A NOTE ON DIMENSION OF WEAK HYPERVECTOR SPACES

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Abstract. In this paper, we study the dimension of weak hypervector spaces. First, we define linearly dependence and independence of vectors and also basis of weak hypervector spaces and then prove some results in this field. Finally, we consider weak subhypervector spaces of such spaces.

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1. Introduction

The concept of hyperstructure was first introduced by Marty [4] in 1934 and has attracted attention of many authors in last decades and has constructed some other structures such as hyperrings, hypergroups, hypermodules, hyperfields, and hypervector spaces. These constructions have been applied to many disciplines such as geometry, hypergraphs, binary relations, combinatorics, codes, cryptography, probability, and etc. A wealth of applications of this concepts is given in [2], [3], [10] and [11].

In 1988, the concept of hypervector space was first introduced by Scafati-Tallini. She studied more properties of this new structure in [7] and [8].

In the present paper, we want to state the concepts of linearly dependence and independence, basis, dimension of weak hypervector spaces and prove some results in this field. In [1], the definitions of these concepts are given and using them some results for anti hypervector spaces are proved. In the mentioned paper, some hypervector spaces have been introduced *dimensionless* that means that the hypervector spaces don't have any collection of linearly independent vectors. Since, in [9], it was proved that a normed weak right or left antidistributive hypervector space is a normed classical vector space, we would like to introduce different definitions of mentioned concepts for weak hypervector spaces that are more universal-such that don't create the dimensionless spaces. This paper is arranged as follows. In Section 3, we identify a certain element of weak hypervector spaces and then introduce a certain class of such spaces that we call them by *normal* and show the defined coordinate of any element in finite dimensional normal hypervector spaces is unique. In Section 4, we consider weak subhypervector spaces and prove some results.

2. Linear dependence and independence of vectors

Definition 2.1. [9] Let $(X, +)$ be an abelian group and F be a field. Then a weak hypervector space is a quadruple $(X, +, o, F)$, where o is a mapping

$$o : F \times X \longrightarrow P_*(X)$$

such that the following conditions are satisfied:

(i) $\forall \mathrm{a} \in \mathrm{F}, \forall \mathrm{x}, \mathrm{y} \in \mathrm{X},\ [\mathrm{ao}(\mathrm{x}+\mathrm{y})] \cap [\mathrm{aox} + \mathrm{aoy}] \neq \emptyset,$

(ii) $\forall \mathrm{a}, \mathrm{b} \in \mathrm{F}, \forall \mathrm{x} \in \mathrm{X},\ [(\mathrm{a}+\mathrm{b})\mathrm{ox}] \cap [\mathrm{aox} + \mathrm{box}] \neq \emptyset,$

(iii) $\forall \mathrm{a}, \mathrm{b} \in \mathrm{F}, \forall \mathrm{x} \in \mathrm{X},\ \mathrm{ao}(\mathrm{box}) = (\mathrm{ab})\mathrm{ox},$

(iv) $\forall \mathrm{a} \in \mathrm{F}, \forall \mathrm{x} \in \mathrm{X},\ \mathrm{ao}(-\mathrm{x}) = (-\mathrm{a})\mathrm{ox} = -(\mathrm{aox}),$

(v) $\forall \mathrm{x} \in \mathrm{X},\ \mathrm{x} \in 1\mathrm{ox}.$

We call (i) and (ii) *weak right* and *left distributive* laws, respectively. Note that the set $ao(box)$ in (3) is of the form $\bigcup_{y \in box} aoy$.

In the following we give some examples of weak hypervector spaces.

Example 2.2. The set $\mathbb{R}^2$ with usual sum and the following scalar product is a weak hypervector space on $\mathbb{R}$.

$$aox = \begin{cases} \overline{Ox} & x \neq 0, \\ \{0\} & x = 0, \end{cases}$$

where $\overline{Ox}$ is the line passing through the origin and the point x.

Example 2.3. The set $\mathbb{R}^2$ with usual sum and the following scalar product is a weak hypervector space on $\mathbb{R}$.

$$aox = \begin{cases} \text{segment } -Ox & x \neq 0 \wedge a < 0 \\ \text{segment } Ox & x \neq 0 \wedge a > 0 \\ \{0\} & x = 0 \vee a = 0 \end{cases},$$

where the segment Ox and the segment $-Ox$ are the closed segments connecting the origin to the point x and $-x$, respectively.

Example 2.4. The set $\mathbb{C}$ with usual sum and the following scalar product is a weak hypervector space on $\mathbb{R}$.

$$aox = \begin{cases} \{re^{i\theta} : 0 \leq r \leq |a||x|, \theta = \arg(x)\} & x \neq 0, \\ \{0\} & x = 0. \end{cases}$$

Example 2.5. The set $\mathbb{C}$ with usual sum and the following scalar product is a weak hypervector space on $\mathbb{R}$.

$$aox = \begin{cases} \{re^{i\theta} : 0 \leq r \leq |a||x|, 0 \leq \theta \leq 2\pi\} & x \neq 0, \\ \{0\} & x = 0. \end{cases}$$

Lemma 2.6. *If X is a weak hypervector space over F, $0 \neq a \in F$ and $x \in X$, then there exists a z in aox such that we have $x \in a^{-1}oz$.*

Proof. Since a is nonzero, from $x \in 1ox$ we obtain $x \in a^{-1}o(aox)$. So there exists a z in aox such that $x \in a^{-1}oz$. ∎

By the above lemma we have the following definition.

Definition 2.7. Let X is a weak hypervector space over F, $a \in F$ and $x \in X$. Essential point of aox, that we denote it by e_{aox}, for $a \neq 0$ is the element of aox such that $x \in a^{-1}oe_{aox}$. For $a = 0$, we define $e_{aox} = 0$.

Remark 2.8. Note that e_{aox} is not unique, necessarily. Hence we denote the set of all these elements by E_{aox}. When in this note we use e_{aox} in an equation, we intend is any element of E_{aox}. See the examples listed above. In Examples 2.3 and 2.4, E_{aox} is equals to the singleton sets

$$\begin{cases} \{-x\} & x \neq 0 \wedge a < 0, \\ \{x\} & x \neq 0 \wedge a > 0, \\ \{0\} & x = 0 \vee a = 0, \end{cases} \quad \text{and} \quad \begin{cases} \{|a|x\} & x \neq 0, \\ \{0\} & x = 0, \end{cases}$$

respectively, but in Examples 2.2 and 2.5, E_{aox} is equals to the sets

$$\begin{cases} \overline{Ox} & x \neq 0, \\ \{0\} & x = 0, \end{cases} \quad \text{and} \quad \begin{cases} \{re^{i\theta} : r = |a||x|, 0 \leq \theta \leq 2\pi\} & x \neq 0, \\ \{0\} & x = 0, \end{cases}$$

respectively.

Definition 2.9. A subset $M = \{x_1, ..., x_n\}$ of X is said to be linearly independent if the equation $0 = \sum_{i=1}^{n} e_{\alpha_i o x_i}$ implies that $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$, where $\alpha_1, ..., \alpha_n$ are scalars. M is said to be linearly dependent if M isn't linearly independent.

An arbitrary subset M of X is linearly independent if every nonempty finite subset of M is linearly independent.

Definition 2.10. A subset M of X is said to be a basis of X if M is linearly independent and spans the elements of X. It means that for any x of X there exists scalars $\alpha_1, ..., \alpha_n$ such that $x = \sum_{i=1}^{n} e_{\alpha_i o x_i}$, where $\{x_1, ..., x_n\}$ is a subset of M. If there exists a finite basis for X, then X is said to be a finite dimensional weak hypervector space.

Definition 2.11. If X is a finite dimensional weak hypervector space, then an ordered basis of X is a finite sequence of vectors that is a basis of X.

Definition 2.12. Let $\{x_1, ..., x_n\}$ be an ordered basis of X and $x \in X$ such that $x = \sum_{i=1}^{n} e_{\alpha_i o x_i}$, where $\alpha_1, ..., \alpha_n$ are scalars. Then $(\alpha_1, ..., \alpha_n)$ is said to be the coordinate of x.

Remark 2.13. By Remark 2.8, it is clear that the coordinate of an element isn't unique, necessarily.

In the follow we introduce a special class of weak hypervector spaces.

Definition 2.14. Let X be a weak hypervector space over F with the following properties

(i) $(\mathrm{E_{a_1ox} + E_{a_2ox}) \cap E_{(a_1+a_2)ox} \neq \emptyset,\ \forall x \in X,\ \forall a_1, a_2 \in F}$,

(ii) $(\mathrm{E_{aox_1} + E_{aox_2}) \cap E_{ao(x_1+x_2)} \neq \emptyset,\ \forall x_1, x_2 \in X,\ \forall a \in F}$.

Then X is called normal weak hypervector space.

Lemma 2.15. *Let X is a weak hypervector space over F, $a, b \in F$ and $x \in X$. Then the following properties hold.*

(i) $x \in E_{1ox}$

(ii) *If $b \neq 0$, then $aoe_{box} = abox$*

(iii) $E_{-aox} = -E_{aox}$

(iv) *If $a \neq 0$, then there exists an $y \in X$ such that $x \in E_{aoy}$.*

(v) *If X is normal, then E_{aox} is singleton.*

Proof. (i) This is obtained from $x \in 1ox$, immediately.

(ii) Since $b \neq 0$, by Definition 2.7 we have $e_{box} \in box$ and $x \in b^{-1}oe_{box}$. The relation $e_{box} \in box$ together with

$$abox = ao(box) = \bigcup_{y \in box} aoy$$

yields that $aoe_{box} \subseteq abox$. Also the relation $x \in b^{-1}oe_{box}$ together with

$$aoe_{box} = abo(b^{-1}oe_{box}) = \bigcup_{y \in b^{-1}oe_{box}} aboy$$

yields that $abox \subseteq aoe_{box}$. Therefore we obtain $aoe_{box} = abox$.

(iii) By part (iv) of Definition 2.1 we have

$$\begin{aligned} E_{-aox} &= \{z; z \in -aox, x \in (-a)^{-1}oz\} \\ &= \{z; -z \in aox, x \in a^{-1}o(-z)\} \\ &= \{-z; z \in aox, x \in a^{-1}oz\} \\ &= -\{z; z \in aox, x \in a^{-1}oz\} = -E_{aox}. \end{aligned}$$

(iv) By setting $y = e_{a^{-1}ox}$, the assertion follows from the parts (i) and (ii).

(v) By normality of X we have $(E_{-aox} + E_{aox}) \cap E_{(-a+a)ox} \neq \emptyset$. Since by Definition 2.7, $E_{(-a+a)ox} = E_{0ox}$ is equal to zero, so we obtain $E_{-aox} + E_{aox} = 0$. Thus the assertion follows from this and part (iii). ∎

Remark 2.16. By part (v) of preceding lemma we can replace E by e in Definition 2.14.

Lemma 2.17. *Let X be a weak hypervector space over F. X is normal if and only if*

$$e_{a_1ox} + e_{a_2ox} = e_{(a_1+a_2)ox}, \ \forall x \in X, \ \forall a_1, a_2 \in F,$$
$$e_{aox_1} + e_{aox_2} = e_{ao(x_1+x_2)}, \ \forall x_1, x_2 \in X, \ \forall a \in F.$$

Proof. It is clear by part (v) of Lemma 2.10 and Remark 2.11. ∎

Remark 2.18. In general, the reverse of part (v) of Lemma 2.15 is not true. As mentioned in Remark 2.8, for any $x \in \mathbb{R}^2$ and $a \in \mathbb{R}$, E_{aox} in Examples 2.3 is singleton, but $\mathbb{R}^2$ is not normal, because for any positive real number a and b we have $E_{box} = E_{aox} = \{x\}$ and so $e_{aox} + e_{box} = \{x\} + \{x\} = \{2x\}$ and $e_{(a+b)ox} = \{x\}$. Therefore,

$$(e_{aox} + e_{box}) \cap e_{(a+b)ox} = \emptyset.$$

Theorem 2.19. *Let X be a normal weak hypervector space. If $\{x_1, ..., x_m\}$ is a basis for X, then every linear independent set of X has at most m elements.*

Proof. Let $\{x_1, ..., x_m\}$ be a basis for X and $\{y_1, ..., y_n\}$ be a linear independent set of X. Thus for any $1 \leq j \leq n$ there exist $\alpha_{1j}, \alpha_{2j}, ..., \alpha_{mj} \in F$ such that $y_j = \sum\limits_{i=1}^{m} e_{\alpha_{ij}ox_i}$. Let $c_1, ..., c_n \in F$. By Lemmas 2.15 and 2.17, we have

$$\begin{aligned} e_{c_1oy_1} + ... + e_{c_noy_n} &= e_{c_1o\sum_{i=1}^m e_{\alpha_{i1}ox_i}} + ... + e_{c_no\sum_{i=1}^m e_{\alpha_{in}ox_i}} \\ &= \sum_{i=1}^{m} e_{c_1o(e_{\alpha_{i1}ox_i})} + ... + \sum_{i=1}^{m} e_{c_no(e_{\alpha_{in}ox_i})} \\ &= \sum_{i=1}^{m} e_{c_1\alpha_{i1}ox_i} + ... + \sum_{i=1}^{m} e_{c_1\alpha_{in}ox_i} \\ &= \sum_{j=1}^{n} e_{c_j\alpha_{1j}ox_1} + ... + \sum_{j=1}^{n} e_{c_j\alpha_{mj}ox_m} \\ &= e_{(\sum_{j=1}^n c_j\alpha_{1j})ox_1} + ... + e_{(\sum_{j=1}^n c_j\alpha_{mj})ox_m}. \end{aligned}$$

If $e_{c_1oy_1} + ... + e_{c_noy_n} = 0$, then $c_1 = ... = c_n = 0$. Since $\{x_1, ..., x_m\}$ is linear independent, by above relation we obtain

$$\sum_{j=1}^{n} c_j\alpha_{1j} = ... = \sum_{j=1}^{n} c_j\alpha_{mj} = 0.$$

Assume, on the contrary, that $n > m$. The remain of proof is the same proof of this lemma in the classical vector space. With the same reason, we can conclude that there exist at least a nonzero c_j and this contradiction completes the proof. ∎

Corollary 2.20. *Let X be a normal and finite dimensional weak hypervector space. Then any two basis of X have equal numbers of elements.*

Proof. It is clear by Theorem 2.19. ∎

Definition 2.21. Let X be a normal and finite dimensional weak hypervector space. The dimension of X is defined the numbers of the elements of the basis of X.

Lemma 2.22. *If X is a finite dimensional normal weak hypervector space, then the coordinate of any element of X is unique.*

Proof. It is clear by part (v) of Lemma 2.15. ∎

Now, we are able to identify the dimension of defined weak hypervector spaces in Examples 2.4 and 2.5.

Example 2.23. We show the dimension of defined weak hypervector space in Example 2.4 is 2. Let i and j be the unit vectors in direction of x-axis and y-axis, respectively. We show $\{i, j\}$ is linearly independent. So let a and b be real such that $e_{aoi} + e_{boj} = 0$. This implies $|a| + |b| = 0$ and then $a = b = 0$. Now, let $x = re^{i\theta}$ be an arbitrary element of C. So, we can write $x = e_{rcos\theta oi} + e_{rsin\theta oj}$. We proved that C is spanned by a linearly independent set that has 2 elements, so its dimension is 2.

Example 2.24. In this example we show the dimension of defined weak hypervector space in Example 2.5 is 1. We show $\{i\}$ is linearly independent. So, let a be real such that $e_{aoi} = 0$. It means that the elements on the circle with radius $|a|$ and center origin are zero. So we obtain $a = 0$.

If $x = re^{i\theta}$ be an arbitrary element of C, then $x = e_{roi}$. Therefore, C is spanned by $\{i\}$, hence its dimension is 1.

3. Weak subhypervector spaces

Definition 3.1. Let X be a weak hypervector space over F. A nonempty subset M of X is called a weak subhypervector space of X, when M satisfies the following properties:

(i) $x + y \in M,\ \forall x, y \in M$,

(ii) $e_{aox} \in M,\ \forall a \in F,\ \forall x \in M$.

Theorem 3.2. *Let X be a weak hypervector space over F and M be a nonempty subset of X. M is a weak subhypervector space of X if and only if for all $a \in F$ and $x, y \in M$ we have $e_{aox} + y \in M$.*

Proof. The necessity part is obvious. For the converse, let $x, y \in M$ and $a \in F$. Since, by assumption, $E_{1ox} + y \subseteq M$ and, by part (i) of Lemma 3.15, $x \in E_{1ox}$ we obtain $x+y \in M$. By part (iii) of Lemma 3.15 since $E_{-1ox} = -E_{1ox}$ so $-x \in E_{-1ox}$ and hence by $E_{-1ox} + x \subseteq M$ we obtain $0 \in M$ and thus $e_{aox} + 0 \in M$. So the proof is completed. ∎

Theorem 3.3. *Let X be a normal weak hypervector space over F and $\emptyset \neq S \subseteq X$. Then the following set is the smallest weak subhypervector space of X containing S:*

$$[S] = \left\{ \sum_{i=1}^{n} e_{a_i o s_i} ; a_i \in F, s_i \in S, n \in \mathbb{N} \right\}.$$

Proof. For any $s \in S$ by part (i) of Lemma 3.15, $s \in E_{1oS}$. So $S \subseteq [S]$. Let $x, y \in [S]$. So, $x = \sum_{i=1}^{n} e_{a_i o s_i}$ and $y = \sum_{j=1}^{m} e_{b_j o t_j}$, for $a_i, b_j \in F$ and $s_i, t_j \in S$. Let $n < m$. We obtain $x + y = \sum_{i=1}^{n+m} e_{c_i o u_i}$ such that, for $1 \leq i \leq n$, $c_i = a_i$, $u_i = s_i$ and, for $n+1 \leq i \leq m+n$, $c_i = b_{i-n}$, $u_i = t_{i-n}$. So, $x + y \in [S]$. Since X is normal, we have $e_{bo\sum_{i=1}^{n} e_{a_i o s_i}} = \sum_{i=1}^{n} e_{boz_{a_i o s_i}}$. By part (ii) of Lemma 3.15, this implies $e_{bo\sum_{i=1}^{n} e_{a_i o s_i}} = \sum_{i=1}^{n} e_{ba_i o s_i}$. Hence, we obtain $e_{bo\sum_{i=1}^{n} e_{a_i o s_i}} \in S$ and then $[S]$ is a weak subhypervector space of X containing S. Now, let M be a weak subhypervector space of X containing S and $x \in S$. So, $x = \sum_{i=1}^{n} e_{a_i o s_i}$, for $a_i \in F$ and $s_i \in S$. Since $S \subseteq M$, we have $e_{a_i o s_i} \in M$ and then $x = \sum_{i=1}^{n} e_{a_i o s_i} \in M$. Thus $[S] \subseteq M$ and so the proof is completed. ∎

Remark 3.4. If $\{M_i\}_{i \in I}$ be a collection of weak subhypervector spaces of X, then $M = \bigcap_{i \in I} M_i$ is a weak subhypervector space, clearly.

It is clear that $[S]$ is equal to intersection of all weak subhypervector spaces of X containing S. We say that $[S]$ is the spanned weak subhypervector space by S.

Proposition 3.5. *Let X be a normal weak hypervector space over F. Then X with the same defined sum and the following scalar product is a classical vector space:*

$$ax = e_{aox},$$

for all $a \in F$ and $x \in X$.

Proof. By part (v) of Lemma 3.115, this defined scalar product is well-defined. Checking the properties of a vector space for X is easy. ∎

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