zeta] \neq 0$, then $\beta(1)/\zeta(1)\big||G : B| = 2^8 \cdot 3^3 \cdot 5^2 \cdot 7$, and hence $\zeta(1) = 2^8$. But $\zeta(1)^2 > |B|$, a contradiction. If $|N : A| = 2$, we have $|B| = 2^7 \cdot 3^2 \cdot 17$. By Lemma 2.1, the Sylow-17 subgroup B_{17} is normal in B, and so B_{17}, *char*, B. Since $H \lhd \lhd G$, then $B_{17} \unlhd G$. Hence $\chi(1)\big||G : B_{17}| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 7$, a contradicion.

Similarly, we can prove that $A/B \ncong A_6$, $L_2(17)$ or $L_2(16)$.

By the similar arguments as Subcase 1.2, we can prove that M/N cannot be isomorphic to one of $L_2(8)$, $U_3(3)$, A_7, $L_3(4)$, A_8, A_9, $S_6(2)$, $L_2(17)$ and $L_2(16)$.

Subcase 1.3. $M/N \cong O_8^-(2)$

In this case, we have $|H/M|\big||Out(O_8^-(2))| = 2$. Thus $|H : M| = 1$ or 2.

If $|H : M| = 1$, then we have $|N| = 4$, and so $N \leq Z(H)$. In this case, $H/N \cong O_8^-(2)$. Therefore, H is a central extension of N by $O_8^-(2)$. Since the Shur Multiplier is 1, then H is a splitting central extension of N by $O_8^-(2)$, and so $H \cong Z_4 \times O_8^-(2)$. Let $\Phi \in Irr(H)$ such that $[\beta_H, \Phi] \neq 0$. Then $\beta(1)/\Phi(1)\big||G : H|$ $= 2^2 \cdot 3 \cdot 5$, and we have $\Phi(1) = 2^{14}$. However, by the structure of H, we see that H has no irreducible character of degree 2^{14}, which leads to a contradiction.

If $|H : M| = 2$, then $|N| = 2$, and hence $N \leq Z(H)$. In this case, M is a splitting central extension of N by $O_8^-(2)$, and so $M \cong Z_4 \times O_8^-(2)$. By [13, Clifford 6.2], M has an irreducible character of degree 2^{13}, a contradiction to the structure of M.

By the same reason as Case 1, we can prove that K/H cannot be isomorphic to one of the groups: A_6, $U_4(2)$, $L_2(7)$, $L_2(8)$, $U_3(3)$, A_7, $L_3(4)$, A_8, $S_6(2)$, $L_2(17)$, $O_8^-(2)$, $A_5 \times A_5$ and $A_6 \times A_6$.

Case 2. $K/H \cong A_7$

In this case, $|G : K| = 1$ or 2. We suppose that $|H| = 2^a \cdot 3^3 \cdot 5 \cdot 17$, where $12 \leq a \leq 13$. Let $\Omega \in Irr(H)$ such that $[\beta_H, \Omega] \neq 0$, then we have $\beta(1)/\Omega(1)\big||G : H|$ $= 2^{16-a} \cdot 3^2 \cdot 5 \cdot 7$, and so $\Omega(1) = 2^a$. But $\Omega(1)^2 > |H|$, a contradiction.

Similarly, we can show that $K/H \ncong A_9$, J_2, A_{10}, $O_8^+(2)$, $L_2(16)$, $S_4(4)$ or $L_4(4)$.

Case 3. $K/H \cong S_8(2)$

In this case, by order comparison we have $H = 1$. Therefore, $G \cong S_8(2)$. Thus we complete the proof of Theorem. ∎

Acknowledgements. This paper was supported by Fundamental Research Funds for the Central Universities (Grant No.XDJK2014C163), Natural Science Foundation Project of CQCSTC (Grant No. cstc2014jcyjA00010), Chongqing Postdoctoral Science Foundation(Grant No. Xm2014029), China Postdoctoral Science Foundation (Grant No. 2014M562264), Natural Science Foundation of China (Grant Nos. 11171364, 11271301, 11471266) and CMEC (Grant Nos. KJ1401006, KJ1401019).

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Accepted: 01.09.2013