THE MODIFIED (w/g)-EXPANSION METHOD AND ITS APPLICATIONS FOR SOLVING THE MODIFIED GENERALIZED VAKHNENKO EQUATION

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Abstract. The modified (w/g)-expansion method for finding traveling wave solutions of nonlinear evolution equations is presented in this paper, which can be thought of as the generalization of the well-known (G'/G)-expansion method given recently by Wang et al. When the w and g are taken special choices, some familiar expansion methods can be obtained. Based on these interesting results, we further give two new forms of expansions via the modified (g'/g^2)-expansion method and the modified (g') expansion method. In order to well illustrate the effectiveness of these two modified expansion methods, they are applied to a modified generalized Vakhnenko equation.

Keywords: the modified (w/g)-expansion method; modified (g'/g^2)-expansion method; modified (g')-expansion method; nonlinear evolution equations; traveling wave solutions; a modified generalized Vakhnenko equation.

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1. Introduction

The investigation of the exact traveling wave solutions of nonlinear evolution equations plays an important role in the study of soliton theory. In recent years, a variety of powerful methods have been applied for constructing exact traveling and solitary wave solutions of nonlinear evolution equations, such as the tanh-function expansion method [1]–[3], Jacobi elliptic function method [4], [5], Exp-function method [6], [7], the F-function expansion method [8], the inverse scattering method [9], Hirota bilinear transformation [10], the Backlund transform method [11], the (G'/G)-expansion method [12], [13], the sine-cosine method [6], [14], the modified simple equation method [15]–[18], the multiple exp-function method [19], [20], the transformed rational function method [21]–[23], the local fractional variation iteration method [24], the local fractional series expansion method [25], the (w/g)-expansion method [26] and so on.

The objective of this paper is to introduce the modified (w/g)-expansion method for finding exact solutions for a modified generalized Vakhnenko equation in mathematical physics. The proposed method is based on the assumption that these exact solutions can be expressed by a polynomial in $(w/g)^i$, $i = 0, 1, ..., m$ and that w, g satisfy the following auxiliary equation

$$(1.1)\quad \left(\frac{w}{g}\right)' = a + b\left(\frac{w}{g}\right) + c\left(\frac{w}{g}\right)^2,$$

namely

$$(1.2)\quad w'g - wg' = ag^2 + bwg + cw^2,$$

where a, b, c are arbitrary constants, while $' = d/d\xi$ and $\xi = k(x - Vt)$, k and V are constants. The degree m of this polynomial can be obtained by considering the homogeneous balance between the highest-order derivatives and the nonlinear terms appearing in the given nonlinear evolution equations. The coefficients of this polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the proposed method. The rest of this paper is organized as follows. In Section 2, we describe a new modified (w/g)-expansion method. In Section 3, we apply this method to solve the modified generalized Vakhnenko equation. In Section 4, Physical explanations of some obtained solutions are obtained. In Section 5, some conclusions are given.

2. Description of a new modified (w/g)- expansion method

For a given nonlinear evolution equation

$$(2.1)\quad P(u, u_t, u_x, u_{xt}, u_{tt}, u_{xx}, ...) = 0,$$

where P is a polynomial in the function $u(x,t)$ and its partial derivatives in which the nonlinear terms are involving. We use the wave transformation

$$(2.2)\quad u(x,t) = u(\xi), \quad \xi = k(x - Vt),$$

where k, V are constants to reduce equation (2.1) into the following ODE:

$$(2.3)\quad Q\left(u^{(r)}, u^{(r+1)}, ...\right) = 0,$$

where $u^{(r)} = \frac{d^r}{d\xi^r}, r \geq 0$, and r is the least order derivatives in the given equation. Setting $u^{(r)} = \upsilon(\xi)$, where $\upsilon(\xi)$ is a new function of ξ, then equation (2.3) reduces to the following ODE:

(2.4) $F(\upsilon, \upsilon', \upsilon'', ...) = 0,$

where $' = d/d\xi$. We further introduce the formal solution of equation (2.4) in the following ansatz:

$$(2.5)\ \ u^{(r)}(\xi) = \upsilon(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{w}{g}\right)^i,$$

where w and g satisfy equation (1.2) and $\alpha_i (i = 0, 1, ..., m)$ are constants to be determined later, while $r \geq 0$. To determine $\upsilon(\xi)$ explicitly, we take the following four steps:

Step 1. Determine the positive integer m in equation (2.5) by balancing the highest-order derivatives and the nonlinear terms in equation (2.4).

Step 2. Substitute (2.5) into (2.4) and collect all terms with the same powers of $(w/g)^i$, $(i = 0, 1, ..., m)$, together, thus the left-hand side of equation (2.4) is converted into a polynomial in $(w/g)^i$. Then set each coefficient of this polynomial to zero, to derive a set of algebraic equations for α_i, k, V.

Step 3. Solve these algebraic equations by the use of Maple or Mathematica to find the values of α_i, k, V.

Step 4. Use the results obtained in the above steps to derive a series of fundamental solutions $\upsilon(\xi)$ of equation (2.4) depending on (w/g). Then we can obtain the exact solutions of equation (2.1) by integrating each of the obtained fundamental solutions $\upsilon(\xi)$ with respect to ξ and r times as follows:

$$(2.6)\ \ u(\xi) = \int^{\xi} \int^{\xi_r} ... \int^{\xi_2} \upsilon(\xi_1) d\xi_1 ... d\xi_{r-1} d\xi_r + \sum_{j=1}^{r} d_j \xi^{r-j},$$

where $d_j (j = 1, ..., r)$ are arbitrary constants.

Remark 1. Let us now examine equation (1.2) carefully as follows:

(1) If we choose $w = g',\ a = -\mu,\ b = -\lambda,\ c = -1$ and $r = 0$, then $u(\xi)$ can be expressed as:

$$(2.7)\ \ u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{g'}{g}\right)^i,$$

where $g(\xi)$ satisfies the linear ODE:

(2.8) $g'' + \lambda g' + \mu g = 0.$

This is just the well-known (G'/G)-expansion method proposed by Wang et al. [12].

(2) If $w = \tanh\xi, g = 1, a = 1, b = 0, c = -1$ and $r = 0$, then

$$(2.9)\quad u(\xi) = \sum_{i=0}^{m} \alpha_i \tanh^i(\xi),$$

which is the well-known tanh-function expansion method, see [1]–[3], [6].

(3) If $w = g'/g, b = 0$ and $r = 0$, then we have the new expansion

$$(2.10)\quad u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{g'}{g^2}\right)^i,$$

where $g(\xi)$ satisfies the nonlinear ODE:

$$(2.11)\quad g''g^2 - 2g(g')^2 = ag^4 + c(g')^2,$$

which is called (g'/g^2)-expansion method and proposed in [26].

(4) If $w = gg'$ and $r = 0$, then we have the new expansion

$$(2.12)\quad u(\xi) = \sum_{i=0}^{m} \alpha_i (g')^i,$$

where $g(\xi)$ satisfies the nonlinear ODE:

$$(2.13)\quad g'' = a + bg' + c(g')^2,$$

which is called g'-expansion method and proposed in [26]. Li et al. [26] have applied the two expansions (2.10) and (2.12) for finding exact solutions to the Vakhnenko equation

$$u_{tx} + u_x^2 + uu_{xx} + u = 0.$$

Remark 2. where $r \geq 1$, the ansatz (2.5) is new and is not reported in [26]. So, in the next section, we apply the two expansions (g'/g^2)-expansion method and the (g')-expansion method using (2.5) with $r \geq 1$ to find new exact solutions of a modified generalized Vakhnenko equation.

3. New solutions of a modified generalized Vakhnenko equation

Consider a modified generalized Vakhnenko equation (mGVE) [27]–[29]:

$$(3.1)\quad \frac{\partial}{\partial x}\left(\wp^2 u + \frac{1}{2}pu^2 + \beta u\right) + q\wp u = 0, \quad \wp = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x},$$

where p, q, β are arbitrary non-zero constants. equation (3.1) can be traced to the well-known Vakhnenko equation (VE) which was initially presented to model high-frequent wave motion in a relaxing medium [29]. Recently, equation (3.1) has been discussed using the (G'/G)-expansion method [27] and using the auxiliary equation method [28]. To calculate the exact solutions for equation (3.1), a sensible step is to transform variables. We introduce two new independent variables X and T defined by:

$$(3.2)\quad x = T + \int_{-\infty}^{X} U(X', T)dX' + x_0, \quad t = X,$$

where $u(x, t) = U(X, T)$ and x_0 is a constant. We introduce a new function W defined by

$$(3.3)\quad W(X, T) = \int_{-\infty}^{X} U(X', T)dX'.$$

Then

$$(3.4)\quad W_X(X, T) = U(X, T), \quad , W_T(X, T) = \int_{-\infty}^{X} U_T(X', T)dX'.$$

It is easy to see that

$$(3.5)\quad \frac{\partial}{\partial T} = \frac{\partial}{\partial x}\frac{\partial x}{\partial T} + \frac{\partial}{\partial t}\frac{\partial t}{\partial T}, \qquad \frac{\partial}{\partial X} = \frac{\partial}{\partial x}\frac{\partial x}{\partial X} + \frac{\partial}{\partial t}\frac{\partial t}{\partial X}.$$

From (3.2) and (3.5) we have

$$(3.6)\quad \frac{\partial}{\partial T} = (1 + W_T)\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial X} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x},$$

and hence $\wp u = U_x$, $\wp^2 u = U_{XX}$. Now, equation (3.1) reduces to

$$(3.7)\quad W_{XXXT} + pW_X W_{XT} + q(1 + W_T)\, W_{XX} + \beta\, W_{XT} = 0.$$

Assume that $W(X, T) = W(\xi)$, where ξ is given by (2.2), then equation (3.7) reduces to the equation

$$(3.8)\quad k^2 V W''' + \frac{1}{2}(p + q)kV(W')^2 + (\beta\, V - q)\, W' = 0,$$

with zero constants of integration. Setting $r = 1$ and $W' = v$, we have $W(\xi) = \int v(\xi)d\xi + d_1$, where $v(\xi)$ satisfies the following ODE:

$$(3.9)\quad k^2 V v'' + \frac{1}{2}(p + q)kV v^2 + (\beta\, V - q)v = 0.$$

Recently, equation (3.9) has been solved in [27] using the (G'/G)-expansion method via the expansion (2.7). Let us now solve equation (3.9) using the expansion (2.10) and (2.12).

3.1. The new modified (g'/g^2) expansion method

Balancing υ'' with υ^2 in equation (3.9) we get $m = 2$. Consequently, the expansion (2.10) reduces to

$$(3.10)\quad \upsilon(\xi) = \alpha_2\left(\frac{g'}{g^2}\right)^2 + \alpha_1\left(\frac{g'}{g^2}\right) + \alpha_0,\ \ \alpha_2 \neq 0.$$

where $g(\xi)$ satisfies equation (2.11). It is easy to see that

$$(3.11)\quad \upsilon'(\xi) = 2\alpha_2 c\left(\frac{g'}{g^2}\right)^3 + \alpha_1 c\left(\frac{g'}{g^2}\right)^2 + 2\alpha_2 a\left(\frac{g'}{g^2}\right) + \alpha_1 a,$$

$$(3.12)\quad \upsilon''(\xi) = 6\alpha_2 c^2\left(\frac{g'}{g^2}\right)^4 + 2\alpha_1 c^2\left(\frac{g'}{g^2}\right)^3 + 8\alpha_2 ac\left(\frac{g'}{g^2}\right)^2 + 2\alpha_1 ac\left(\frac{g'}{g^2}\right) + 2\alpha_2 a^2.$$

Substituting (3.10)–(3.12) into (3.9) and collecting all terms with the same powers of $(g'/g^2)^i$, $i = 0, 1, 2, 3, 4$ together, the left hand side of equation (3.9) is converted into a polynomial in $(g'/g^2)^i$. Setting each coefficient of this polynomial to zero, we get the following algebraic equations:

$$(3.13)\quad 0 : 2\alpha_2 a^2 k^2 V + \frac{1}{2}(p+q)kV\alpha_0^2 + (V\beta - q)\alpha_0 = 0,$$

$$(3.14)\quad 1 : 2\alpha_1 ack^2 V + \alpha_0\alpha_1(p+q)kV + (V\beta - q)\alpha_1 = 0,$$

$$(3.15)\quad 2 : 8\alpha_2 ack^2 V + \frac{1}{2}(p+q)kV\alpha_1^2 + \alpha_0\alpha_2(p+q)kV + (V\beta - q)\alpha_2 = 0,$$

$$(3.16)\quad 3 : 2\alpha_1 c^2 k^2 V + \alpha_1\alpha_2(p+q)kV = 0,$$

$$(3.17)\quad 4 : 6\alpha_2 c^2 k^2 V + \frac{1}{2}(p+q)kV\alpha_2^2 = 0.$$

Solving the above algebraic equations, we have the following results:

Case 1.

$$(3.18)\quad \alpha_2 = -\frac{12c^2k}{p+q},\ \ \alpha_1 = 0,\ \ \alpha_0 = -\frac{12ack}{p+q},\ \ V = \frac{q}{\beta - 4ack^2},\ \beta \neq 4ack^2$$

Case 2.

$$(3.19)\quad \alpha_2 = -\frac{12c^2k}{p+q},\ \ \alpha_1 = 0,\ \ \alpha_0 = -\frac{4ack}{p+q},\ \ V = \frac{q}{\beta + 4ack^2},\ \beta \neq -4ack^2$$

where $p+q \neq 0$. It is well known [26], the solution of equation (2.11) is given, and hence g'/g^2 has the form:

(1) If $ac > 0$, then

$$(3.20)\quad \frac{g'}{g^2} = \sqrt{\frac{a}{c}}\left[\frac{c_1\cos(\sqrt{ac}\,\xi) + c_2\sin(\sqrt{ac}\,\xi)}{c_1\sin(\sqrt{ac}\,\xi) - c_2\cos(\sqrt{ac}\,\xi)}\right],$$

(2) If $ac < 0$, then

$$(3.21)\quad \frac{g'}{g^2} = \frac{1}{2c}\left[2\sqrt{|ac|} - \frac{4c_1\sqrt{|ac|}\exp\left(2\xi\sqrt{|ac|}\right)}{c_1\exp\left(2\xi\sqrt{|ac|}\right) - c_2}\right],$$

(3) If $a = 0$, $c \neq 0$, then

$$(3.22)\quad \frac{g'}{g^2} = \frac{-\,c_1}{c\,(c_1\xi + c_2)},$$

where c_1 and c_2 are arbitrary constants. Now, we have the following exact solutions of equation (3.9)

(i) If $ac > 0$, then for case1, we have

$$(3.23)\quad v(\xi) = \frac{-12kac}{p+q}\left[\frac{c_1\cos(\sqrt{ac}\,\xi) + c_2\sin(\sqrt{ac}\,\xi)}{c_1\sin(\sqrt{ac}\,\xi) - c_2\cos(\sqrt{ac}\,\xi)}\right]^2 - \frac{12kac}{p+q},$$

and consequently, we get

$$(3.24)\quad W(\xi) = \frac{-12kac}{p+q}\int\left[\frac{c_1\cos(\sqrt{ac}\,\xi) + c_2\sin(\sqrt{ac}\,\xi)}{c_1\sin(\sqrt{ac}\,\xi) - c_2\cos(\sqrt{ac}\,\xi)}\right]^2 d\xi - \frac{12kac}{p+q}\xi + d_1,$$

where d_1 is a constant of integration. Simplifying (3.24), we get two values of $W(\xi)$ as follows:

The first value is

$$(3.25)\quad W_1(\xi) = \frac{-12k\sqrt{ac}}{p+q}\tan\left(\xi_1 + \sqrt{ac}\,\xi\right) + d_1,$$

where $\xi_1 = \tan^{-1}\left(\frac{c_1}{c_2}\right)$, $c_1^2 + c_2^2 \neq 0$, and ξ is given by

$$(3.26)\quad \xi = k(X - VT) = k\left(t - \frac{qT}{\beta - 4ack^2}\right), \quad T = x - W_1(\xi) - x_0.$$

In this case, equation (3.1) has the general periodic solution

$$(3.27)\quad u_1(x,t) = W_{1x}(\xi) = \frac{-12k^2ac}{p+q}\sec^2\left(\xi_1 + \sqrt{ac}\,\xi\right).$$

The second value is

$$(3.28)\quad W_2(\xi) = \frac{-12k\sqrt{ac}}{p+q}\cot\left(\xi_2 - \sqrt{ac}\,\xi\right) + d_1,$$

where $\xi_2 = \cot^{-1}\left(\frac{c_1}{c_2}\right)$, $c_1^2 + c_2^2 \neq 0$, and ξ is given by (3.26), while T has the form

(3.29) $T = x - W_2(\xi) - x_0.$

In this case, equation (3.1) has the general periodic solution

(3.30) $u_2(x,t) = W_{2x}(\xi) = \dfrac{-12k^2ac}{p+q}\csc^2\left(\xi_2 - \sqrt{ac}\,\xi\right),$

(ii) If $a\,c < 0$, then for case 1, we have

(3.31) $v(\xi) = \dfrac{-12k}{p+q}\left[\sqrt{|ac|} - \dfrac{2c_1\sqrt{|ac|}\exp\left(2\xi\sqrt{|ac|}\right)}{c_1\exp\left(2\xi\sqrt{|ac|}\right) - c_2}\right]^2 - \dfrac{12kac}{p+q},$

and, consequently, we get

(3.32)

$$W(\xi) = \frac{-12k\,|ac|}{p+q}\int\left[\frac{c_1\exp\left(\xi\sqrt{|ac|}\right) + c_2\exp\left(-\xi\sqrt{|ac|}\right)}{c_1\exp\left(\xi\sqrt{|ac|}\right) - c_2\exp\left(-\xi\sqrt{|ac|}\right)}\right]^2 d\xi - \frac{12kac}{p+q}\xi + d_1.$$

Setting $c_1 = \frac{1}{2}(A+B)$, $c_2 = \frac{-1}{2}(A-B)$, $\phi = \xi\sqrt{|ac|}$, where A and B are constants, then (3.32) reduces to

(3.33) $W(\xi) = \dfrac{-12k\,|ac|}{p+q}\displaystyle\int\left[\frac{A\sinh\phi + B\cosh\phi}{A\cosh\phi + B\sinh\phi}\right]^2 d\xi + \dfrac{12k\,|ac|}{p+q}\xi + d_1.$

Using the formulas (8), (10), (12) and (14) obtained in Peng [30], we have respectively the following general forms of soliton solutions:

(1) If$|A| > |B|$, then

(3.34) $W_1(\xi) = \dfrac{12k}{p+q}\sqrt{|a\,c|}\tanh[\phi + sgn(AB)\psi_1] + d_1,$

and, consequently, the solution of equation (3.1) has the form:

(3.35) $u_1(x,t) = W_{1x}(\xi) = \dfrac{12k^2\,|a\,c|}{p+q}\sec h^2[\phi + sgn(AB)\psi_1],$

where

(3.36) $\xi = k(X - VT) = k\left(t - \dfrac{qT}{\beta - 4ack^2}\right), \quad T = x - W_1(\xi) - x_0,$

and$\psi_1 = \tanh^{-1}\left(\frac{|B|}{|A|}\right)$, $sgn(AB)$ is the sign function. Note that the solution (3.35) is in agreement with the obtained in [27].

(2) If $|B| > |A| \neq 0$, then

$$(3.37)\ \ W_2(\xi) = \frac{12k\sqrt{|a\,c|}}{p+q}\coth[\phi + sgn(AB)\psi_2] + d_1,$$

and consequently, the solution of equation (3.1) has the form

$$(3.38)\ \ u_2(x,t) = W_{2x}(\xi) = \frac{-12k^2\,|a\,c|}{p+q}\csc h^2[\phi + sgn(AB)\psi_2],$$

where

$$(3.39)\ \ \xi = k(X - VT) = k\left(t - \frac{qT}{\beta - 4ack^2}\right),\quad T = x - W_2(\xi) - x_0,$$

and $\psi_2 = \coth^{-1}\left(\frac{|B|}{|A|}\right)$.

(3) If $|B| > |A| = 0$, then

$$(3.40)\ \ W_3(\xi) = \frac{12k\sqrt{|a\,c|}}{p+q}\coth\phi + d_1,$$

and, consequently, the solution of equation (3.1) has the form

$$(3.41)\ \ u_3(x,t) = W_{3x}(\xi) = \frac{-12k^2\,|ac|}{p+q}\csc h^2\phi,$$

where

$$(3.42)\ \ \xi = k(X - VT) = k\left(t - \frac{qT}{\beta - 4ack^2}\right),\quad T = x - W_3(\xi) - x_0,$$

(4) If $|A| = |B|$, then

$$(3.43)\ \ W_4(\xi) = d_1,$$

and, consequently, the solution of equation (3.1) is the zero solution

$$(3.44)\ \ u_4(x,t) = W_{4x}(\xi) = 0,$$

where

$$(3.45)\ \ \xi = k(X - VT) = k\left(t - \frac{qT}{\beta - 4ack^2}\right),\quad T = x - W_4(\xi) - x_0.$$

(iii) If $a = 0$, $c \neq 0$, then for case 1, we have

$$(3.46)\ \ \upsilon(\xi) = \frac{-12k}{p+q}\left(\frac{c_1}{c_1\xi + c_2}\right)^2,$$

and, consequently, we get

$$W_5(\xi) = \frac{12k}{p+q}\left(\frac{c_1}{c_1\xi + c_2}\right) + d_1. \tag{3.47}$$

The solution of equation (3.1) in this case has the form:

$$u_5(x,t) = W_{5x}(\xi) = \frac{-12k^2}{(p+q)}\left(\frac{c_1^2}{(c_1\xi + c_2)^2}\right), \tag{3.48}$$

where

$$\xi = k(X - VT) = k\left(t - \frac{qT}{\beta}\right), \quad T = x - W_5(\xi) - x_0, \tag{3.49}$$

Similarly, we can find the exact solutions for case 2, which are omitted here for simplicity.

3.2. The new modified (g')-expansion method

Setting $m = 2$, in (2.12) we have the formal solution of equation (3.9) in the form

$$v(\xi) = \alpha_2 (g')^2 + \alpha_1 (g') + \alpha_0, \quad \alpha_2 \neq 0, \tag{3.50}$$

where g satisfies equation (2.13). It is easy to see that

$$v'(\xi) = 2\alpha_2 c (g')^3 + (2\alpha_2 b + \alpha_1 c)(g')^2 + (2\alpha_2 a + \alpha_1 b)(g') + \alpha_1 a, \tag{3.51}$$

$$\begin{aligned} v''(\xi) = {} & 6\alpha_2 c (g')^4 + (10\alpha_2 bc + 2\alpha_1 c^2)(g')^3 \\ & + (4\alpha_2 b^2 + 3\alpha_1 bc + 8\alpha_2 ac)(g')^2 \\ & + (6\alpha_2 ab + 2\alpha_1 ac + \alpha_1 b^2)(g') + (2\alpha_2 a^2 + \alpha_1 ab). \end{aligned} \tag{3.52}$$

Substituting (3.50)-(3.52) into (3.9) and collecting all terms with the same powers of $(g')^i$, $(i = 0, 1, 2, 3, 4)$ together, the left-hand side of equation (3.9) is converted into a polynomial in $(g')^i$. Setting each coefficient of this polynomial to zero, we get the following algebraic equations:

$$0: k^2 V\left(2\alpha_2 a^2 + \alpha_1 ab\right) + \frac{1}{2}(p+q)kV\alpha_0^2 + (\beta V - q)\alpha_0 = 0, \tag{3.53}$$

$$1: k^2 V\left(6\alpha_2 ab + 2\alpha_1 ac + \alpha_1 b^2\right) + \alpha_1\alpha_0(p+q)kV + (\beta V - q)\alpha_1 = 0, \tag{3.54}$$

$$\begin{aligned} 2: k^2 V\left(4\alpha_2 b^2 + 3\alpha_1 bc + 8\alpha_2 ac\right) + \tfrac{1}{2}(p+q)kV\alpha_1^2 + \\ \alpha_2\alpha_0(p+q)kV + (\beta V - q)\alpha_2 = 0, \end{aligned} \tag{3.55}$$

$$3: k^2 V(10\alpha_2 bc + 2\alpha_1 c^2) + \alpha_2\alpha_1(p+q)kV = 0, \tag{3.56}$$

$$4: 6k^2 V\alpha_2 c^2 + \frac{1}{2}(p+q)kV\alpha_2^2 = 0. \tag{3.57}$$

Solve these algebraic equations using the Maple or Mathematica we get the following cases:

Case 1.

$$\alpha_2 = \frac{-12kc^2}{p+q},\ \alpha_1 = \frac{-12kbc}{p+q},\ \alpha_0 = \frac{-2k}{p+q}\left(b^2+2ac\right),\ V = \frac{q}{4ack^2+\beta-k^2b^2}. \tag{3.58}$$

Case 2.

$$\alpha_2 = \frac{-12kc^2}{p+q},\ \alpha_1 = \frac{-12kbc}{p+q},\ \alpha_0 = \frac{-3kb^2}{p+q},\ b = -2\sqrt{ac},\ V = \frac{q}{\beta}. \tag{3.59}$$

Case 3.

$$\alpha_2 = \frac{-12kc^2}{p+q},\ \ \alpha_1 = 0,\ \ \alpha_0 = \frac{-12ack}{p+q},\ \ b = 0,\ \ V = \frac{q}{\beta - 4ack^2}. \tag{3.60}$$

Case 4.

$$\alpha_2 = \frac{-12kc^2}{p+q},\ \ \alpha_1 = 0,\ \ \alpha_0 = \frac{-4ack}{p+q},\ \ b = 0,\ \ V = \frac{q}{\beta + 4ack^2}. \tag{3.61}$$

It is well-known [26], the solution of equation (2.13) is given, and hence $g'(\xi)$ has the forms:

If $\Delta = 4ac - b^2 < 0$, then

$$g'(\xi) = \frac{1}{2c}\left[\sqrt{-\Delta}\tanh\left(\frac{-\xi}{2}\sqrt{-\Delta}\right) - b\right]. \tag{3.62}$$

If $\Delta = 4ac - b^2 > 0$, then

$$g'(\xi) = \frac{1}{2c}\left[\sqrt{\Delta}\tan\left(\frac{\xi}{2}\sqrt{\Delta}\right) - b\right]. \tag{3.63}$$

If $\Delta = 4ac - b^2 = 0$, then

$$g'(\xi) = \frac{-1}{c}\left(\frac{1}{\xi} + \frac{b}{2}\right). \tag{3.64}$$

Now, we deduce the following exact solutions of equation (3.9) as follows:

(i) If $\Delta < 0$, then for case 1, we have

$$\upsilon(\xi) = \frac{-3k\Delta}{p+q}\sec h^2\left(\frac{-\xi}{2}\sqrt{-\Delta}\right) + \frac{2k\Delta}{p+q}, \tag{3.65}$$

and consequently, we get

$$W_1(\xi) = \int \upsilon(\xi)d\xi + d_1, \tag{3.66}$$

which can be written as

$$W_1(\xi) = \frac{-6k\sqrt{-\Delta}}{p+q}\tanh\left(\frac{-\xi}{2}\sqrt{-\Delta}\right) + \frac{2k\Delta}{p+q}\xi + d_1. \tag{3.67}$$

Now, the solution of equation (3.1) in this case is the soliton solution:

$$(3.68)\quad u_1(x,t) = W_{1x}(\xi) = \frac{k^2\Delta}{p+q}\left[2 - 3\sec h^2\left(\frac{-\xi}{2}\sqrt{-\Delta}\right)\right],$$

where

$$(3.69)\quad \xi = k(X - VT) = k\left(t - \frac{qT}{4ack^2 + \beta - k^2b^2}\right),\ \ T = x - W_1(\xi) - x_0.$$

(ii) If $\Delta > 0$, then for case 1, we have

$$(3.70)\quad \upsilon(\xi) = \frac{-3k\Delta}{p+q}\sec^2\left(\frac{\xi}{2}\sqrt{\Delta}\right) + \frac{2k\Delta}{p+q},$$

and consequently, we get

$$(3.71)\quad W_2(\xi) = \int \upsilon(\xi)d\xi + d_1,$$

which can be written as

$$(3.72)\quad W_2(\xi) = \frac{-6k\sqrt{\Delta}}{p+q}\tan\left(\frac{\xi}{2}\sqrt{\Delta}\right) + \frac{2k\Delta}{p+q}\xi + d_1.$$

Now, the solution of equation (3.1) in this case is the soliton solution:

$$(3.73)\quad u_2(x,t) = W_{2x}(\xi) = \frac{k^2\Delta}{p+q}\left[2 - 3\sec^2\left(\frac{\xi}{2}\sqrt{\Delta}\right)\right],$$

where

$$(3.74)\quad \xi = k(X - VT) = k\left(t - \frac{qT}{4ack^2 + \beta - k^2b^2}\right),\ \ T = x - W_2(\xi) - x_0.$$

(iii) If $\Delta = 0$, then for case 1, we have

$$(3.75)\ \ \upsilon(\xi) = \left(\frac{-12k}{p+q}\right)\frac{1}{\xi^2},$$

and consequently, we get

$$(3.76)\quad W_3(\xi) = \int \upsilon(\xi)d\xi + d_1,$$

which can be written as

$$(3.77)\quad W_2(\xi) = \left(\frac{12k}{p+q}\right)\frac{1}{\xi} + d_1.$$

Now, the solution of equation (3.1) in this case is the soliton solution:

$$(3.78)\quad u_3(x,t) = W_{3x}(\xi) = \left(\frac{-12k^2}{p+q}\right)\frac{1}{\xi^2},$$

where

$$(3.79)\quad \xi = k(X - VT) = k\left(t - \frac{qT}{\beta}\right),\ \ T = x - W_3(\xi) - x_0.$$

Similarly, we can find exact solutions for the other cases, which are omitted here for simplicity.

4. Physical explanations of some obtained solutions

In this section, we will present some graphs for three types of solutions of equation (3.1) namely hyperbolic, trigonometric and rational function solutions by selecting some special values of the parameters in the exact solutions using the mathematical software Maple, which can be shown below in Figures 1–6. From these explicit solutions, we see that the results (3.27),(3.30) and (3.73) are periodic solutions, the results (3.35) and (3.68) are bell-shaped soliton solutions, the results (3.38) and (3.41) are singular bell-shaped soliton solutions while the results (3.48) and (3.78) are rational function solutions.

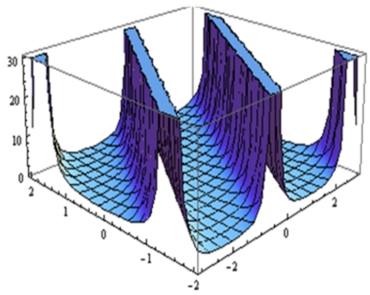

Fig1. The plot of the solution (3.27) when $k=a=c=1, p=q=-6, \beta=-2, \xi_1=0$.

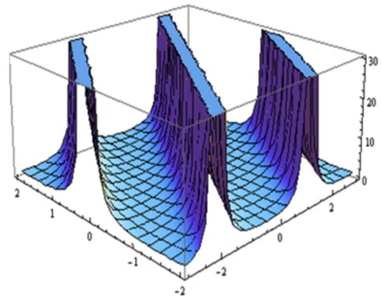

Fig2. The plot of the solution (3.30) when $k=a=c=1, p=q=-6, \beta=-2, \xi_2=0$.

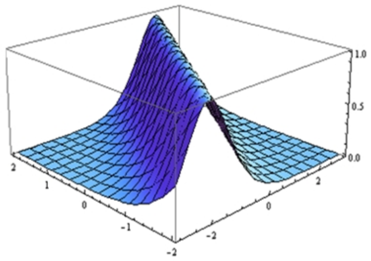

Fig3. The plot of the solution (3.35) when $k=c=1, a=-1, p=q=6, \beta=2, \psi_1=0$.

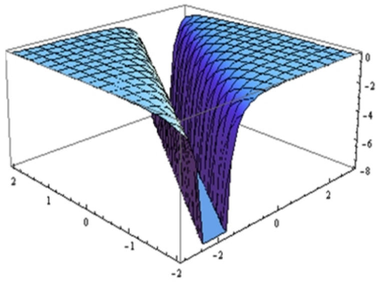

Fig4. The plot of the solution (3.38) when $k=c=1, a=-1, p=q=6, \beta=2, \psi_2=0$.

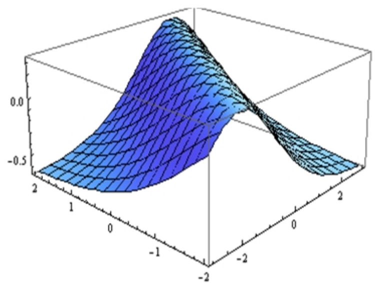

Fig5. The plot of the solution (3.68) when $k=p=1, q=2, \Delta=-1, \beta=3$.

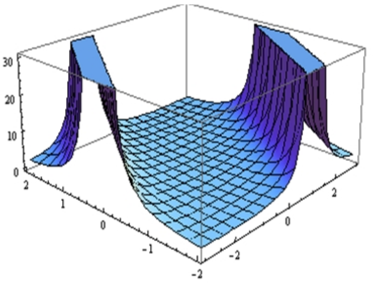

Fig6. The plot of the solution (3.73) when $k=p=1, q=2, \Delta=1, \beta=1$.

The plot of some solutions

5. Some conclusions

In this paper, we have introduced and presented the new modified (w/g)-expansion method when w and g are taken special values and then we proposed two expansion methods via the modified (g'/g^2)-expansion method and the modified (g')-expansion method. We have applied these two methods to find the exact solutions for the modified generalized Vakhnenko equation (3.1) such as the periodic, soliton and rational function solutions. Finally, with the aid of Maple or Mathematica we have assured the correctness of the obtained solutions by putting them back into the original equation (3.1).

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