A THEOREM ABOUT FINITE GROUPS WITH SPECIAL CONJUGACY CLASSES[1]

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Abstract. Let G be a finite group. G has the property that for any conjugacy class length, G has exactly two conjugacy classes having such length. The paper classifies all possible stucture of the finite group G under a condiction that G' is nilpotent.

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1. Introduction

In 1973, F. Mark ([1]) began to investigate the "S_3-conjecture", where S_3 is the only finite group whose conjugacy classes all have different length. Many authors also studied this problem in special cases ([3], [4]). In 1994, J.P. Zhang ([5]) proved the conjecture is true if the group is a finite solvable group. In 2004, C.M. Boner and M.B. Ward ([2]) investigated the similar problem that if G is a finite group with exactly two conjugacy classes of the same length, and G' is nilpotent, then $G \cong Z_2, D_{10}, A_4$. This paper continues this kind of problems. We will prove that if G is a finite group with the property P^*(P^* will be defined later), and G' is nilpotent, then $G \cong Z_2$, $G \cong Z_2 \times S_3$ or $G \cong Z_3 \rtimes Z_4$.

For the sake of convenience, we first give some notations. Let $x \in G$, we denote by $o(x)$ the order of x, x^G denotes the conjugacy class containing x, $|x^G|$ denotes the length of the conjugacy class of x, A char G means that A is a character subgroup of G. Let $\mathrm{Irr}(G)$ be the set of all irreducible characters of G, $\mathrm{Irr}^*(G)$ be the set of nonlinear irreducible characters of G. We call a finite group G having property P^* if G has property that for any conjugacy class length, G has exactly two conjugacy classes having such length. We assume that all groups in this paper are finite groups. All other symbols used in this paper are standard and the readers can consult [7] for more information.

We begin our work with the following lemmas.

2. Preliminaries

Lemma 2.1. ([5]) *Suppose that G is a finite solvable group. If the conjugacy classes of G all have different length, then $G \cong S_3$.*

In the following proof, we suppose that G is a finite group with property P^*, and G' is nilpotent.

Lemma 2.2. *$Z(G) \cong Z_2$ and G/G' is an Abelian 2-group.*

Proof. Because G contains exactly two conjugacy classes of length 1, and the conjugacy classes length of any elements of $Z(G)$ is 1, we have that $Z(G) \cong Z_2$. Let $z \in Z(G)$, $z \neq 1$, $x \in G$, and $o(x)$ is odd, then the length of conjugacy classes of x and zx are equal. $o(x)$ is odd implies $C_G(x) = C_G(x^2)$, which concludes that the conjugacy length of x and x^2 are equal too. Since G has property P^*, two elements among x, x^2, zx must be conjugate. The order of x and zx are different implies that x is conjugate to x^2. Therefore, there exists $g \in G$, such that $x^2 = x^g$.

So we have that $x = x^{-1}x^g = [x, g] \in G'$, from which we know that G/G' is an Abelian 2-group.

Lemma 2.3. *Let $P_2 \in Syl_2(G)$. If P_2 is non-abelian, then $G' = Z_2H, P_2' = Z_2$, where $Z_2 = Z(G)$, H is a $2'$-Hall normal subgroup of G.*

Proof. By Lemma 2.2, G/G' is an Abelian 2-group. Since G' is nilpotent, we can set $G' = L \times H$, where $L \leq P_2$. Then $P_2' \leq L \leq P_2$. Since L char $G' \lhd G$, we have that $L \lhd G$ and $L \cap Z(P_2) \neq \emptyset$. Also, since H char $G' \lhd G$, $H \lhd G$ and the elements of L can commute with the elements of H. Thus $Z(G) = Z_2 = L \cap Z(P_2) \leq L$.

Now, we prove $L = Z_2$. Otherwise, since $L \lhd G$, L consists of conjugacy classes of G, $|L| = |Z_2| + |a^G| + ..., +|b^G|$, where $a, b \in L$. Let $|a^G|$ be the minimal length of the conjugacy classes of non-unit elements of L. Clearly, the G-conjugacy classes length of L is power of 2. If $|a^G| \geq 4$,we have a contradiction immediately since $|Z_2| = 2$. Therefore, $|a^G| = 2$. But G has another conjugacy class of length 2, let it be x^G, that is $|x^G| = 2$. If $x \in L$, then $|x^G| + |a^G| = 4$. Since G has the property P^*, the other conjugacy classes length of L is at least 4, with the same reason we have a contradiction. If $x \notin L$, it is easy to know that $|x^G| = |(zx)^G|$ and $|a^G| = |(za)^G|$, where $z \in Z(G)$. G has property P^* implies that x is conjugate to zx, and a is conjugate to za. Because $|a^G| = 2$, we can conclude that for any element g of G, $a^g = az$ or $a^g = a$. With the same reason we have that $x^g = zx$, or $x^g = x$. Furthermore, it is easy to prove for any $g \in G$, $(ax)^g = ax$, or $(ax)^g = zax$. So $|(ax)^G| = 2$ or $ax \in Z(G)$. Obviously, $ax \notin Z(G)$. We assert that ax is not conjugate to a. Otherwise, there exists $g \in G$ such that $a = (ax)^g$, then $a = (ax)^g = ax$ or $a = (ax)^g = zax$, both of them are impossible. In the same way, we can prove that ax is not conjugate to x. Therefore, $(ax)^G$ is the third conjugacy class with length 2 except a^G, x^G, it contradicts to the fact that G has property P^*.

Therefore, $|L| = 2$, and $P_2' = L = Z_2$.

Lemma 2.4. *P_2 is an Abelian subgroup.*

Proof. Let $|P_2| = 2^n$, if P_2 is not an Abelian subgroup, by Lemma 2.3, $G' = Z_2H$, $P_2' = Z_2$. Let N is a minimal normal subgroup of G, and $N \leq H$. Since $H \leq G'$, and G' is nilpotent, N is a p-group and $N \leq Z(H)$. Let $M = P_2N$. We consider a quotient group $\bar{M} = M/Z_2 = \bar{P}_2\bar{N} \leq \bar{G} = G/Z_2$, where $\bar{P}_2 = P_2/Z_2$, $\bar{N} = NZ_2/Z_2$. Clearly, $\bar{N}$ is a minimal normal subgroup of $\bar{M}$. If $\bar{M} = M/Z_2$ is Abelian, then for any $x \in P_2$, $y \in N$, such that $y^{-1}y^x \in Z_2 \cap N = 1$ since $N \lhd G$. Therefore $xy = yx$ for any $x \in P_2$, $y \in N$. Since $N \leq Z(H)$, we have $N \leq Z(G) = Z_2$, a contradiction. Therefore, $\bar{M} = M/Z_2$ is non- Abelian.

$\bar{P}_2$ is Abelian and $\bar{N}$ is a minimal normal subgroup of $\bar{M}$ impliy that $\bar{M}' = \bar{N}$. Let $\chi \in Irr(\bar{M})$, $\chi(1) > 1$. Since $\bar{N}$ is an Abelian normal subgroup of $\bar{M}$, we have $\chi(1) \,|\, |\bar{M} : \bar{N}| = 2^{n-1}$(Ref [6]). Also, since $\bar{P}_2$ is Abelian, for any $a \in \bar{P}_2$, $(|a^{\bar{M}}|, \chi(1)) = 1$. Therefore, $\chi(a) = 0$, or $\rho(a) = \omega I$(Ref[6]), where ρ is an irreducible representation of $\bar{M}$ which affords χ, I is a $\chi(1) \times \chi(1)$ identity matrix.

If $\rho(a) = \omega I$, since $\bar{M}' = \bar{N}$, for any $g \in \bar{N}$, $a^{-1}a^g = h \in \bar{N}$. Therefore, $\rho(a) = \rho(a)\rho(h)$, and $\rho(h) = I$ follows. So we have that $h \in ker\chi$.

(i) if $h \neq 1$, then $1 < \bar{N} \cap ker\chi \lhd \bar{M}$. $\bar{N}$ is a minimal normal subgroup of $\bar{M}$, implies that $\bar{N} \leq ker\chi$. It means that χ may be viewed as a irreducible character of $\bar{M}/\bar{N} \cong \bar{P}_2$. But $\bar{P}_2$ is Abelian, hence χ is one degree irreducible character, it contradicts to $\chi(1) > 1$.

(ii) if $h = 1$, then $a \in C_{\bar{G}}(\bar{N})$. Let $C_{\bar{G}}(\bar{N}) = \bar{K} \times \bar{H}(\ \bar{K} \leq \bar{P}_2)$, then $\bar{K}$ char $C_{\bar{G}}(\bar{N}) \lhd \bar{G}$, we have $\bar{K} \lhd \bar{G}$. So $KZ_2 \lhd G$. The elements of H and KZ_2 can commute since H is $2'$-Hall normal subgroup of G. Thus any $x \in KZ_2$, $|x^G|$ is power of 2. By class equation, $2^s = |KZ_2| = 2 + 2^{n_1} + ..., +2^{n_i}$, $n_1 \leq n_2 \leq ..., \leq n_i$. if $n_1 \geq 2$, we have a contradiction immediately. If $n_1 = 1$, then KZ_2 contains an element u with $|u^G| = 2$. So G contains another element v with $|v^G| = 2$ since G has property P^*. It is easy to see that $H \leq C_G(v)$ and $v \in KZ_2$. Hence $2^s = |KZ_2| = 2 + 2 + 2 + 2^{n_3} + ..., +2^{n_i}$, $2 \leq n_3 \leq ..., \leq n_i$, a contradiction.

Therefore, for any $a \in \bar{P}_2$, $a \neq 1$, $\chi(a) = 0$.

Now, we restrict χ on $\bar{P}_2$, we have

$$(\chi, \chi)_{\bar{P}_2} = \frac{1}{|\bar{P}_2|} \sum_{a \in \bar{P}_2} \chi(a)\chi(a^{-1}) = \frac{\chi^2(1)}{|\bar{P}_2|}.$$

Therefore, $\chi^2(1) = |\bar{P}_2|(\chi, \chi)_{\bar{P}_2} = 2^{n-1}(\chi, \chi)_{\bar{P}_2}$. Let $\chi = n_1\theta_1 + n_2\theta_2 + ... + n_t\theta_t$, where $\theta_i \in Irr(\bar{P}_2)$ and $n_i \geq 1$. Since $\theta_i(1) = 1$, $\chi(1) = n_1 + n_2 + ... + n_t$, by orthogonality relation of irreducible character, $(\chi, \chi)_{\bar{P}_2} = n_1^2 + n_2^2 + ... + n_t^2 \geq n_1 + n_2 + ... + n_t = \chi(1)$, that is $\chi^2(1) = 2^{n-1}(\chi, \chi)_{\bar{P}_2} \geq 2^{n-1}\chi(1)$. Thus, we can conclude that $\chi(1) \geq 2^{n-1}$. Since $\chi(1)$ is power of 2, we have $(\chi, \chi)_{\bar{P}_2}$ is power of 2.

$\bar{M}' = \bar{N}$ means that the number of one degree irreducible characters of $\bar{M}'$ is $|\bar{M}/\bar{M}'| = |\bar{P}_2| = 2^{n-1}$. Let $|N| = p^m$, then

$$2^{n-1}p^m = |\bar{M}| = |\bar{M}/\bar{M}'| + \sum_{\chi \in Irr^*(\bar{M})} \chi^2(1) = 2^{n-1} + 2^{n-1} \sum_{\chi \in Irr^*(\bar{M})} (\chi, \chi)_{\bar{P}_2}.$$

Therefore, $p^m = 1 + \sum_{\chi \in Irr^*(\bar{M})} (\chi, \chi)_{\bar{P}_2}$. Since $(\chi, \chi)_{\bar{P}_2} \geq \chi(1) \geq 2^{n-1}$ and $(\chi, \chi)_{\bar{P}_2}$ is power of 2, $p^m = 1 + 2^{n-1} \sum_i 2^{n_i}$. Hence $2^{n-1} \,|\, p^m - 1$.

We have known before that $N \lhd G$, and $N \leq Z(H)$. So for any $x \in N$, $|x^G|$ is power of 2.

Clearly, $|x^G| = |(zx)^G|$, they are not conjugate because $o(x) \neq o(zx)$. Since $x \in N$ and $zx \notin N$ and G has property P^*, the distinct conjugacy classes of elements of N have distinct length.

So $p^m = |N| = 1 + |x_1^G| + |x_2^G| + ... + |x_s^G| = 1 + 2^{m_1} + 2^{m_2} + ... + 2^{m_s}$, $m_1 < m_2 < ... < m_s$. Because $2^{n-1} \,|\, p^m - 1$ and $|P_2| = |\bar{P}_2||Z_2| = 2^n$, we have $s = 1$, $m_1 = n - 1$.That is $p^m - 1 = 2^{n-1}$. But $N = 1 \cup x^G$, it implies that $\bar{P}_2$ acts fixed-point-free on N by conjugacy. By [7] 7.24, $\bar{P}_2$ is a cyclic group or generalized quaternion group. Since $\bar{P}_2$ is Abelian, $\bar{P}_2 = P_2/Z(G)$ is a a cyclic group, which implies that P_2 is Abelian. It contradicts to our hypothesis of P_2 non-Abelian.

In fact, we have proven that P_2 is an Abelian subgroup.

3. Main theorem

Theorem 3.1. *Suppose that G is a finite group with property P^*, and G' is nilpotent, then $G \cong Z_2$, $G \cong Z_2 \times S_3$, or $G \cong Z_3 \rtimes Z_4$.*

Proof. Since G has property P^*, $Z(G) = Z_2 = \langle z \rangle$, $o(z) = 2$. Clearly, $G \cong Z_2$, if G is Abelian.

If G is not Abelian, let $\sum_1 = \{x^G | x \in G, zx \in x^G\}$, $\sum_2 = \{x^G | x \in G, zx \notin x^G\}$. Then $G = \sum_1 \cup \sum_2$. Clearly, $x^G \in \sum_2$ if and only if $(zx)^G \in \sum_2$. Let $\sum_2 = \sum_{21} \cup \sum_{22}$, where $\sum_{21} = \{x_1^G, x_2^G, ..., x_l^G\}$, $\sum_{22} = \{(zx_1)^G, (zx_2)^G, ..., (zx_l)^G\}$. Because $|x_i^G| = |(zx_i)^G|$, $i = 1, 2, ..., l$, and G has property P^*, the length of conjugacy classes of $\sum_{21}$ are distinct.

Firstly, we assert that $\sum_1 = \emptyset$. Otherwise, let $x^G \in \sum_1$. By definition of $\sum_1$, $zx \in x^G$, so there exists $g \in G$ such that $zx = x^g$. We decompose x as production of a 2-element a and a $2'$- element b, $x = ab = ba$. Let $o(a) = 2^s$, $o(b) = t$, $(2, t) = 1$. Since $(zab)^t = ((ab)^g)^t = za^t = (a^t)^g$, if let $y = a^t$, then $zy \in y^G$. Since a is a 2-element and t is odd, y is a 2-element. Let $P_2 \in Syl_2(G)$, $y \in P_2$. Since P_2 is Abelian, $|y^G|$ is a odd number. On the other hand, since $zy \in y^G$, $Z(G)$ acts on conjugacy class y^G. Obviously, the action is faithful. Hence $2 \,|\, |y^G|$, it is a contradiction. Thus $\sum_1 = \emptyset$.

Now, considering $\overline{G} = G/Z(G)$, we will prove that every conjugacy classes in $G/Z(G)$ has distinct length.

Let $\bar{x} \in G/Z(G)$, since $\sum_1 = \emptyset$, $zx \notin x^G$. It is easy to see that $\overline{C_G(x)} \leq C_{\overline{G}}(\bar{x})$. Let $\bar{g} \in C_{\overline{G}}(\bar{x})$, then we have $x^g = x$, or $x^g = xz$. By $\sum_1 = \emptyset$, we can conclude that

$x^g = x$, that is $g \in C_G(x)$. Therefore, $C_{\overline{G}}(\bar{x}) = \overline{C_G(x)}$. We have known that the conjugacy classes of $\sum_{21}$ has distinct length. So if let $\sum_{21}^* = \{(\bar{x})^{\overline{G}} \mid x^G \in \sum_{21}\}$, then the conjugacy classes of $\sum_{21}^*$ has distinct length too. Also since $G/Z(G) = \sum_{21}^* = \{(\bar{x})^{\overline{G}} \mid x^G \in \sum_2\}$, by Lemma 2.1, $G/Z(G) \cong S_3$. Therefore, G is a group with order 12. But all finite groups of order 12 are the following:

$$Z_{12},\ Z_2 \times Z_2 \times Z_3,\ A_4,\ G \cong Z_2 \times S_3,\ G \cong Z_3 \rtimes Z_4.$$

It is easy to check that, $G \cong Z_2 \times S_3$ or $G \cong Z_3 \rtimes Z_4$. The proof is completed.

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