THE EXPLICIT EXPRESSION OF THE DRAZIN INVERSE OF SUMS OF TWO MATRICES AND ITS APPLICATION

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Abstract. In this paper, we give explicit expressions of $(P \pm Q)_d$ of two matrices P and Q, in terms of P, Q, P_d and Q_d, under the condition that $PQ = P^2$, and apply the result to finding an explicit representation for the Drazin inverse of a 2×2 block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ under some conditions.

Keywords: Drazin inverse, group inverse, representation, block matrix, idempotent matrix.

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1. Introduction

Recently, the representations of the Drazin inverse of matrices have been widely investigated (see, for example, [2], [10], [11], [15], [16], [17] and the literature mentioned below). In [12], Meyer and Rose presented a representation for the Drazin inverse of $\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$ in terms of sub-blocks A, B, D and their Drazin inverses. And in [1] later, Campbell and Meyer suggested to find an explicit representation for the Drazin inverse of a 2×2 block matrix in relation to its sub-blocks. Since then, a lot of special cases of this problem have been studied (see, for example, [3],

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[4], [5], [6], [7], [8], [13] and references therein). But no one can solve the general problem up to the present.

These investigations motivate us to deal with a representation for the Drazin inverse of a 2×2 block matrix by exploiting an explicit expression of the Drazin inverse of sums of two matrices. The paper is organized as follows. In this section, we will introduce some notions and lemmas. In Section 2, we will present these explicit expressions of differences and sums of two matrices P and Q under the conditions $PQ = P^2$. In Section 3, we will deduce an explicit representation for the Drazin inverse of the 2×2 block matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ under the conditions $AB = 0$ and $D^2 = \frac{1}{2}CB$ in terms of its sub-blocks and A_d and D_d.

Throughout this paper the symbol $\mathbb{C}^{m\times n}$ stands for the set of $m \times n$ complex matrices and $I \in \mathbb{C}^{n\times n}$ stands for the unit matrix. Let $A \in \mathbb{C}^{n\times n}$, the Drazin inverse, denoted by A_d, of matrix A is defined as the unique matrix satisfying

$$A^{k+1}A_d = A^k, \quad A_dAA_d = A_d, \quad AA_d = A_dA$$

where $k = \text{Ind}(A)$ is the index of A. In particular, if $\text{Ind}(A) = 1$, then A_d is called the group inverse, denoted by A_g, of A (see [1], [9], [14]). Apparently, if A is nonsingular, then $\text{Ind}(A) = 0$, otherwise $\text{Ind}(A) \geq 1$, especially $\text{Ind}(0) = 1$. If A is nilpotent, then $A_d = 0$. If X is nonsingular and $B = XAX^{-1}$, then $B_d = XA_dX^{-1}$. For convenience, we write $A^\pi = I - AA_d$ and

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.1}$$

where $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{n\times m}, C \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{m\times m}$.

Also, we need two functions: the ceiling function $\lceil x \rceil$, the smallest integer greater than or equal to x, and the floor function $\lfloor x \rfloor$, the largest integer less than or equal to x.

Now, we list and deduce some lemmas.

Lemma 1.1 [12, Theorem 3.2] *Let M be given by* (1.1) *with $C = 0$, $t = \text{Ind}(A)$ and $l = \text{Ind}(D)$. Then*

$$M_d = \begin{bmatrix} A_d & S \\ 0 & D_d \end{bmatrix}$$

where

$$S = A^\pi \sum_{n=0}^{t-1} A^n B D_d^{n+2} + \sum_{n=0}^{l-1} A_d^{n+2} B D^n D^\pi - A_d B D_d.$$

Lemma 1.2 [8, Theorem 3.1] *Let M be given by* (1.1) *with $D = 0$, $t = \text{Ind}(A)$ and $r = \text{Ind}(BC)$. If $AB = 0$, then*

$$M_d = \begin{bmatrix} XA & (BC)_dB \\ CX & 0 \end{bmatrix}$$

where

$$X = (BC)^{\pi} \sum_{n=0}^{r-1} (BC)^n A_d^{2n+2} + \sum_{n=0}^{\lceil \frac{t}{2} \rceil - 1} (BC)_d^{n+1} A^{2n} A^{\pi}.$$

Lemma 1.3 *Let $P, Q \in \mathbb{C}^{n \times n}$ with $s = \mathrm{Ind}(P)$ and $h = \mathrm{Ind}(Q)$. If $PQ = 0$, then* (i) *for any positive integer $n \geq 2$,*

$$(P+Q)^n = P^n + Q^n + \sum_{i=1}^{n-1} Q^i P^{n-i}. \tag{1.2}$$

(ii)[10, Theorem 2.1]

$$(P+Q)_d = \sum_{n=0}^{s-1} Q_d^{n+1} P^n P^{\pi} + Q^{\pi} \sum_{n=0}^{h-1} Q^n P_d^{n+1}.$$

Proof. (i) When $n = 2$, obviously, $(P+Q)^2 = P^2 + Q^2 + QP$ holds. Assume that (1.2) holds for $k \geq 2$, namely, $(P+Q)^k = P^k + Q^k + \sum_{i=1}^{k-1} Q^i P^{k-i}$, $k \geq 2$. Then for $k+1$,

$$(P+Q)^{k+1} = (P^k + Q^k + \sum_{i=1}^{k-1} Q^i P^{k-i})(P+Q) = P^{k+1} + Q^{k+1} + \sum_{i=1}^{k} Q^i P^{k+1-i}.$$

Hence, by induction, (1.2) holds for any positive integer n.

Remark 1: If $P^r = 0$ in Lemma 1.3(ii), then $(P+Q)_d = \sum_{n=0}^{r-1} Q_d^{n+1} P^n$. ∎

The following lemma generalizes [1, Theorem 7.8.4(iii)].

Lemma 1.4 *Let $P \in \mathbb{C}^{n \times m}$, $Q \in \mathbb{C}^{m \times n}$, then $(PQ)_d = P(QP)_d^2 Q$.*

Lemma 1.5 *Let $P, Q \in \mathbb{C}^{n \times n}$. If $PQ = P^2$, then for any positive integer n,*

$$(Q-P)^n = Q^{n-1}(Q-P). \tag{1.3}$$

Proof. When $n = 1$, obviously, (1.3) holds. Assume that (1.3) holds for k, namely, $(Q-P)^k = Q^{k-1}(Q-P)$. Then for $k+1$,

$$(Q-P)^{k+1} = Q^{k-1}(Q-P)(Q-P) = Q^k(Q-P).$$

Hence, by induction, (1.3) holds for any positive integer n. ∎

2. The Drazin inverse of differences and sums of two matrices

In this section, we will investigate how to express $[I - Q + (WP)^k]_d$ for any positive integer k, and $(P \pm Q)_d$ under some conditions. We begin with the following theorem.

Theorem 2.1 *Let $P, Q \in \mathbb{C}^{n\times n}$ with* $\mathrm{Ind}(P) = s$. *If $PQ = P$ and Q is idempotent, then*
(i)

$$(2.1) \qquad (I - Q + P^k)_d = \begin{cases} (I-Q)\sum_{n=0}^{\lceil \frac{s}{k}\rceil - 1} P^{kn}P^{\pi} + QP_d^k, & \text{if } k < s; \\ (I-Q)P^{\pi} + QP_d^k, & \text{if } k \geq s. \end{cases}$$

(ii)

$$(2.2) \qquad [I - Q + (QP)^k]_d = I - Q + QP_d^k.$$

Proof. (i) Since $Q = Q^2$, $h = \mathrm{Ind}(I - Q) = 1$. From $PQ = P$, we have $P^k(I - Q) = 0$ for any positive integer k, and $(P^k)^{\pi} = P^{\pi}$. Thus, by Lemma 1.3(ii), we get

$$(2.3) \qquad [I - Q + P^k]_d = (I-Q)\sum_{n=0}^{s_0-1} P^{kn}P^{\pi} + QP_d^k$$

where $s_0 = \mathrm{Ind}P^k$.

Note that $n \geq \lceil \frac{s}{k} \rceil$ implies $kn \geq s$ and that when $kn \geq s$, $P^{kn}P^{\pi} = 0$. So (2.3) becomes

$$[I - Q + P^k]_d = (I-Q)\sum_{n=0}^{\lceil \frac{s}{k}\rceil - 1} P^{kn}P^{\pi} + QP_d^k.$$

Consequently, (2.1) holds.

(ii) Replacing P with QP in the proof of (i) yields

$$(2.4) \qquad [I - Q + (QP)^k]_d = (I-Q)\sum_{n=0}^{\lceil \frac{l}{k}\rceil - 1} [(QP)^{kn} - (QP)^{kn+1}(QP)_d] + Q(QP)_d^k$$

where $l = \mathrm{Ind}(QP)^k$.

Since $PQ = P$, by Lemma 1.4,

$$(QP)_d = Q(PQ)_d^2 P = QP_d^2 P = QP_d.$$

In general, by induction, we can easily show that for any positive integer k,

$$(2.5) \qquad (QP)_d^k = QP_d^k.$$

By the above equation and Lemma 1.4,

$$P_d = (PQ)_d = P(QP)_d^2 Q = PQP_d^2 Q = P_d Q.$$

Since $(I - Q)(QP)^i = 0$ for any positive integer i, by (2.4) and (2.5), we reach (2.2). ∎

Now, we present the expression of $(P \pm Q)_d$.

Theorem 2.2 *Let* $P, Q \in \mathbb{C}^{n\times n}$ *with* $\mathrm{Ind}(P) = s \geq 1$ *and* $\mathrm{Ind}(Q) = r$. *If* $PQ = P^2$ *and* $h = \mathrm{Ind}(P-Q) \geq 1$, *then* $h \leq r+1$ *and*
(i)

$$(P-Q)_d = Q_d^2P - Q_d = Q_d^2(P-Q). \tag{2.6}$$

(ii)

$$(P-Q)(P-Q)_d = Q_d(Q-P). \tag{2.7}$$

(iii)

$$\begin{aligned}(P+Q)_d &= \frac{1}{2}Q_d + \sum_{n=0}^{s-1} 2^{n-1}Q_d^{n+1}P^nP^\pi + Q^\pi \sum_{n=0}^{h-1} 2^{-(n+2)}Q^nP_d^{n+1} \\ &\quad +2^{-(h+1)}Q^\pi Q^{h-1}P_d^h.\end{aligned} \tag{2.8}$$

In particular, when $h = r+1$,

$$(P+Q)_d = \frac{1}{2}Q_d + \sum_{n=0}^{s-1} 2^{n-1}Q_d^{n+1}P^nP^\pi + Q^\pi \sum_{n=0}^{r-1} 2^{-(n+2)}Q^nP_d^{n+1}. \tag{2.9}$$

Proof. Since $\mathrm{Ind}(P) = s \geq 1$, there exists a nonsingular matrix W_1 such that

$$P = W_1 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} W_1^{-1} \quad \textit{and} \quad P_d = W_1 \begin{bmatrix} P_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} W_1^{-1} \tag{2.10}$$

where P_1 is nonsingular and P_2 is nilpotent with $P_2^s = 0$. Partitioning $W_1^{-1}QW_1$ conformably with $W_1^{-1}PW_1$, we have

$$Q = W_1 \begin{bmatrix} Q_1 & Q_4 \\ Q_3 & Q_2 \end{bmatrix} W_1^{-1}.$$

Since $PQ = P^2$, we can deduce $P_1 = Q_1$, $Q_4 = 0$ and $P_2Q_2 = P_2^2$. Thus

$$P - Q = W_1 \begin{bmatrix} 0 & 0 \\ -Q_3 & P_2 - Q_2 \end{bmatrix} W_1^{-1} \text{ and } Q_d = W_1 \begin{bmatrix} P_1^{-1} & 0 \\ H & (Q_2)_d \end{bmatrix} W_1^{-1} \tag{2.11}$$

where H is some matrix obtained by using Lemma 1.1.

Since $P_2Q_2^{k-1} = P_2^k = 0, k \geq \max\{s, 2\}$, we have

$$\begin{aligned}(Q_2^sP_2)^2 &= Q_2^s(P_2Q_2^s)P_2 = 0, \\ P_2(Q_2)_d &= P_2Q_2^s(Q_2)_d^{s+1} = 0, \\ (Q_2^sP_2)_d &= 0.\end{aligned}$$

By (1.3) and Remark 1, therefore, we have

$$(P_2-Q_2)_d^{s+1} = (-1)^s(Q_2^sP_2 - Q_2^{s+1})_d = (-1)^s[-(Q_2)_d^{s+1} + (Q_2)_d^{s+2}P_2]$$

and then, by Lemma 1.5,

$$\begin{aligned}(P_2-Q_2)_d &= (P_2-Q_2)_d^{s+1}(P_2-Q_2)^s\\ &= (-1)^{2s-1}[-(Q_2)_d^{s+1}+(Q_2)_d^{s+2}P_2](Q_2^{s-1}P_2-Q_2^s)\\ &= (Q_2)_d^2P_2-(Q_2)_d,\\ (P_2-Q_2)_d^2 &= [(Q_2)_d^2P_2]^2-(Q_2)_d^2P_2(Q_2)_d-(Q_2)_d(Q_2)_d^2P_2+(Q_2)_d^2\\ &= (Q_2)_d^2-(Q_2)_d^3P_2.\end{aligned}$$

By Lemma 1.1, we get that

$$\begin{aligned}(P-Q)_d &= W_1\begin{bmatrix}0 & 0\\ -(P_2-Q_2)_d^2Q_3 & (P_2-Q_2)_d\end{bmatrix}W_1^{-1}\\ &= W_1\begin{bmatrix}0 & 0\\ -[(Q_2)_d^2P_2-(Q_2)_d]^2Q_3 & (Q_2)_d^2P_2-(Q_2)_d\end{bmatrix}W_1^{-1}\\ &= (Q_d^2P-Q_d)P^\pi-(Q_d^2-Q_d^3P)P^\pi QPP_d\\ &= Q_d^2P-Q_d-(Q_d^2P-Q_d)PP_d-(Q_d^2-Q_d^3P)(QPP_d-P_dPQPP_d)\\ &= Q_d^2P-Q_d-(Q_d^2P-Q_d)PP_d-Q_d^2(QPP_d-P^2P_d)\\ &= Q_d^2P-Q_d \;=\; Q_d^2(P-Q).\end{aligned} \tag{2.12}$$

Since $\mathrm{Ind}(Q)=r$, $Q_dQ^{r+1}=Q^r$. By (1.3) and (2.12),

$$\begin{aligned}(P-Q)_d(P-Q)^{r+2} &= Q_d^2(P-Q)^{r+3} &&= (-1)^{r+2}Q_d^2Q^{r+2}(P-Q)\\ &= (-1)^rQ^r(P-Q) &&= (P-Q)^{r+1}.\end{aligned}$$

Hence $\mathrm{Ind}(P-Q)=\mathrm{Ind}(Q-P)\le r+1$.

(ii) By (i) and Lemma 1.5,

$$(P-Q)(P-Q)_d=(P-Q)_d(P-Q)=Q_d^2(P-Q)^2=Q_d(Q-P).$$

(iii) By (i) and (ii),

$$(Q-P)_d^2=Q_d^2(Q-P)(Q-P)_d=Q_d^3(Q-P).$$

Generally, for any positive integer n, by induction,

$$(Q-P)_d^n \;=\; Q_d^{n+1}(Q-P). \tag{2.13}$$

From $PQ=P^2$, it follows easily that $P(P-Q)=0$ and $PQ^k=P^{k+1}$ for

$k \geq 1$. Thus, by Lemma 1.2 and (1.3), (2.7) and (2.13), we have

$$
\begin{aligned}
&(P+Q)_d = [2P + (Q-P)]_d \\
&= \sum_{n=0}^{s-1} (Q-P)_d^{n+1} (2P)^n P^\pi + (Q-P)^\pi \sum_{n=0}^{h-1} (Q-P)^n (2P)_d^{n+1} \\
&= \sum_{n=0}^{s-1} 2^n Q_d^{n+2} (Q-P) P^n P^\pi + (Q^\pi + Q_d P) \left[\frac{1}{2} P_d + \sum_{n=1}^{h-1} 2^{-(n+1)} Q^{n-1} (Q-P) P_d^{n+1} \right] \\
&= \sum_{n=0}^{s-1} 2^n (Q_d^{n+1} P^n - Q_d^{n+2} P^{n+1}) P^\pi + \frac{1}{2} (Q^\pi + Q_d P) P_d \\
&\qquad + Q^\pi \sum_{n=1}^{h-1} 2^{-(n+1)} Q^{n-1} (Q-P) P_d^{n+1} \\
&= Q_d P^\pi + \sum_{n=1}^{s-1} 2^n Q_d^{n+1} P^n P^\pi - \sum_{n=1}^{s-1} 2^{n-1} Q_d^{n+1} P^n P^\pi + \frac{1}{2} (Q^\pi + Q_d P) P_d \\
&\qquad + Q^\pi \sum_{n=1}^{h-1} 2^{-(n+1)} Q^n P_d^{n+1} - Q^\pi \sum_{n=1}^{h-2} 2^{-(n+2)} Q^n P_d^{n+1} - 2^{-2} Q^\pi P_d \\
&= \frac{1}{2} Q_d + \frac{1}{2} Q_d P^\pi + \sum_{n=1}^{s-1} 2^{n-1} Q_d^{n+1} P^n P^\pi \\
&\qquad + 2^{-h} Q^\pi Q^{h-1} P_d^h + Q^\pi \sum_{n=1}^{h-2} 2^{-(n+2)} Q^n P_d^{n+1} + 2^{-2} Q^\pi P_d \\
&= \frac{1}{2} Q_d + \sum_{n=0}^{s-1} 2^{n-1} Q_d^{n+1} P^n P^\pi + Q^\pi \sum_{n=0}^{h-1} 2^{-(n+2)} Q^n P_d^{n+1} + 2^{-(h+1)} Q^\pi Q^{h-1} P_d^h.
\end{aligned}
$$

When $h = r + 1$, (2.9) follows from (2.8). ∎

Remark 2. If $h = 0$ in Theorem 2.2, then $P - Q$ is nonsingular and therefore $P = 0$ since $PQ = P^2$. Similarly, if $s = 0$, then P is nonsingular and so $P = Q$. Thus Theorem 2.2 is trivial for the two special cases.

If P is idempotent in Theorem 2.2, then $P^k P^\pi = 0, k \geq 0$. So we have the following result.

Corollary 2.1 *Let $P, Q \in \mathbb{C}^{n\times n}$ with $r = \mathrm{Ind}(Q)$ and $h = \mathrm{Ind}(P - Q) \geq 1$. If $PQ = P$ and P is idempotent, then $h \leq r + 1$ and*

$$(P+Q)_d = Q_d - \frac{1}{2} Q_d P + Q^\pi \sum_{n=0}^{h-1} 2^{-(n+2)} Q^n P + 2^{-(h+1)} Q^\pi Q^{h-1} P$$

In particular, when $h = r + 1$,

$$(P+Q)_d = Q_d - \frac{1}{2} Q_d P + Q^\pi \sum_{n=0}^{r-1} 2^{-(n+2)} Q^n P$$

If P and Q are both idempotent in Theorem 2.2, then

$$(P+Q)_d = Q + \frac{1}{4}P - \frac{3}{4}QP.$$

Since

$$(P+Q)^2(P+Q)_d = (P+Q)\left(\frac{1}{2}P + Q - \frac{1}{2}QP\right) = P+Q,$$

we have the result below.

Corollary 2.2 *Let $P, Q \in \mathbb{C}^{n\times n}$ be idempotent. If $PQ = P$, then*

$$(P-Q)_d = QP - Q \quad \textit{and} \quad (P+Q)_g = Q + \frac{1}{4}P - \frac{3}{4}QP.$$

Theorem 2.3 *Let $P, Q \in \mathbb{C}^{n\times n}$. If $PQ = P^2$ and $QP = Q^2$, then*

$$(P+Q)_d = \frac{1}{4}(P_d + Q_d) \qquad \textit{and} \qquad (P-Q)_d = 0.$$

Proof. Premultiplying $QP = Q^2$ by Q_d^2 yields $Q_d P = QQ_d$ and then $Q_d P P_d = QQ_d P_d$. So, by Lemma 1.4, we get

$$QP_d = QPP_d^2 = Q^2(PQ)_d = Q^2P(QP)_d^2Q = Q^3Q_d^4Q = QQ_d = Q_dP,$$

and then $Q_dQP_d = Q_d$. Thus for $n \geq 1$,

$$(2.14) \qquad Q_dP^{n-1}(I - PP_d) \;=\; (Q_d - Q_dPP_d)P^{n-1} \;=\; 0,$$

$$(2.15) \qquad (I - QQ_d)Q^nP_d \;=\; (I - QQ_d)Q^{n-1}Q_dP \;=\; 0.$$

Since $PQ = P^2$, by Theorem 2.2(iii), we have (2.8) and put (2.14) and (2.15) in (2.8). As a result,

$$(P+Q)_d = \frac{1}{2}Q_d + \frac{1}{4}(I - Q_dQ)P_d = \frac{1}{2}Q_d + \frac{1}{4}(P_d - Q_d) = \frac{1}{4}(P_d + Q_d).$$

Since $(P-Q)^2 = P^2 - PQ - QP + Q^2 = 0$, $(P-Q)_d = 0$. ■

3. The Drazin inverse of 2×2 block matrices

In this section, we turn our attention to the representation for the Drazin inverse of a 2-by-2 block matrix M given by (1.1), in terms of its sub-blocks and A_d and D_d, by Theorem 2.2.

Theorem 3.1 *Let M be given by* (1.1) *with* $t = \mathrm{Ind}(A)$, $s = \mathrm{Ind}\left(\begin{bmatrix} A & \frac{1}{2}B \\ C & 0 \end{bmatrix}\right)$, $q = \mathrm{Ind}\left(\begin{bmatrix} 0 & \frac{1}{2}B \\ 0 & D \end{bmatrix}\right)$, $h = \mathrm{Ind}\left(\begin{bmatrix} A & 0 \\ C & -D \end{bmatrix}\right)$ *and* $r = \mathrm{Ind}(BC)$. *If* $D^2 = \frac{1}{2}CB$

and $AB = 0$, then $h \leq s+1$ and

$$M_d = \frac{1}{2}\begin{bmatrix} L & BD_d^2 \\ 2K(1,r)+S(t) & 0 \end{bmatrix} + \sum_{n=1}^{q-1} 2^{n-1}\begin{bmatrix} \frac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix}$$
$$-\frac{1}{3}(1-4^{-\lceil\frac{h}{2}\rceil})G(0,t) - \frac{1}{6}(1-4^{-\lfloor\frac{h}{2}\rfloor})G(1,t) + \sum_{n=2}^{h-1} 2^{-(n+2)}G(n,n)$$
$$+2^{-(h+1)}[G(h-1,h-1)-G(h-1,t)]. \tag{3.1}$$

In particular, when $h = s+1$,

$$M_d = \frac{1}{2}\begin{bmatrix} L & BD_d^2 \\ 2K(1,r)+S(t) & 0 \end{bmatrix} + \sum_{n=1}^{q-1} 2^{n-1}\begin{bmatrix} \frac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix}$$
$$-\frac{1}{3}(1-4^{-\lceil\frac{s}{2}\rceil})G(0,t) - \frac{1}{6}(1-4^{-\lfloor\frac{s}{2}\rfloor})G(1,t) + \sum_{n=2}^{s-1} 2^{-(n+2)}G(n,n) \tag{3.2}$$

where $\sum\limits_{n=k}^{m} = 0$ whenever $m < k$ and

$$L = (2I - BD_d^2C)A_d + BK(1,r-1)A_d + BD_d^2S(t)A - \frac{1}{2}BD_dS(t), \tag{3.3}$$

$$S(n) = \sum_{k=0}^{\lceil\frac{n}{2}\rceil-1} D_d^{2k+2}CA^{2k}A^{\pi}, \tag{3.4}$$

$$G(n,m) = \begin{cases} \begin{bmatrix} \frac{1}{2}BD_d^2S(m)A & 0 \\ D_dS(m)A & 0 \end{bmatrix}, & n \text{ is even;} \\ \\ \begin{bmatrix} \frac{1}{2}BD_d^3S(m-2)A^2 & 0 \\ D_d^2S(m-2)A^2 & 0 \end{bmatrix}, & n \text{ is odd,} \end{cases} \tag{3.5}$$

$$K(n,m) = D^{\pi}\sum_{k=0}^{m-1} D^{n+2k-1}CA_d^{n+2k+1}. \tag{3.6}$$

Proof. Assume that

$$W = \begin{bmatrix} Y_0 & Y_1 \\ Y_2 & 0 \end{bmatrix} = \begin{bmatrix} Y_0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y_1 \\ Y_2 & 0 \end{bmatrix} := W_1 + W_2$$

where $Y_0Y_1 = 0$. So, for $k \geq 1$,

$$W_1^k = \begin{bmatrix} Y_0^k & 0 \\ 0 & 0 \end{bmatrix}, \qquad W_2^{2k} = \begin{bmatrix} (Y_1Y_2)^k & 0 \\ 0 & (Y_2Y_1)^k \end{bmatrix},$$
$$W_2^{2k-1} = \begin{bmatrix} 0 & (Y_1Y_2)^{k-1}Y_1 \\ (Y_2Y_1)^{k-1}Y_2 & 0 \end{bmatrix}.$$

and then

$$
\begin{aligned}
W_2^{2k}W_1^{n-2k} &= \begin{bmatrix} (Y_1Y_2)^kY_0^{n-2k} & 0 \\ 0 & 0 \end{bmatrix}, \\
W_2^{2k-1}W_1^{n-2k+1} &= \begin{bmatrix} 0 & 0 \\ (Y_2Y_1)^{k-1}Y_2Y_0^{n-2k+1} & 0 \end{bmatrix}, \\
HW_2^{2j}W_1^k &= 0, \quad \text{for } j > 1, \qquad (3.7)
\end{aligned}
$$

where $H = \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix}$ partitioned conformably to W. For $n \geq 2$, by Lemma 1.3(i) and (3.7),

$$
\begin{aligned}
HW^n &= H(W_1^n + W_2^n + \sum_{i=1}^{n-1} W_2^i W_1^{n-i}) = HW_2^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} HW_2^{2k-1}W_1^{n-2k+1} \\
&= HW_2^n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} H \begin{bmatrix} 0 & 0 \\ (Y_2Y_1)^{k-1}Y_2Y_0^{n-2k+1} & 0 \end{bmatrix}. \qquad (3.8)
\end{aligned}
$$

Rewrite M as

$$
M = \begin{bmatrix} A & \frac{1}{2}B \\ C & 0 \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2}B \\ 0 & D \end{bmatrix} := P + Q.
$$

From the conditions that $2D^2 = CB$ and $AB = 0$, it follows $PQ = Q^2$, namely $Q^TP^T = (Q^T)^2$ where the symbol F^T stands for the transpose of a matrix F. Note that $\text{Ind}\left(\begin{bmatrix} A_1 & 0 \\ A_3 & A_2 \end{bmatrix}\right) \geq 1$ whenever square matrix A_1 or A_2 is singular by [12, Theorem 2.1]. So $q \geq 1$. If A is nonsingular, then $B = 0$ since $AB = 0$. Thus $D^2 = 0$ and then D is singular. So $h \geq 1$. Therefore, applying Theorem 2.2 to M^T, we have $h = \text{Ind}(P - Q) \leq \text{Ind}(P) + 1 = s + 1$ and

$$
\begin{aligned}
M_d &= \frac{1}{2}P_d + Q^\pi \sum_{n=0}^{q-1} 2^{n-1}Q^nP_d^{n+1} + \sum_{n=0}^{h-1} 2^{-(n+2)}Q_d^{n+1}P^nP^\pi \\
&\quad + 2^{-(h+1)}Q_d^hP^{h-1}P^\pi. \qquad (3.9)
\end{aligned}
$$

Obviously, for $n \geq 1$,

$$
Q^n = \begin{bmatrix} 0 & \frac{1}{2}BD^{n-1} \\ 0 & D^n \end{bmatrix}, \quad Q_d^n = \begin{bmatrix} 0 & \frac{1}{2}BD_d^{n+1} \\ 0 & D_d^n \end{bmatrix}, \quad Q^\pi = \begin{bmatrix} I & -\frac{1}{2}BD_d \\ 0 & D^\pi \end{bmatrix}.
$$

By Lemma 1.4,

$$
\begin{aligned}
C\left(\frac{1}{2}BC\right)_d &= C\left(\frac{1}{2}BC\right)_d^2\left(\frac{1}{2}BC\right) = \left(\frac{1}{2}CB\right)_d C = D_d^2C, \qquad (3.10) \\
C(BC)_dB &= \frac{1}{2}D_d^2CB = D_dD, \qquad (3.11) \\
D_dC(BC)^\pi &= D_dC(I - (BC)_dBC) = D_dC - D_d^2DC = 0 \qquad (3.12)
\end{aligned}
$$

So, by Lemma 1.2 and (3.11),

$$P_d = \begin{bmatrix} XA & (BC)_d B \\ CX & 0 \end{bmatrix}, \qquad P^\pi = \begin{bmatrix} (BC)^\pi - XA^2 & 0 \\ -CXA & D^\pi \end{bmatrix}$$

where

$$X = \left(\frac{1}{2}BC\right)^\pi \sum_{m=0}^{r-1} \left(\frac{1}{2}BC\right)^m A_d^{2m+2} + \sum_{m=0}^{\lceil \frac{t}{2} \rceil -1} \left(\frac{1}{2}BC\right)_d^{m+1} A^{2m} A^\pi$$

and $AX = A_d$.

Now, taking $W = P$, we have $W_1 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$ and $W_2 = \begin{bmatrix} 0 & \frac{1}{2}B \\ C & 0 \end{bmatrix}$, and, accordingly, $Y_0 = A, Y_1 = \frac{1}{2}B$, and $Y_2 = C$. Thus,

$$(Y_2 Y_1)^{k-1} = \left(\frac{1}{2}CB\right)^{k-1} = D^{2k-2}.$$

Taking $H = Q_d^{n+1}$ in (3.8) for $n \geq 2$ yields

$$\begin{aligned} Q_d^{n+1} P^n P^\pi &= Q_d^{n+1} W_2^n P^\pi + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} 0 & \frac{1}{2}BD_d^{n+2} \\ 0 & D_d^{n+1} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ D^{2k-2}CA^{n-2k+1} & 0 \end{bmatrix} P^\pi \\ &= Q_d^{n+1} W_2^n P^\pi + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} \frac{1}{2}BD_d^{n-2k+4}CA^{n-2k+1}[(BC)^\pi - XA^2] & 0 \\ D_d^{n-2k+3}CA^{n-2k+1}[(BC)^\pi - XA^2] & 0 \end{bmatrix} \quad (3.13) \\ &= Q_d^{n+1} W_2^n P^\pi + \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor -1} \begin{bmatrix} \frac{1}{2}BD_d^{n-2k+2}CA^{n-2k-1}A^\pi & 0 \\ D_d^{n-2k+1}CA^{n-2k-1}A^\pi & 0 \end{bmatrix}. \end{aligned}$$

since $A[(BC)^\pi - XA^2] = AA^\pi$ and $k \leq n/2$.

In order to continue conveniently the above computation, we write

$$V(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor -1} D_d^{n-2k+1} CA^{n-2k-1} A^\pi, \quad n \geq 2,$$

and consider it in terms of the parity of n as follows. If $n = 2w+1, w \geq 1$, then since

$$S((2w+1)-2) = \sum_{k=0}^{\lceil \frac{2w-1}{2} \rceil -1} D_d^{2k+2} CA^{2k} A^\pi = \sum_{k=0}^{w-1} D_d^{2k+2} CA^{2k} A^\pi,$$

we have

$$\begin{aligned} V(2w+1) &= \sum_{k=0}^{w-1} D_d^{2w-2k+2} CA^{2w-2k} A^\pi = \sum_{i=0}^{w-1} D_d^{2i+4} CA^{2i+2} A^\pi \\ &= D_d^2 S((2w+1)-2) A^2. \end{aligned}$$

If $n = 2w, w \geq 1$, then

$$\begin{aligned} V(2w) &= \sum_{k=0}^{w-1} D_d^{2w-2k+1} C A^{2w-2k-1} A^{\pi} = \sum_{i=0}^{w-1} D_d^{2i+3} C A^{2i+1} A^{\pi} \\ &= D_d S(2w) A. \end{aligned}$$

Hence,

$$V(n) = \begin{cases} D_d S(n) A, & n \text{ is even;} \\ D_d^2 S(n-2) A^2, & n \text{ is odd} \end{cases}$$

and then for $n \geq 2$,

$$G(n,n) = \begin{bmatrix} \frac{1}{2} B D_d V(n) & 0 \\ V(n) & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} B D_d^{n-2k+2} C A^{n-2k-1} A^{\pi} & 0 \\ D_d^{n-2k+1} C A^{n-2k-1} A^{\pi} & 0 \end{bmatrix} \tag{3.14}$$

We also need to deal with $Q_d^{n+1} W_2^n P^{\pi}$. Likewise, we consider it in terms of the parity of n as follows. When $n = 2k$,

$$\begin{aligned} Q_d^{2k+1} W_2^{2k} P^{\pi} &= \begin{bmatrix} 0 & \frac{1}{2} B D_d^{2k+2} \\ 0 & D_d^{2k+1} \end{bmatrix} \begin{bmatrix} (\frac{1}{2} BC)^k & 0 \\ 0 & D^{2k} \end{bmatrix} \begin{bmatrix} (BC)^{\pi} - X A^2 & 0 \\ -CXA & D^{\pi} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} B D_d^2 C X A & 0 \\ -D_d C X A & 0 \end{bmatrix}, \end{aligned}$$

and when $n = 2k-1$, by (3.12),

$$\begin{aligned} Q_d^{2k} W_2^{2k-1} P^{\pi} &= \begin{bmatrix} 0 & \frac{1}{2} B D_d^{2k+1} \\ 0 & D_d^{2k} \end{bmatrix} \begin{bmatrix} 0 & (\frac{1}{2} BC)^{k-1} B \\ D^{2k-2} C & 0 \end{bmatrix} \begin{bmatrix} (BC)^{\pi} - X A^2 & 0 \\ -CXA & D^{\pi} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} B D_d^3 C X A^2 & 0 \\ -D_d^2 C X A^2 & 0 \end{bmatrix}. \end{aligned}$$

By (3.10) and (3.12),

$$\begin{aligned} D_d C X A &= D_d C \sum_{k=0}^{\lceil \frac{t}{2} \rceil - 1} \left(\frac{1}{2} BC \right)_d^{k+1} A^{2k+1} A^{\pi} = D_d \sum_{k=0}^{\lceil \frac{t}{2} \rceil - 1} D_d^{2k+2} C A^{2k+1} A^{\pi} \\ &= D_d S(t) A, \\ D_d^2 C X A^2 &= D_d^2 \sum_{k=0}^{\lceil \frac{t}{2} \rceil - 1} D_d^{2k+2} C A^{2k+2} A^{\pi} = D_d^2 \sum_{k=0}^{\lceil \frac{t}{2} \rceil - 2} D_d^{2k+2} C A^{2k+2} A^{\pi} \\ &= D_d^2 \sum_{k=0}^{\lceil \frac{t-2}{2} \rceil - 1} D_d^{2k+2} C A^{2k+2} A^{\pi} = D_d^2 S(t-2) A^2. \end{aligned}$$

Thus

$$Q_d^{n+1} W_2^n P^{\pi} = -G(n,t). \tag{3.15}$$

Hence, putting (3.14) and (3.15) in (3.13) yields

$$Q_d^{n+1}P^nP^\pi = G(n,n) - G(n,t) \tag{3.16}$$

where $n \geq 2$. Especially,

$$Q_d^hP^{h-1}P^\pi = G(h-1,h-1) - G(h-1,t). \tag{3.17}$$

By (3.12),

$$\begin{aligned}\frac{1}{4}Q_dP^\pi + \frac{1}{8}Q_d^2PP^\pi &= \frac{1}{4}\begin{bmatrix} 0 & \frac{1}{2}BD_d^2 \\ 0 & D_d \end{bmatrix}\begin{bmatrix} (BC)^\pi - XA^2 & 0 \\ -CXA & D^\pi \end{bmatrix} \\ &\quad + \frac{1}{8}\begin{bmatrix} 0 & \frac{1}{2}BD_d^3 \\ 0 & D_d^2 \end{bmatrix}\begin{bmatrix} A & \frac{1}{2}B \\ C & 0 \end{bmatrix}\begin{bmatrix} (BC)^\pi - XA^2 & 0 \\ -CXA & D^\pi \end{bmatrix} \\ &= \frac{1}{4}\begin{bmatrix} -\frac{1}{2}BD_d^2CXA & 0 \\ -D_dCXA & 0 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} -\frac{1}{2}BD_d^3CXA^2 & 0 \\ -D_d^2CXA^2 & 0 \end{bmatrix} \\ &= -2^{-2}G(0,t) - 2^{-3}G(1,t). \end{aligned} \tag{3.18}$$

Hence, we can obtain, by (3.16) and (3.18),

$$\begin{aligned}\sum_{n=0}^{h-1} 2^{-(n+2)}Q_d^{n+1}P^nP^\pi &= \frac{1}{4}Q_dP^\pi + \frac{1}{8}Q_d^2PP^\pi + \sum_{n=2}^{h-1} 2^{-(n+2)}[G(n,n) - G(n,t)] \\ &= \sum_{n=2}^{h-1} 2^{-(n+2)}G(n,n) - \sum_{n=0}^{h-1} 2^{-(n+2)}G(n,t). \end{aligned} \tag{3.19}$$

In particular, when $h < 3$, the first sum $\sum\limits_{n=2}^{h-1} = 0$ in (3.19).

Note that by (3.5), $G(1,t) = G(3,t) = \cdots = G(2k-1,t) = \cdots$ and $G(0,t) = G(2,t) = \cdots = G(2k,t) = \cdots$. Thus

$$\begin{aligned}\sum_{n=0}^{h-1} 2^{-(n+2)}G(n,t) &= \sum_{k=0}^{\lceil \frac{h}{2} \rceil - 1} 2^{-(2k+2)}G(2k,t) + \sum_{k=0}^{\lfloor \frac{h}{2} \rfloor - 1} 2^{-(2k+3)}G(2k+1,t) \\ &= \frac{1}{3}(1 - 4^{-\lceil \frac{h}{2} \rceil})G(0,t) + \frac{1}{6}(1 - 4^{-\lfloor \frac{h}{2} \rfloor})G(1,t). \end{aligned} \tag{3.20}$$

Next, taking $W = P_d$, we have $W_1 = \begin{bmatrix} XA & 0 \\ 0 & 0 \end{bmatrix}$ and $W_2 = \begin{bmatrix} 0 & (BC)_dB \\ CX & 0 \end{bmatrix}$, and, accordingly, $Y_0 = XA, Y_1 = (BC)_dB$, and $Y_2 = CX$. Thus

$$(XA)^i = X(AX)^{i-1}A = XA_d^{i-1}A, \text{ for } i \geq 1,$$

and, by $AB = 0$ and Lemma 1.4,

$$Y_2Y_1 = CX(BC)_dB = C\left(\frac{1}{2}BC\right)_d(BC)_dB = \left(\frac{1}{2}CB\right)_d = D_d^2.$$

Note that

$$Q^\pi Q^n = \begin{bmatrix} I & -\frac{1}{2}BD_d \\ 0 & I-DD_d \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2}BD^{n-1} \\ 0 & D^n \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2}BD^{n-1}D^\pi \\ 0 & D^nD^\pi \end{bmatrix}.$$

Therefore, taking $H = Q^n$ in (3.8) for $n \geq 1$ yields

$$\begin{aligned} Q^\pi Q^n P_d^{n+1} &= Q^\pi Q^n W_2^{n+1} \\ &+ \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \begin{bmatrix} 0 & \frac{1}{2}BD^{n-1}D^\pi \\ 0 & D^nD^\pi \end{bmatrix}\begin{bmatrix} 0 & 0 \\ D_d^{2k-2}CX(XA)^{n-2k+2} & 0 \end{bmatrix} \\ &= Q^\pi Q^n W_2^{n+1} + \begin{bmatrix} \frac{1}{2}BD^{n-1}D^\pi CX^2AA_d^{n-1} & 0 \\ D^nD^\pi CX^2AA_d^{n-1} & 0 \end{bmatrix}. \end{aligned} \tag{3.21}$$

Consider $Q^\pi Q^n W_2^{n+1}$ as follows. When $n = 2k$,

$$Q^\pi Q^n W_2^{2k+1} = \begin{bmatrix} 0 & \frac{1}{2}BD^{n-1}D^\pi \\ 0 & D^nD^\pi \end{bmatrix}\begin{bmatrix} 0 & (Y_1Y_2)^kY_1 \\ D_d^{2k}Y_2 & 0 \end{bmatrix} = 0.$$

And when $n = 2k-1$,

$$Q^\pi Q^n W_2^{2k} = \begin{bmatrix} 0 & \frac{1}{2}BD^{n-1}D^\pi \\ 0 & D^nD^\pi \end{bmatrix}\begin{bmatrix} (Y_1Y_2)^k & 0 \\ 0 & D_d^{2k} \end{bmatrix} = 0.$$

Hence

$$Q^\pi Q^n W_2^{n+1} = 0. \tag{3.22}$$

By (3.10) and (3.11),

$$\begin{aligned} X &= (BC)^\pi A_d^2 + (BC)^\pi \sum_{k=1}^{r-1} \frac{1}{2}BD^{2k-2}CA_d^{2k+2} + \sum_{k=0}^{\lceil \frac{t}{2} \rceil -1} \frac{1}{2}BC(\frac{1}{2}BC)_d^{m+2}A^{2k}A^\pi \\ &= (BC)^\pi A_d^2 + \frac{1}{2}BD^\pi \sum_{k=1}^{r-1} D^{2k-2}CA_d^{2k+2} + \frac{1}{2}B\sum_{k=0}^{\lceil \frac{t}{2} \rceil -1} D_d^{2k+4}CA^{2k}A^\pi \\ &= (BC)^\pi A_d^2 + \frac{1}{2}BD^\pi \sum_{k=1}^{r-1} D^{2k-2}CA_d^{2k+2} + \frac{1}{2}BD_d^2S(t) \end{aligned}$$

and then

$$\begin{aligned} CX &= D^\pi CA_d^2 + D^\pi D^2 \sum_{k=1}^{r-1} D^{2k-2}CA_d^{2k+2} + D^2D_d^2S(t) \\ &= D^\pi \sum_{k=0}^{r-1} D^{2k}CA_d^{2k+2} + S(t) \\ &= K(1,r) + S(t), \end{aligned} \tag{3.23}$$

$$D^\pi CX = K(1,r), \tag{3.24}$$

$$\begin{aligned} XA &= (BC)^\pi A_d + \tfrac{1}{2}BD^\pi \textstyle\sum_{k=1}^{r-1} D^{2k-2}CA_d^{2k+1} + \tfrac{1}{2}BD_d^2S(t)A \\ &= \left(I - \tfrac{1}{2}BD_d^2C\right)A_d + \tfrac{1}{2}BD^\pi \textstyle\sum_{k=0}^{r-2} D^{2k}CA_d^{2k+3} + \tfrac{1}{2}BD_d^2S(t)A \\ &= \left(I - \tfrac{1}{2}BD_d^2C\right)A_d + \tfrac{1}{2}BK(1,r-1)A_d + \tfrac{1}{2}BD_d^2S(t)A, \end{aligned} \tag{3.25}$$

$$D^\pi CX^2A = K(1,r)XA = K(1,r)A_d. \tag{3.26}$$

By Lemma 1.4, (3.23) and (3.24),

$$\begin{aligned} Q^\pi P_d &= \begin{bmatrix} I & -\tfrac{1}{2}BD_d \\ 0 & D^\pi \end{bmatrix} \begin{bmatrix} XA & (BC)_dB \\ CX & 0 \end{bmatrix} \\ &= \begin{bmatrix} XA - \tfrac{1}{2}BD_dCX & (BC)_dB \\ D^\pi CX & 0 \end{bmatrix} \\ &= \begin{bmatrix} XA - \tfrac{1}{2}BD_dS(t) & \dfrac{1}{2}BD_d^2 \\ K(1,r) & 0 \end{bmatrix}, \end{aligned} \tag{3.27}$$

and, by (3.21), (3.22) and (3.26)

$$Q^\pi Q^n P_d^{n+1} = \begin{bmatrix} \tfrac{1}{2}BD^{n-1}K(1,r)A_dA_d^{n-1} & 0 \\ D^nK(1,r)A_dA_d^{n-1} & 0 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix}. \tag{3.28}$$

Consequently, by (3.27), (3.28),(3.23) and (3.25),

$$\begin{aligned} &\frac{1}{2}P_d + Q^\pi \sum_{n=0}^{q-1} 2^{n-1}Q^nP_d^{n+1} \\ &= \frac{1}{2}\begin{bmatrix} XA & \dfrac{1}{2}BD_d^2 \\ CX & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} XA - \dfrac{1}{2}BD_dS(t) & \dfrac{1}{2}BD_d^2 \\ K(1,r) & 0 \end{bmatrix} \\ &+ \sum_{n=1}^{q-1} 2^{n-1} \begin{bmatrix} \dfrac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} L & BD_d^2 \\ 2K(1,r)+S(t) & 0 \end{bmatrix} + \sum_{n=1}^{q-1} 2^{n-1} \begin{bmatrix} \tfrac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix} \end{aligned} \tag{3.29}$$

where

$$L = (2I - BD_d^2C)A_d + BK(1,r-1)A_d + BD_d^2S(t)A - \frac{1}{2}BD_dS(t).$$

Especially, when $q = 1$, the sum $\sum\limits_{n=1}^{q-1} = 0$ in (3.29).

Putting (3.17), (3.19), (3.20), (3.22) and (3.29) in (3.9) yields (3.1).

If $h = s+1$, then (3.9) becomes

$$M_d = \frac{1}{2}P_d + Q^\pi \sum_{n=0}^{q-1} 2^{n-1} Q^n P_d^{n+1} + \sum_{n=0}^{s-1} 2^{-(n+2)} Q_d^{n+1} P^n P^\pi.$$

So, putting (3.19), (3.20), (3.22) and (3.29) in the above equation yields (3.2). ∎

Adding some restrictions to D, we have the following results.

Corollary 3.3 *Let M be given by* (1.1). *If $D^2 = CB = 0$ and $AB = 0$, then*

$$M_d = \begin{bmatrix} A_d + BCA_d^3 + BDCA_d^4 & 0 \\ CA_d^2 + DCA_d^3 & 0 \end{bmatrix} \tag{3.30}$$

Proof. Assume that indices of matrices mentioned are the same as those in Theorem 3.1. Since $D^2 = 0$, then $D_d = 0$. Putting $D_d = 0$ in (3.3) $\sim$ (3.6), we have

$$\begin{array}{rclrcl} S(n) & = & 0, & K(1,m) & = & CA_d^2, \\ G(n,m) & = & 0, & K(2,m) & = & DCA_d^3, \\ L & = & 2A_d + BCA_d^3, & K(n,m) & = & 0 \ (n \geq 3). \end{array}$$

Substituting the above equations in (3.1) yields (3.30). ∎

Corollary 3.4 *Let M be given by* (1.1) *with $t = \mathrm{Ind}(A)$, $s = \mathrm{Ind}\left(\begin{bmatrix} A & \frac{1}{2}B \\ C & 0 \end{bmatrix}\right)$, $q = \mathrm{Ind}\left(\begin{bmatrix} 0 & \frac{1}{2}B \\ 0 & D \end{bmatrix}\right)$ and $h = \mathrm{Ind}\left(\begin{bmatrix} A & 0 \\ C & -D \end{bmatrix}\right)$. If $D^2 = D = \frac{1}{2}CB$ and $AB = 0$, then $h \leq s+1$ and*

$$\begin{aligned} M_d = {} & \frac{1}{2}\begin{bmatrix} H & BD \\ 2(I-D)CA_d^2 + S(t) & 0 \end{bmatrix} + 2^{-(h+1)}[G(h-1,h-1) - G(h-1,t)] \\ & -\frac{1}{3}(1-4^{-\lceil \frac{h}{2} \rceil})G(0,t) - \frac{1}{6}(1-4^{-\lfloor \frac{h}{2} \rfloor})G(1,t) + \sum_{n=2}^{h-1} 2^{-(n+2)}G(n,n). \end{aligned} \tag{3.31}$$

In particular, when $h = s+1$,

$$\begin{aligned} M_d & = \frac{1}{2}\begin{bmatrix} H & BD \\ 2(I-D)CA_d^2 + S(t) & 0 \end{bmatrix} - \frac{1}{3}(1-4^{-\lceil \frac{s}{2} \rceil})G(0,t) \\ & -\frac{1}{6}(1-4^{-\lfloor \frac{s}{2} \rfloor})G(1,t) + \sum_{n=2}^{s-1} 2^{-(n+2)}G(n,n) \end{aligned} \tag{3.32}$$

where

$$H = (2I - BDC)A_d + qB(I-D)CA_d^3 + BS(t)A - \frac{1}{2}BS(t),$$

$$S(n) = \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} DCA^{2k}A^{\pi},$$

$$G(n,m) = \begin{cases} \begin{bmatrix} \frac{1}{2}BS(m)A & 0 \\ S(m)A & 0 \end{bmatrix}, & n \text{ is even;} \\ \\ \begin{bmatrix} \frac{1}{2}BS(m-2)A^2 & 0 \\ S(m-2)A^2 & 0 \end{bmatrix}, & n \text{ is odd.} \end{cases}$$

Proof. Since $D = D^2$, then $\mathrm{Ind}(D) = 1$ and $D_d = D$ and then $K(n,m) = 0$, $n \geq 2$, and $K(1,m) = (I-D)CA_d^2$. Clearly,

$$S(n) = \sum_{k=0}^{\lceil \frac{n}{2} \rceil - 1} DCA^{2k}A^{\pi}, \tag{3.33}$$

$$G(2k,m) = \begin{bmatrix} \frac{1}{2}BS(m)A & 0 \\ S(m)A & 0 \end{bmatrix}; \; G(2k-1,m) = \begin{bmatrix} \frac{1}{2}BS(m-2)A^2 & 0 \\ S(m-2)A^2 & 0 \end{bmatrix}, \tag{3.34}$$

$$L = (2I - BDC)A_d + B(I-D)CA_d^3 + BS(t)A - \frac{1}{2}BS(t). \tag{3.35}$$

By [12, Theorem 2.1], $1 \leq q \leq 2$. Thus when $q = 2$,

$$\sum_{n=1}^{q-1} 2^{n-1} \begin{bmatrix} \frac{1}{2}BK(n,r)A_d & 0 \\ K(n+1,r) & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}B(I-D)CA_d^3 & 0 \\ 0 & 0 \end{bmatrix}.$$

However when $q = 1$, $\sum_{n=1}^{q-1} = 0$. Therefore, Define

$$\begin{aligned} H &= L + (q-1)B(I-D)CA_d^3 \\ &= (2I - BDC)A_d + qB(I-D)CA_d^3 + BS(t)A - \frac{1}{2}BS(t). \end{aligned} \tag{3.36}$$

Putting (3.33) $\sim$ (3.36) in (3.1) and (3.2), respectively, yields (3.31) and (3.32). ■

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