INCLUSION RESULTS ON SUBCLASSES OF STARLIKE AND CONVEX FUNCTIONS ASSOCIATED WITH STRUVE FUNCTIONS

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Abstract. The present investigation our goal is to determine necessary and sufficient condition for Struve functions belonging to the classes $\mathcal{J}^*_\lambda(\alpha,\beta)$ and $\mathcal{G}^*_\lambda(\alpha,\beta)$.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$$

of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad z \in \mathbb{U}. \tag{1}$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and also univalent in $\mathbb{U}$. Denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions whose non-zero coefficients from second on, is given by

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$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \tag{2}$$

Also, for functions $f \in \mathcal{A}$ given by (1) and $g \in \mathcal{A}$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad z \in \mathbb{U}.$$

A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$, if and only if $\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{U})$. This function class is denoted by $\mathcal{S}^*(\alpha)$. We also write $\mathcal{S}^*(0) \equiv \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. A function $f \in \mathcal{A}$ is said to be convex of order α $(0 \leq \alpha < 1)$ if and only if $\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U})$. This class is denoted by $\mathcal{K}(\alpha)$. Further, $\mathcal{K}(0) = \mathcal{K}$, the well-known standard class of convex functions. It is an established fact that $f \in \mathcal{K}(\alpha) \Longleftrightarrow zf' \in \mathcal{S}^*(\alpha)$.

It is well known that the special functions (series) play an important role in geometric function theory, especially in the solution by de Branges [6] of the famous Bieberbach conjecture. There is an extensive literature dealing with geometric properties of different types of special functions, especially for the generalized, Gaussian hypergeometric functions [5], [7], [12] and the Bessel functions [1], [2], [3], [8].

We recall here the Struve function of order p (see [10], [15]), denoted by $\mathcal{H}_p$ is given by

$$\mathcal{H}_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+\frac{3}{2})\,\Gamma(p+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \forall z \in \mathbb{C} \tag{3}$$

which is the particular solution of the second order non-homogeneous differential equation

$$z^2\omega''(z) + z\omega'(z) + (z^2 - p^2)\omega(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} \tag{4}$$

where p is unrestricted real(or complex) number. The solution of the non-homogeneous differential equation

$$z^2\omega''(z) + z\omega'(z) - (z^2 + p^2)\omega(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{1}{2})} \tag{5}$$

is called the modified Struve function of order p and is defined by the formula

$$\mathcal{L}_p(z) = -ie^{-ip\pi/2}\mathcal{H}_p(iz) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\frac{3}{2})\ \Gamma(p+n+\frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \forall z \in \mathbb{C}.$$

Let the second order non-homogeneous linear differential equation [15] (also see [10] and references cited therein),

$$z^2\omega''(z) + bz\omega'(z) + [cz^2 - p^2 + (1-b)p]\omega(z) = \frac{4(z/2)^{p+1}}{\sqrt{\pi}\Gamma(p+\frac{b}{2})} \tag{6}$$

where $b, p, c \in \mathbb{C}$ which is natural generalization of Struve equation. It is of interest to note that when $b = c = 1$, then we get the Struve function (3) and for $c = -1, b = 1$ the modified Struve function (5). This permit us to study Struve and modified Struve functions. Now, denote by $w_{p,b,c}(z)$ the generalized Struve function of order p given by

$$w_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (c)^n}{\Gamma(n+\frac{3}{2})\ \Gamma(p+n+\frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \forall z \in \mathbb{C}$$

which is the particular solution of the differential equation (6).Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in $\mathbb{U}$. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z) = 2^p\sqrt{\pi}\Gamma\left(p+\frac{b+2}{2}\right)\ z^{\frac{-p-1}{2}}\ \omega_{p,b,c}\ (\sqrt{z}),\quad \sqrt{1}=1.$$

By using well known Pochhammer symbol (or the shifted factorial) defined, in terms of the familiar Gamma function, by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & (n=0), \\ a(a+1)(a+2)\cdots(a+n-1) & (n \in \mathbb{N} = \{1,2,3,\ldots\}) \end{cases}$$

we can express $u_{p,b,c}(z)$ as

$$\begin{aligned} u_{p,b,c}(z) &= \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(m)_n\ (3/2)_n}\ z^n \\ &= b_0 + b_1 z + b_2 z^2 + ... + b_n z^n + ..., \end{aligned}$$

where $m = \left(p + \frac{b+2}{2}\right) \neq 0, -1, -2, \ldots$. This function is analytic on $\mathbb{C}$ and satisfies the second-order inhomogeneous linear differential equation

$$4z^2u''(z) + 2(2p+b+3)zu'(z) + (cz+2p+b)u(z) = 2p+b.$$

For convenience, throughout in the sequel, we use the following notations

$$\begin{aligned} w_{p,b,c}(z) &= w_p(z), \\ u_{p,b,c}(z) &= u_p(z), \\ m &= p + \frac{b+2}{2} \end{aligned}$$

and for if $c < 0, m > 0 (m \neq 0, -1, -2, \dots)$ let,

$$zu_p(z) \;=\; z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} z^n = z + \sum_{n=2}^{\infty} b_{n-1} z^n$$

and

$$\Psi(z) = z(2 - u_p(z)) \;=\; z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} z^n \tag{7}$$

In this paper, we introduce two new subclasses of $\mathcal{S}$ namely $\mathcal{J}_\lambda(\alpha, \beta)$ and $\mathcal{G}_\lambda(\alpha, \beta)$ to discuss some inclusion properties.

For some α $(0 \leq \alpha < 1)$, λ $(0 \leq \lambda \leq 1)$, $\beta > 0$ and functions of the form (1), we let $\mathcal{J}_\lambda(\alpha, \beta)$ be the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re\left(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - \alpha\right) > \beta \left|\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\right|$$

and $\mathcal{G}_\lambda(\alpha, \beta)$ the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re\left(\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - \alpha\right) > \beta \left|\frac{zf'(z) + z^2 f''(z)}{(1-\lambda)z + \lambda z f'(z)} - 1\right|.$$

Also denote $\mathcal{J}^*_\lambda(\alpha, \beta) = \mathcal{J}_\lambda(\alpha, \beta) \cap \mathcal{T}$ and $\mathcal{G}^*_\lambda(\alpha, \beta) = \mathcal{G}_\lambda(\alpha, \beta) \cap \mathcal{T}$, the subclasses of $\mathcal{T}$.

Example 1 [4] For some $\alpha(0 \leq \alpha < 1)$, $\beta > 0$ and choosing $\lambda = 1$ and functions of the form (2), we let $\mathcal{TS}_\mathcal{P}(\alpha, \beta)$ be the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re\left(\frac{zf'(z)}{f(z)} - \alpha\right) > \beta \left|\frac{zf'(z)}{f(z)} - 1\right|$$

and $\mathcal{UCT}(\alpha, \beta)$ the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \alpha\right) > \beta \left|\frac{zf''(z)}{f'(z)}\right|.$$

Note that $\mathcal{TS}_\mathcal{P}(\alpha, 0) \equiv \mathcal{T}^*(\alpha, 0)$ and $\mathcal{UCT}(\alpha, 0) \equiv C(\alpha)$[11], further $\mathcal{TS}_\mathcal{P}(0, \beta) \equiv \mathcal{TS}_p(\beta)$ and $\mathcal{UCT}(0, \beta) \equiv UCT(\beta)$[13]

Example 2 For some $\alpha(0 \leq \alpha < 1)$, $\beta > 0$ and choosing $\lambda = 0$ and functions of the form (2), we let

(i) $\mathcal{USD}(\alpha, \beta)$ the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re\left(f'(z) - \alpha\right) > \beta \left|f'(z) - 1\right|$$

and

(ii) $\mathcal{UCD}(\alpha, \beta)$ the subclass of $\mathcal{S}$ of satisfying the analytic criteria

$$\Re\left((zf'(z))' - \alpha\right) > \beta \left|(zf'(z))' - 1\right|.$$

Suitably specializing the parameters we get the various subclasses studied in [9] and see the references cited therein.(also see [4], [14])

Recently, Yagmur and Orhan [15] (see [10]) have determined various sufficient conditions for the parameters p, b and c such that the functions $u_{p,b,c}(z)$ or $z \to zu_{p,b,c}(z)$ to be univalent, starlike, convex and close to convex in the open unit disk. Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see [5], [7], [12]) and by work of Baricz [1], [2], [3]. In this paper,we we obtain sufficient condition for function $h(z)$, given by

$$\begin{aligned} h_\mu(z) &= (1-\mu)zu_p(z) + \mu z{u_p}'(z) \\ &= z + \sum_{n=2}^{\infty}(1+n\mu-\mu)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}z^n. \end{aligned} \tag{8}$$

where $0 \leq \mu \leq 1$ in the present investigation our goal is to determine sufficient condition for function $h_\mu(z)$ belonging to the classes $\mathcal{J}_\lambda(\alpha, \beta)$ and $\mathcal{G}_\lambda(\alpha, \beta)$.

2. Main results and their consequences

We recall the following necessary and sufficient conditions for the functions $f \in \mathcal{J}^*_\lambda(\alpha, \beta)$, $f \in \mathcal{G}^*_\lambda(\alpha, \beta)$ and the subclasses stated in the Examples 1 and 2 which are relevant for our study.

Lemma 1 *A function $f(z)$ of the form* (1) *is in*

(i) *the class $\mathcal{J}_\lambda(\alpha, \beta)$ if*

$$\sum_{n=2}^{\infty}[n(1+\beta) - \lambda(\alpha+\beta)]|a_n| \leq 1-\alpha. \tag{9}$$

(ii) *the class $\mathcal{G}_\lambda(\alpha, \beta)$ if*

$$\sum_{n=2}^{\infty}n[n(1+\beta) - \lambda(\alpha+\beta)]|a_n| \leq 1-\alpha. \tag{10}$$

The above sufficient conditions are also necessary for functions f of the form (2).

Lemma 2 *A function $f(z)$ of the form* (2) *is in*

(i) *the class $\mathcal{TS}_\mathcal{P}(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty}[n(1+\beta) - (\alpha+\beta)]|a_n| \leq 1-\alpha.$$

(ii) *the class $\mathcal{UCT}(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty}n[n(1+\beta) - (\alpha+\beta)]|a_n| \leq 1-\alpha.$$

Lemma 3 *A function $f(z)$ of the form (2) is in*

(i) *the class $\mathcal{USD}(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(1+\beta)|a_n| \leq 1-\alpha.$$

(ii) *the class $\mathcal{UCD}(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} n^2(1+\beta)|a_n| \leq 1-\alpha.$$

Theorem 1 *If $c<0, m>0(m\neq 0,-1,-2,...$ then $h_\mu(z)\in\mathcal{J}_\lambda(\alpha,\beta)$ if*

$$\begin{aligned}\mu(1+\beta)u_p''(1) &+[(2\mu+1)(1+\beta)-\mu\lambda(\alpha+\beta)]u_p'(1)\\ &+[(1+\beta)-\lambda(\alpha+\beta)]u_p(1)\\ &\leq 2-\alpha(1+\lambda)+\beta(1-\lambda).\end{aligned} \tag{11}$$

Proof. Since $zu_p(z)=z+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}z^n$,

$$u_p(1)-1 = \sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}, \tag{12}$$

and differentiating $zu_p(z)$ with respect to z and taking $z=1$ we have

$$\begin{aligned}zu_p'(z)+u_p(z) &= 1+\sum_{n=2}^{\infty} n\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}z^{n-1}\\ u_p'(1)+u_p(1)-1 &= \sum_{n=2}^{\infty} n\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}.\end{aligned}$$

Further, differentiating $zu_p'(z)+u_p(z)$ with respect to z and taking $z=1$, we get

$$zu_p''(z)+2u_p'(z) = \sum_{n=2}^{\infty} n(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}z^{n-2}$$

$$u_p''(1)+2u_p'(1) = \sum_{n=2}^{\infty} n(n-1)\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}. \tag{13}$$

Since $h_\mu(z)\in\mathcal{J}_\lambda(\alpha,\beta)$, by virtue of Lemma 1 and (9) it suffices to show that

$$\sum_{n=2}^{\infty}(1+n\mu-\mu)[n(1+\beta)-\lambda(\alpha+\beta)]\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right)\leq 1-\alpha. \tag{14}$$

Now, let

$$S(n,\lambda,\beta,\alpha) = \sum_{n=2}^{\infty}(1+n\mu-\mu)[n(1+\beta)-\lambda(\alpha+\beta)]\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right)$$

$$\begin{aligned} S(n,\lambda,\beta,\alpha) &= \mu(1+\beta)\sum_{n=2}^{\infty} n^2\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right) \\ &+ [(1+\beta)(1-\mu)-\lambda\mu(\alpha+\beta)]\sum_{n=2}^{\infty} n\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right) \\ &- \lambda(\alpha+\beta)(1-\mu)\sum_{n=2}^{\infty}\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right). \end{aligned}$$

Writing $n^2 = n(n-1)+n$, we get

$$\begin{aligned} S(n,\lambda,\beta,\alpha) &= \mu(1+\beta)\sum_{n=2}^{\infty} n(n-1)\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right) \\ &+ [(1+\beta)-\lambda\mu(\alpha+\beta)]\sum_{n=2}^{\infty} n\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right) \\ &- \lambda(\alpha+\beta)(1-\mu)\ \sum_{n=2}^{\infty}\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right) \end{aligned}$$

From (12), (13), 13 and taking $z = 1$, we get

$$\begin{aligned} S(n,\lambda,\beta,\alpha) &\leq \mu(1+\beta)u_p''(1) + [(1+\beta)-\lambda\mu(\alpha+\beta)](u_p'(1)+u_p(1)-1) \\ &\quad -\lambda(\alpha+\beta)(1-\mu)(u_p(1)-1) \\ &= \mu(1+\beta)u_p''(1) + [(2\mu+1)(1+\beta)-\lambda\mu(\alpha+\beta)]u_p'(1) \\ &\quad +[(1+\beta)-\lambda(\alpha+\beta)(u_p(1)-1) \end{aligned}$$

But this expression is bounded above by $1-\alpha$ if (11) holds.

Thus, the proof is complete. ∎

Theorem 2 *If $c < 0, m > 0(m \neq 0,-1,-2,...$ then $zu_p(z) \in \mathcal{J}_\lambda(\alpha,\beta)$ if*

(15) $$(1+\beta)u_p'(1) + [(1+\beta)-\lambda(\alpha+\beta)]u_p(1) \leq 2-\alpha(1+\lambda)+\beta(1-\lambda).$$

Proof. By virtue of Lemma 1 of (9), it suffices to show that

$$\sum_{n=2}^{\infty}[n(1+\beta)-\lambda(\alpha+\beta)]\left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}}\right) \leq 1-\alpha.$$

Since $h_0(z) = zu_p(z)$, hence by taking $\mu = 0$ in (14) we get the above inequality. Hence by taking $\mu = 0$ in the Theorem 1, we get the desired result given in 15. ■

Theorem 3 *If $c < 0, m > 0(m \neq 0, -1, -2, ...$ then $zu_p(z) \in \mathcal{G}_\lambda(\alpha, \beta)$ if*

$$\begin{aligned} &(1+\beta)u_p''(1) + [3(1+\beta) - \lambda(\alpha+\beta)]u_p'(1) \\ &\quad + [(1+\beta) - \lambda(\alpha+\beta)]u_p(1) \\ &\quad \leq 2 - \beta(\lambda-1) - \alpha(\lambda+1). \end{aligned} \tag{16}$$

Proof. By virtue of Lemma 1 of (10), it suffices to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta) - \lambda(\alpha+\beta)] \left(\frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} \right) \leq 1 - \alpha.$$

By definition $zu_p(z) \in \mathcal{G}_\lambda(\alpha, \beta) \Leftrightarrow zu_p'(z) \in \mathcal{J}_\lambda(\alpha, \beta)$. That is by taking $\mu = 1$ we have $h_1(z) = zu_p'(z) \in \mathcal{J}_\lambda(\alpha, \beta)$, hence by taking $\mu = 1$ in the Theorem 1, we get the desired result given in 16. ■

Remark 1 The above conditions (11) and (16) are also necessary for functions $\Psi(z)$ given by (7) and of the form

$$\begin{aligned} h_\mu^*(z) &= (1-\mu)\Psi(z) + \mu\Psi'(z) \\ &= z - \sum_{n=2}^{\infty} (1 + n\mu - \mu) \frac{(-c/4)^{n-1}}{(m)_{n-1}\ (3/2)_{n-1}} z^n \end{aligned}$$

is in the classes $\mathcal{J}_\lambda^*(\alpha, \beta)$ and $\mathcal{G}_\lambda^*(\alpha, \beta)$ respectively.

Further, by taking $\lambda = 0(\text{or})\lambda = 1$ in Theorems 2 and 3, we state the following corollaries without proof.

Corollary 1 *If $c < 0, m > 0(m \neq 0, -1, -2, ...)$ then $z(2 - u_p(z))$,*

(i) *is in $\mathcal{TS}_\mathcal{P}(\alpha, \beta)$ if and only if*

$$(1+\beta)u_p'(1) + (1-\alpha)u_p(1) \leq 2(1-\alpha).$$

(ii) *is in $\mathcal{UCT}(\alpha, \beta)$ if and only if*

$$(1+\beta)u_p''(1) + (3 - 2\beta - \alpha)u_p'(1) + (1-\alpha)u_p(1) \leq 2(1-\alpha).$$

Corollary 2 *If* $c < 0, m > 0(m \neq 0, -1, -2, ...)$ *then* $z(2 - u_p(z))$,

(i) *is in* $\mathcal{USD}(\alpha, \beta)$ *if and only if*

$$(1+\beta)[u_p'(1) + u_p(1)] \leq 2 - \alpha + \beta.$$

(ii) *is in* $\mathcal{UCD}(\alpha, \beta)$ *if and only if*

$$(1+\beta)[u_p''(1) + 3u_p'(1) + u_p(1)] \leq 2 - \alpha + \beta.$$

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