HOPF MODULES IN THE WEAK YETTER-DRINFELD CATEGORIES

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Abstract. Suppose that L is a weak Hopf algebra over the field k with a bijective antipode and H is a weak Hopf algebra in the weak Yetter-Drinfeld category ${}_L^L\mathcal{YD}$. We prove that the fundamental theorem for right H-Hopf modules in ${}_L^L\mathcal{YD}$.

Keywords: Weak Hopf algebra, Yetter-Drinfeld module, Fundamental theorem.

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1. Introduction

Weak Hopf algebras have been proposed by G. Bohm, F. Nill and K. Szlachanyi as a generalization of ordinary Hopf algebras in the following sense: the defining axioms are the same, but the multiplicativity of the counit and the comultiplicativity of the unit are replaced by weaker axioms. The initial motivation to study weak Hopf algebras is their connection with the theory of algebra extension [1], and another important application of weak Hopf algebras is that they provide a natural framework for the study of dynamical twists in Hopf algebras [2].

Just like finite-dimensional Hopf algebras, finite-dimensional weak Hopf algebra also obey the mathematical beauty of giving rise to a self-dual notion: the dual of it can be canonically endowed with a weak Hopf algebra structure. The notion of a weak Yetter-Drinfeld category ${}_L^L\mathcal{YD}$ over a weak Hopf algebra L has been introduced by Bohm in [3], and further studied by Caenepeel in [4]. The paper [5] proves that if H is a finite-dimensional weak Hopf algebra in the category ${}_L^L\mathcal{YD}$ over a weak Hopf algebra L, then its linear dual H^* is also a weak Hopf algebra in ${}_L^L\mathcal{YD}$.

In this paper, we prove the fundamental theorem for right H-Hopf modules in ${}_L^L\mathcal{YD}$. We also show that if H is dimensional, its dual H^* has a right H-Hopf module structure which is not analogous to usual one.

2. Preliminaries

A weak Hopf algebra is a vector space L with the structure of an associative unital algebra (L, m, μ) with multiplication $m : L \otimes L \longrightarrow L$ and unit $1 \in L$ and a coassociative coalgebra (L, Δ, ε) with comultiplication $\Delta : L \longrightarrow L \otimes L$ and counit $\varepsilon : L \longrightarrow k$ such that

(i) The comultiplication Δ is a (not necessarily unit-preserving) homomorphism of algebras such that

$$(\Delta \otimes id)\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1). \tag{2.1}$$

(ii) The counit satisfies the following identity

$$\varepsilon(kgl) = \varepsilon(kg_1)\varepsilon(g_2 l) = \varepsilon(kg_2)\varepsilon(g_1 l), \ \forall k, g, l \in L. \tag{2.2}$$

(iii) There is a linear map $S_L : L \longrightarrow L$ called an antipode, such that, for all $l \in L$,

$$m(id \otimes S_L)\Delta(l) = (\varepsilon \otimes id)(\Delta(1)(l \otimes 1)), \tag{2.3}$$

$$m(S_L \otimes id)\Delta(l) = (id \otimes \varepsilon)((1 \otimes l)\Delta(1)), \tag{2.4}$$

$$S_L(l) = S_L(l_1) l_2 S_L(l_3). \tag{2.5}$$

The linear map defined in (2.3) and (2.4) are called target and source counital maps and denoted by ε_t and ε_s respectively:

$$\begin{aligned} \varepsilon_t(l) &= \varepsilon(1_{(1)} l) 1_{(2)} = \varepsilon(S_L(l) 1_{(1)}) 1_{(2)}, \\ \varepsilon_s(l) &= 1_{(1)} \varepsilon(l 1_{(2)}) = 1_{(1)} \varepsilon(1_{(2)} S_L(l)). \end{aligned} \tag{2.6}$$

For all $l \in L$, we have

$$\begin{aligned} l_1 \otimes \varepsilon_t(l_2) &= 1_{(1)} l \otimes 1_{(2)}, \\ \varepsilon_s(l_1) \otimes l_2 &= 1_{(1)} \otimes l 1_{(2)}, \end{aligned} \tag{2.7}$$

$$\begin{aligned} l_1 \otimes \varepsilon_s(l_2) &= l 1_{(1)} \otimes S_L(1_{(2)}), \\ \varepsilon_t(l_1) \otimes l_2 &= S_L(1_{(1)}) \otimes 1_{(2)} l. \end{aligned} \tag{2.8}$$

We will briefly recall the necessary definitions and notions on the weak Hopf algebras.

Definition 2.1. An algebra H is a left L-comodule algebra if H is a left L-comodule via $x \longmapsto \sigma_H(x) = x^{-1} \otimes x^0$ such that

(1) $\sigma_H(xy) = \sigma_H(x)\sigma_H(y) = x^{-1}y^{-1} \otimes x^0 y^0$,

(2) $1^{-1} \otimes x1^0 = \varepsilon_s(x^{-1}) \otimes x^0 \quad \forall x \in H$.

Applying (2), we have the following equality

$$\begin{aligned} 1^{-1}\otimes 1^{0} &= \varepsilon_s(1^{-1})\otimes 1^{0} = S_L(1_{(2)})\otimes 1_{(1)} \longrightarrow 1,\\ &= S_L(1_{(2)})\otimes \varepsilon_t(1_{(1)}) \longrightarrow 1,\\ &= 1_{(1)}\otimes 1_{(2)} \longrightarrow 1 \end{aligned}$$

Definition 2.2. An algebra H is a left L-module algebra if H is a left L-module via $l\otimes x \longmapsto l \longrightarrow x$ such that

(1) $l \longrightarrow xy = (l_1 \longrightarrow x)(l_2 \longrightarrow y)$,

(2) $l \longrightarrow 1 = \varepsilon_t(l) \longrightarrow 1, \quad \forall x,\ y \in H,\ l\in L$,

the second equality is equivalent to $\varepsilon_t(l) \longrightarrow x = (l \longrightarrow 1)x$.

Definition 2.3. An algebra H is a left L-module coalgebra if H is a left L-module via $l\otimes x \longmapsto l \longrightarrow x$ such that

(1) $\Delta(l \longrightarrow x) = (l \longrightarrow x)_1 \otimes (l \longrightarrow x)_2 = (l_1 \longrightarrow x_1)\otimes (l_2 \longrightarrow x_2)$,

(2) $\varepsilon_s(l) \longrightarrow x = x_1\varepsilon(l \longrightarrow x_2), \quad \forall l\in L,\ x\in H$,

the second equation is equivalent to

$$\varepsilon(lk \longrightarrow h) = \varepsilon(lk_2)\varepsilon(k_1 \longrightarrow h),\ \varepsilon(\varepsilon_s(l) \longrightarrow h) = \varepsilon(l \longrightarrow h),\ l,\ k\in L, h\in H.$$

Definition 2.4. An algebra H is a left L-comodule coalgebra if H is a left L-comodule via $x \longmapsto \sigma_H(x) = x^{-1}\otimes x^{0}$ such that

(1) $x^{-1}\otimes (x^0)_1 \otimes (x^0)_2 = x_1^{-1}x_2^{-1}\otimes x_1^0\otimes x_2^0$,

(2) $\varepsilon(x^0)x^{-1} = \varepsilon(x^0)\varepsilon_t(x^{-1}) \quad \forall x\in H$.

3. Weak Hopf algebras in weak Yetter-Drinfeld category

Let L be a weak Hopf algebra with a bijective antipode S_L. We recall that the weak Yetter-Drinfeld category ${}^{L}_{L}\mathcal{YD}$ is the braided monoidal categories whose objects V are both left L-comodules and satisfy the following conditions:

(1) $\sigma_V(v) = v^{-1}\otimes v^{0} \in L\otimes_t V = \{1_{(1)}l\otimes 1_{(2)} \longrightarrow v | \forall l\in L, v\in V\}$,
(2) $l_1v^{-1}\otimes l_2 \longrightarrow v^0 = (l_1 \longrightarrow v)^{-1}l_2 \otimes (l_1 \longrightarrow v)^0 i.e.,$
(3) $\sigma_V(l \longrightarrow v) = (l \longrightarrow v)^{-1}\otimes (l \longrightarrow v)^0 = l_1v^{-1}S_L(l_3)\otimes l_2 \longrightarrow v^0$,

where the L-module action is denoted by $l \longrightarrow v$ for $l\in L,\ v\in V$ and the L-comodule structure map is denoted by $\sigma_V:\ V \longrightarrow L\otimes V$. We use the following notation:

$$\sigma_V(v) = v^{-1}\otimes v^{0}, \quad (\Delta\otimes id)\sigma_V(v) = (id\otimes\sigma_V)\sigma_V(v) = v^{-2}\otimes v^{-1}\otimes v^{0}).$$

The braiding $\tau = \tau_{V,\,W}: \quad V \otimes_t W \longrightarrow W \otimes_t V$ in this category is given by

$$\begin{aligned}\tau(1_{(1)} \longrightarrow v \otimes 1_{(2)} \longrightarrow w) &= v^{-1} \longrightarrow w \otimes v^0, \\ \tau^{-1}(1_{(1)} \longrightarrow w \otimes 1_{(2)} \longrightarrow v) &= v^0 \otimes S_L^{-1}(v^{-1}) \longrightarrow w.\end{aligned}$$

Let $V \in {}_L^L\mathcal{YD}$. Then, for all $v \in V$, we have

$$\begin{aligned}\varepsilon_s(v^{-1}) \otimes v^0 &= S_L(1_{(2)}) \otimes 1_{(1)} \longrightarrow v, \\ \varepsilon_t(v^{-1}) \otimes v^0 &= S_L(1_{(1)}) \otimes 1_{(2)} \longrightarrow v, \forall v \in V.\end{aligned}$$

In [5], Shen Bing-liang introduces the definition of weak Hopf algebra in the weak Yetter-Drinfeld category ${}_L^L\mathcal{YD}$. Moreover, they have showed that if H is a finite-dimensional weak Hopf algebra in ${}_L^L\mathcal{YD}$, then its dual H^* is a weak Hopf algebra in ${}_L^L\mathcal{YD}$.

Definition 3.1. Let L be a weak Hopf algebra with a bijective antipode S_L. An object $H \in {}_L^L\mathcal{YD}$ is called a weak bialgebra in this category if it is both an algebra and a coalgebra satisfying the following conditions:

$$\begin{aligned}(1)\quad \Delta(xy) &= x_1(x_2^{-1} \longrightarrow y_1) \otimes x_2^0 y_2, \\ \varepsilon(xyz) &= \varepsilon(xy_1)\varepsilon(y_2 z) = \varepsilon(x(y_1^{-1} \longrightarrow y_2))\varepsilon(y_1^0 z), \\ \Delta^2(1) &= 1_1 \otimes 1_2 1'_1 \otimes 1'_2 = 1_1 \otimes (1_2^{-1} \longrightarrow 1'_1)1_2^0 \otimes 1'_2.\end{aligned}$$

(2) H is both a left L-module algebra, L-comodule algebra, L-module coalgebra and L-comodule coalgebra.

(3) there exist an antipode $S: \ H \longrightarrow H$ (here S is both left L-linear and L-colinear i.e., S is a morphism in the category of ${}_L^L\mathcal{YD}$) satisfying

$$\begin{aligned}x_1 S(x_2) &= \varepsilon((x^{-1} \longrightarrow 1_1)x^0)1_2, \\ S(x_1)x_2 &= 1_1\varepsilon((1_2^{-1} \longrightarrow x)1_2^0), \\ S(x_1)x_2S(x_3) &= S(x), \quad \forall x \in H.\end{aligned}$$

Similar to the notation of weak Hopf algebra, we denote

$$\varepsilon_t(x) = \varepsilon((x^{-1} \longrightarrow 1_1)x^0)1_2, \qquad \varepsilon_s(x) = 1_1\varepsilon((1_2^{-1} \longrightarrow x)1_2^0).$$

According to the definitions of $\varepsilon_t,\ \varepsilon_s$ we can obtain explicit expressions for these coproducts

$$\Delta(\varepsilon_t(x)) = \varepsilon_t(x)1_1 \otimes 1_2, \qquad \Delta(\varepsilon_s(x)) = 1_1 \otimes 1_2\varepsilon_s(x).$$

Furthermore, for $x \in H$,

$$\begin{aligned}\varepsilon(\varepsilon_t(x)) &= \varepsilon((x^{-1} \longrightarrow 1)x^0), \\ &= \varepsilon((\varepsilon_t(x^{-1}) \longrightarrow 1)x^0), \\ &= \varepsilon((S_L(1_{(1)}) \longrightarrow 1)(1_{(2)} \longrightarrow x)), \\ &= \varepsilon((\varepsilon_t(1_{(1)}) \longrightarrow 1)(1_{(2)} \longrightarrow x)), \\ &= \varepsilon((1_{(1)} \longrightarrow 1)(1_{(2)} \longrightarrow x)), \\ &= \varepsilon(x),\end{aligned}$$

in a similar way we can compute $\varepsilon(\varepsilon_s(x)) = \varepsilon(x)$. Applying Definition 3.1, one obtains immediately the following identities

$$\begin{aligned} \varepsilon(x\varepsilon_t(y)) &= \varepsilon(xy_1S(y_2)) = \varepsilon(xy_1)\varepsilon(y_2S(y_3)) = \varepsilon(xy), \\ \varepsilon(\varepsilon_s(x)y) &= \varepsilon(S(x_1)x_2y) = \varepsilon(S(x_1)x_2)\varepsilon(x_3y) = \varepsilon(xy). \end{aligned}$$

As S is both left L-linear and L-colinear, we can easily check that ε_t and ε_s are also both left L-linear and L-colinear. Moreover it is both an anti-algebra map and an anti-coalgebra map, that is

$$\begin{aligned} Sm &= m\tau_{H,H}(S \otimes S),\ i.e., S(xy) = (x^{-1} \longrightarrow S(y))S(x^0), \quad x,\ y \in H, \\ \Delta S &= (S \otimes S)\tau_{H,H}\Delta,\ i.e., \Delta(S(x)) = (x_1^{-1} \longrightarrow S(x_2)) \otimes S(x_1^0). \end{aligned}$$

In this paper, we will always assume that the antipode S is bijective. The composite-inverse S^{-1} satisfies

$$\begin{aligned} S^{-1}m &= m(S^{-1} \otimes S^{-1})\tau^{-1},\ i.e., \\ S^{-1}(xy) &= S^{-1}(y^0)(S_L^{-1}(y^{-1}) \longrightarrow S^{-1}(x)), \\ \Delta S^{-1} &= (S^{-1} \otimes S^{-1})\tau^{-1}\Delta,\ i.e., \\ \Delta(S^{-1}(x)) &= S^{-1}(x_2^0) \otimes S_L^{-1}(x_2^{-1}) \longrightarrow S^{-1}(x_1). \end{aligned}$$

Proposition 3.2. *Suppose H is a weak Hopf algebra in ${}_L^L\mathcal{YD}$, the following identities hold*

$$\varepsilon_t \circ S = S \circ \varepsilon_S, \quad \varepsilon_s \circ S = S \circ \varepsilon_t.$$

Proof. For $x \in H$ we have

$$\begin{aligned} \varepsilon_t \circ S(x) &= [S(x)]_1 S([S(x)]_2) = (x_1^{-1} \longrightarrow S(x_2))S(S(x_1^0)), \\ &= S(S(x_1)x_2) = S \circ \varepsilon_S(x). \end{aligned}$$

In a similar way, one can verify $\varepsilon_s \circ S = S \circ \varepsilon_t$. ∎

As a preparation for the theorem below, we notice that Proposition 3.2 has counterparts involving the antipode,

$$\varepsilon_t(x) = \varepsilon(S(x)1_1)1_2, \qquad \varepsilon_s(x) = 1_1\varepsilon(1_2S(x)).$$

As a matter of fact,

$$\varepsilon_t(x) = \varepsilon(\varepsilon_t(x)1_1)1_2 = \varepsilon(x_1S(x_2)1_1)1_2 = \varepsilon(\varepsilon_s(x_1)S(x_2)1_1)1_2 = \varepsilon(S(x)1_1)1_2.$$

The second equality can be proven analogously. Applying Proposition 3.2, one can obtain

$$\begin{aligned} \varepsilon_s(x) &= (S \circ \varepsilon_t \circ S^{-1})(x) = S(1_{(2)}\varepsilon(S(S^{-1}(x))1_{(1)})) = S(1_{(2)})\varepsilon(x1_{(1)}), \\ \varepsilon_t(x) &= (S \circ \varepsilon_s \circ S^{-1})(x) = S(1_{(1)}\varepsilon(1_{(2)}S(S^{-1}(x))) = S(1_{(1)})\varepsilon(1_{(2)}x). \end{aligned}$$

In a similar way, we can verify the first equation.

Proposition 3.3. Suppose H is a weak Hopf algebra in ${}^L_L\mathcal{YD}$. For all $x \in H$ we have the identities

$$x_1 \otimes \varepsilon_s(x_2) = x1_1 \otimes S(1_2), \quad \varepsilon_t(x_1) \otimes x_2 = S(1_1) \otimes 1_2 x.$$

Proof. Using the above definitions, one obtains

$$\begin{aligned} x_1 \otimes \varepsilon_s(x_2) &= x_1 \otimes S(1_2)\varepsilon(x_2 1_1), \\ &= x_1(x_2^{-1} \longrightarrow 1_{1'})\varepsilon(x_2^0 1_{2'} 1_1)S(1_2), \\ &= x_1(x_2^{-1} \longrightarrow 1_1)\varepsilon(x_2^0 1_2)S(1_3), \\ &= (h1_1)_1\varepsilon((h1_1)_2) \otimes S(1_2), \\ &= h1_1 \otimes S(1_2), \end{aligned}$$

$$\begin{aligned} \varepsilon_t(x_1) \otimes x_2 &= S(1_1)\varepsilon(1_2 x_1) \otimes x_2, \\ &= S(1_1)\varepsilon(1_2 1_{1'}(1_{2'}^{-1} \longrightarrow x_1)) \otimes 1_{2'}^0 x_2, \\ &= S(1_1)\varepsilon(1_2(1_3^{-1} \longrightarrow x_1)) \otimes 1_3^0 x_2, \\ &= S(1_1) \otimes \varepsilon((1_2 x)_1)(1_2 x)_2, \\ &= S(1_1) \otimes 1_2 x. \blacksquare \end{aligned}$$

4. Hopf modules in the Yetter-Drinfeld categories

Since a weak Hopf algebra H in the weak Yetter-Drinfeld categories ${}^L_L\mathcal{YD}$ is both algebra and coalgebra, one can consider modules and comodules over H. As in the theory of Hopf algebras, an H-Hopf module is an H-module which is also an H-comodule such that these two structures are compatible (the action "commutes" with coaction):

Definition 4.1. Let H be a weak Hopf algebra in ${}^L_L\mathcal{YD}$. A right H-Hopf module M in ${}^L_L\mathcal{YD}$ such that it is both a right H-module and a right H-comodule via $\rho_M : M \longrightarrow M \otimes H$, $\rho_M(m) = m_0 \otimes m_1$ and the following equations hold:

(1) $\rho_M(mh) = m_0(m_1^{-1} \longrightarrow h_1) \otimes m_1^0 h_2, m \in M,\ h \in H,$
(2) $\sigma_M(mh) = m^{-1}h^{-1} \otimes m^0 h^0, m \in M,\ h \in H,$
(3) $m^{-1} \otimes (m^0)_0 \otimes (m^0)_1 = m_0^{-1} m_1^{-1} \otimes m_0^0 \otimes m_1^0 m \in M,$
(4) $l \longrightarrow (mh) = (l_1 \longrightarrow m)(l_2 \longrightarrow h), l \in L,\ m \in M,\ h \in H,$
(5) $\rho_M(l \longrightarrow m) = (l_1 \longrightarrow m_0)(l_2 \longrightarrow m_1), l \in L,\ m \in M.$

We remark that $M \otimes_t H$ is a right H-module by $(m \otimes h)x = m(h^{-1} \longrightarrow x_1) \otimes h^0 x_2$ and a right H-comodule $\rho_{M\otimes H}(m \otimes h) = m_0 \otimes m_1^{-1} \longrightarrow h_1 \otimes m_1^0 h_2$. The condition (1) means that the H-comodule structure $\rho_M : M \longrightarrow M \otimes H$ is H-linear, or equivalently the H-module structure map $\varphi_M : M \otimes H \longrightarrow M$ is H-colinear.

Also, (2) (resp. (4))$\Longleftrightarrow \varphi_M$ is L-colinear (resp. L-linear); (3)(resp. (5))$\Longleftrightarrow \rho_M$ is L-colinear (resp. L-linear).

Example 1. H itself is a right H-Hopf module (in ${}_L^L\mathcal{YD}$) in the natural way. If V is an object in ${}_L^L\mathcal{YD}$, then so is $V \otimes_t H$ by $l \longrightarrow (v \otimes h) = (l_1 \longrightarrow v) \otimes (l_2 \longrightarrow h)$ and $\sigma_{V\otimes H}(v \otimes h) = v^{-1}h^{-1} \otimes v^0 h^0$. It is also both a right H-module and a right H-comodule by $(v \otimes h)x = v \otimes hx$ and $\rho_{V\otimes H}(v \otimes h) = v \otimes h_1 \otimes h_2$. One easily checks that $V \otimes_t H$ is an H-Hopf module.

Lemma 4.2. *If H is a weak Hopf algebra in ${}_L^L\mathcal{YD}$ and M a right H-Hopf module in ${}_L^L\mathcal{YD}$, we define $M^{coH} = \{m \in M | \rho_M(m) = m1_1 \otimes 1_2\}$ is a L-submodule, then*

(1) *M^{coH} is a L-submodule.*

(2) *M^{coH} is a L-subcomodule of M, so $M^{coH} \in {}_L^L\mathcal{YD}$.*

Proof. (1) Let $n \in M^{coH}$, then

$$\begin{aligned}
\rho_M(l \longrightarrow n) &= (l_1 \longrightarrow n1_1) \otimes (l_2 \longrightarrow 1_2),\\
&= (l_1 \longrightarrow n)(l_2 \longrightarrow 1_1) \otimes (l_3 \longrightarrow 1_2),\\
&= [(l_1 \longrightarrow n) \otimes 1]\Delta(\varepsilon_t(l_2) \longrightarrow 1),\\
&= ((1_{(1)}l \longrightarrow n) \otimes 1)\Delta(1_{(2)} \longrightarrow 1),\\
&= [1_{(1)} \longrightarrow (l \longrightarrow n)][1_{(2)} \longrightarrow (1_{(1')} \longrightarrow 1_1)] \otimes 1_{(2')} \longrightarrow 1_2,\\
&= (l \longrightarrow n)1_1 \otimes 1_2.
\end{aligned}$$

Hence $l \longrightarrow n \in M^{coH}$.

(2) Applying $1^{-1} \otimes 1^0 = 1_{(1)} \otimes (1_{(2)} \longrightarrow 1)$ and $\varepsilon_t(l) \longrightarrow x = (l \longrightarrow 1_H)x$, we obtain

$$\begin{aligned}
1^{-1} \otimes (1^0)_1 \otimes (1^0)_2 &= 1_{(1)} \otimes (1_{(2)} \longrightarrow 1_1) \otimes 1_2,\\
&= 1_{(1)} \otimes (\varepsilon_t(1_{(2)}) \longrightarrow 1_1) \otimes 1_2,\\
&= 1_{(1)} \otimes (1_{(2)} \longrightarrow 1)1_1 \otimes 1_2,\\
&= 1^{-1} \otimes 1^0 1_1 \otimes 1_2.
\end{aligned}$$

For $n \in M^{coH}$ we do a calculation:

$$\begin{aligned}
n^{-1} \otimes (n^0)_0 \otimes (n^0)_1 &= n_0^{-1} n_1^{-1} \otimes n_0^0 \otimes n_1^0,\\
&= n^{-1} 1_1^{-1} 1_2^{-1} \otimes n^0 1_1^0 \otimes 1_2^0,\\
&= n^{-1} 1^{-1} \otimes n^0 (1^0)_1 \otimes (1^0)_2,\\
&= n^{-1} 1^{-1} \otimes n^0 1^0 1_1 \otimes 1_2,\\
&= n^{-1} \otimes n^0 1_1 \otimes 1_2.
\end{aligned}$$

This implies that $n^{-1} \otimes n^0 \in L \otimes M^{coH}$, so $M^{coH} \in {}_L^L\mathcal{YD}$. ∎

Theorem 4.3. *If H is a weak Hopf algebra in ${}_L^L\mathcal{YD}$ and M a right H-Hopf module in ${}_L^L\mathcal{YD}$, M^{coH} is defined as above. Then*

(1) *Let* $P(m) = m_0 S(m_1)$, $m \in M$, *then* $P(m) \in M^{coH}$. *If* $n \in M^{coH}$ *and* $h \in H$, *then* $\rho_M(nh) = nh_1 \otimes h_2$ *and* $P(nh) = n\varepsilon_t(h)$.

(2) *The map* $F : M^{coH} \otimes_t H \longrightarrow M$, $F(n \otimes h) = nh$ *is an isomorphism of Hopf modules, the inverse map is given by* $G(m) = P(m_0)m_1$.

Proof. (1) Since ε_t is a left L-comodule map we have $\sigma(h_1 S(h_2)) = h^{-1} \otimes \varepsilon_t(h^0)$. Applying $x_1 \otimes \varepsilon_s(x_2) = x 1_1 \otimes S(1_2)$, we obtain

$$\begin{aligned} \rho_M(P(m)) &= m_0(m_1^{-1} m_2^{-1} \longrightarrow S(m_3)) \otimes m_1^0 S(m_2^0), \\ &= m_0(m_1^{-1} \longrightarrow S(m_2)) \otimes \varepsilon_t(m_1^0), \\ &= m_0[S(m_1)]_1 \otimes S^{-1} \circ \varepsilon_s([S(m_1)]_2), \\ &= m_0 S(m_1)1_1 \otimes S^{-1}(S(1_2)), \\ &= m_0 S(m_1)1_1 \otimes 1_2. \end{aligned}$$

If $n \in M^{coH}$ and $h \in H$, then

$$\begin{aligned} \rho(nh) &= n1_1(1_2^{-1} \longrightarrow h_1) \otimes 1_2^0 h_2 = nh_1 \otimes h_2, \\ P(nh) &= nh_1 S(h_2) = n\varepsilon_t(h). \end{aligned}$$

(2) Since

$$\begin{aligned} F(l \longrightarrow (n \otimes h)) &= F((l_1 \longrightarrow n) \otimes (l_2 \longrightarrow h)), \\ &= (l_1 \longrightarrow n)(l_2 \longrightarrow h), \\ &= l \longrightarrow nh, \\ &= l \longrightarrow F(n \otimes h), \end{aligned}$$

then F is a left H-linear map. F is also left L-colinear by the following equality

$$\sigma(F(n \otimes h)) = \sigma(nh) = n1_{(1)}h^{-1} \otimes 1_{(2)}h^0 = (id \otimes F)\sigma(n \otimes h),$$

clearly F is right H-linear. It is also right H-colinear by (1). Now we have

$$\begin{aligned} GF(n \otimes h) &= P(nh_1) \otimes h_2 = n\varepsilon_t(h_1) \otimes h_2, \\ &= n \otimes \varepsilon_t(h_1)h_2 = n \otimes S(1_1)1_2 h, \\ &= n \otimes h, \\ FG(m) &= m_0 S(m_1) m_2 = m_0 \varepsilon_s(m_1), \\ &= [m_0\varepsilon_s(m_1)]_0 \varepsilon([m_0\varepsilon_s(m_1)]_1), \\ &= m_0(m_1^{-1} \longrightarrow 1_1)\varepsilon(m_1^0 1_2 \varepsilon_s(m_2)), \\ &= m_0 \varepsilon(m_1 \varepsilon_s(m_2)), \\ &= m_0 \varepsilon(m_1 1_1 S(1_2)), \\ &= m_0 \varepsilon(m_1), \\ &= m. \end{aligned}$$

∎

Example. Let H be a weak Hopf algebra in ${}^L_L\mathcal{YD}$. $M = H$ is defined as a right H-Hopf module by Δ. Then $M^{CoH} = \{\varepsilon_t(H)|h \in H\}$.

5. Application

By [5] we make H^* into a weak Hopf algebra in ${}^L_L\mathcal{YD}$. H^* has the contragredient left L-module structure, ie.,

$$(l \longrightarrow f)(h) = f(S_L(l) \longrightarrow h), \quad l \in L,\ f \in H^*,\ \ h \in H.$$

Also, since H is a finite-dimensional left L-comodule, H^* has the transposed right L-comodule structure and so it becomes a left L-comodule via $\sigma_{H^*} : H^* \longrightarrow L \otimes H^*$, $\sigma_{H^*}(f) = f^{-1} \otimes f^0$, where

$$f^0(h)f^{-1} = f(h^0)S_L^{-1}(h^{-1}), \quad h \in H.$$

Now assume that H is finite-dimensional, we will show that H^* becomes a right H-Hopf module in ${}^L_L\mathcal{YD}$. First, H^* is a right H-module by

$$(fh)(x) = f(hx), \quad f \in H^*, \quad h \in H.$$

Second, H^* is a right H-comodule using the identification $\theta_H : H^* \otimes H \cong Hom(H, H)$, $\theta_H(f \otimes h)(l) = f(h^{-1} \longrightarrow l)h^0$ as follows:

$$\rho_{H^*} :\ H^* \longrightarrow H^* \otimes H \cong Hom(H,\ H), \quad \rho_{H^*}(f)(x) = f(x_1)S(x_2).$$

That is, $\rho_{H^*}(f) = f_0 \otimes f_1$ means

$$f(x_1)S(x_2) = \rho_{H^*}(f)(x) = \theta_H(f_0 \otimes f_1)(x) = f_0(f_1^{-1} \longrightarrow x)f_1^0, \quad x \in H. \tag{5.1}$$

Proposition 5.1. *H^* is a right H-comodule by $\theta_{H\otimes H}$.*

Proof. First, we check that H-comodule via above using $\theta_{H\otimes H}$. Applying σ_H to (5.1) we obtain

$$f_1^{-1} \otimes f_0(f_1^{-2} \longrightarrow x)f_1^0 = x_2^{-1} \otimes f(x_1)S(x_2^0). \tag{5.2}$$

Now for $f \in H^*$, $\ x \in H$, we have

$$\begin{aligned}
\theta_{H\otimes H}((f_0)_0 \otimes (f_0)_1 \otimes f_1)(x) &= (f_0)_0((f_0)_1^{-1}f_1^{-1} \longrightarrow x)(f_0)_1^0 \otimes f_1^0,\\
&= f_0((f_1^{-1} \longrightarrow x)_1)S((f_1^{-1} \longrightarrow x)_2) \otimes f_1^0,\\
&= f_0(f_1^{-2} \longrightarrow x_1)(f_1^{-1} \longrightarrow S(x_2)) \otimes f_1^0,\\
&= (f_1^{-1} \longrightarrow S(x_2)) \otimes f_0(f_1^{-2} \longrightarrow x_1)f_1^0\\
&= (x_2^{-1} \longrightarrow S(x_3)) \otimes f(x_1)S(x_2^0),\\
&= \Delta_H(f(x_1)S(x_2)),\\
&= \Delta_H(f_0(f_1^{-1} \longrightarrow x)f_1^0),\\
&= f_0(f_1^{-1} \longrightarrow x)(f_1^0)_1 \otimes (f_1^0)_2,\\
&= f_0((f_1)_1^{-1}(f_1)_2^{-1} \longrightarrow x)(f_1)_1^0 \otimes (f_1)_2^0,\\
&= \theta_{H\otimes H}(f_0 \otimes (f_1)_1 \otimes (f_1)_2)(x).
\end{aligned}$$

It implies that $(f_0)_0 \otimes (f_0)_1 \otimes f_1 = f_0 \otimes (f_1)_1 \otimes (f_1)_2$.

According to $\sigma_V(l \longrightarrow v) = l_1 v^{-1} S_L(l_3) \otimes l_2 \longrightarrow v^0$, we calculate the following equality

$$\begin{aligned}
&(1_{(1)} \longrightarrow f_0)((1_{(2)} \longrightarrow f_1)^{-1} \longrightarrow x)(1_{(2)} \longrightarrow f_1)^0\\
&\quad = f_0((S_L(1_{(1)})1_{(2)} f_1^{-1} S_L(1_{(4)})) \longrightarrow x)(1_{(3)} \longrightarrow f_1^0),\\
&\quad = f_0((\varepsilon_s(1_{(1)}) f_1^{-1} S_L(1_{(3)})) \longrightarrow x)(1_{(2)} \longrightarrow f_1^0),\\
&\quad = f_0((1_{(1')} f_1^{-1} S_L(1_{(2)})) \longrightarrow x)(1_{(1)}1_{(2')} \longrightarrow f_1^0),\\
&\quad = f_0((1_{(1)} f_1^{-1} S_L(1_{(2)})) \longrightarrow x)(1_{(3)} \longrightarrow f_1^0),\\
&\quad = f_0((1 \longrightarrow f_1)^{-1} \longrightarrow x)(1 \longrightarrow f_1)^0,\\
&\quad = f_0(f_1^{-1} \longrightarrow x) f_1^0,
\end{aligned}$$

so we obtain $(1_{(1)} \longrightarrow f_0) \otimes (1_{(2)} \longrightarrow f_1) = f_0 \otimes f_1$.

Applying ε to (5.1), we obtain

$$\begin{aligned}
f(x) &= f_0(f_1^{-1} \longrightarrow x)\varepsilon(f_1^0),\\
&= f_0(\varepsilon_t(f_1^{-1})\varepsilon(f_1^0) \longrightarrow x),\\
&= f_0(S_L(1_{(1)} \longrightarrow x))\varepsilon(1_{(2)} \longrightarrow f_1),\\
&= (1_{(1)} \longrightarrow f_0)(x)\varepsilon(1_{(2)} \longrightarrow f_1),\\
&= f_0(x)\varepsilon(f_1),\\
&= (f_0\varepsilon(f_1))(x).
\end{aligned}$$

Hence $(id \otimes \varepsilon)\rho_{H^*}(f) = f$, thus H^* becomes a right H-comodule. ∎

Theorem 5.2. *With the notation as above, then H^* is a right H-Hopf module in ${}_L^L\mathcal{YD}$. Moreover, $(H^*)^{coH} = \{f \in H^* | f(x_1)x_2 = f(1_1 x)1_2, \quad x \in H\}$.*

Proof. Now, we prove that H^* is a right H-Hopf module. First, we will show that $(fh)_0 \otimes (fh)_1 = f_0(f_1^{-1} \longrightarrow h_1) \otimes f_1^0 h_2$, since for $x \in H$,

$$\begin{aligned}
&\theta_H \ \ (f_0(f_1^{-1} \longrightarrow h_1) \otimes f_1^0 h_2)(x)\\
&\quad = (f_0(f_1^{-1} \longrightarrow h_1))((f_1^0 h_2)^{-1} \longrightarrow x)((f_1^0 h_2)^0),\\
&\quad = (f_0(f_1^{-2} \longrightarrow h_1))(f_1^{-1} h_2^{-1} \longrightarrow x) f_1^0 h_2^0,\\
&\quad = f_0((f_1^{-2} \longrightarrow h_1)(f_1^{-1} h_2^{-1} \longrightarrow x)) f_1^0 h_2^0,\\
&\quad = f_0(f_1^{-1} \longrightarrow (h_1(h_2^{-1} \longrightarrow x)) f_1^0 h_2^0,\\
&\quad = f((h_1(h_2^{-1} \longrightarrow x))_1) S((h_1(h_2^{-1} \longrightarrow x))_2) h_2^0,\\
&\quad = f(h_1(h_2^{-2} h_3^{-2} \longrightarrow x_1)) S(h_2^{-1} h_3^{-1} \longrightarrow x_2) S(h_2^0) h_3^0,\\
&\quad = f(h_1(h_2^{-2} \longrightarrow x_1)) S(h_2^{-1} \longrightarrow x_2)\varepsilon_s(h_2^0),\\
&\quad = f(h_1((\varepsilon_s(h_2))^{-2} \longrightarrow x_1)) S((\varepsilon_s(h_2))^{-1} \longrightarrow x_2)(\varepsilon_s(h_2))^0,\\
&\quad = f(h1_1(1_2^{-2} \longrightarrow x_1))(1_2^{-1} \longrightarrow S(x_2)) S(1_2^0),\\
&\quad = f(h1_1(1_2^{-1} \longrightarrow x_1)) S(1_2^0 x_2),\\
&\quad = (f \cdot h)(1_1(1_2^{-1} \longrightarrow x_1)) S(1_2^0 x_2),\\
&\quad = (f \cdot h)(x_1) S(x_2),\\
&\quad = [(f \cdot h)_0 \otimes [(f \cdot h)_1](x).
\end{aligned}$$

To verify that $\sigma_{H^*}(fh) = f^{-1}h^{-1} \otimes f^0h^0$ for $f \in H^*,\ h \in H$, we compute for $x \in H$

$$\begin{aligned}
f^{-1}h^{-1}f^0h^0(x) &= f^{-1}h^{-1}f^0(h^0x),\\
&= f(h^0x^0)S_L^{-1}(h^{-1}x^{-1})h^{-2},\\
&= f((1_{(1)} \longrightarrow h)x^0)S_L^{-1}(x^{-1})1_{(2)}\\
&= f((1_{(1')}1_{(1)} \longrightarrow h)(1_{(2')} \longrightarrow x^0))S_L^{-1}(x^{-1})1_{(2)},\\
&= f((1_{(1)} \longrightarrow h)(\varepsilon_t(1_{(2)}) \longrightarrow x^0))S_L^{-1}(x^{-1})1_{(3)},\\
&= f((1_{(1)} \longrightarrow h)(1_{(2)} \longrightarrow 1)x^0)S_L^{-1}(x^{-1})1_{(3)},\\
&= f((1_{(1')} \longrightarrow h)(1_{(2')} \longrightarrow (1_{(1)} \longrightarrow 1))x^0)S_L^{-1}(x^{-1})1_{(2)},\\
&= f(h(1_{(1)} \longrightarrow 1)x^0)S_L^{-1}(x^{-1})1_{(2)},\\
&= f(h(\varepsilon_t(1_{(1)}) \longrightarrow x^0))S_L^{-1}(x^{-1})1_{(2)},\\
&= f(h(S(1_{(1)}) \longrightarrow x^0))S_L^{-1}(x^{-1})1_{(2)},\\
&= f(h(S(S^{-1}(1_{(2)})) \longrightarrow x^0)S_L^{-1}(x^{-1})S_L^{-1}(1_{(1)}),\\
&= f(h(1_{(2)} \longrightarrow x^0))S_L^{-1}(1_{(1)}x^{-1}),\\
&= f(hx^0)S_L^{-1}(x^{-1}),\\
&= (fh)(x^0)S_L^{-1}(x^{-1}),\\
&= (fh)^{-1}(fh)^0(x).
\end{aligned}$$

Next, we want to check $(l \longrightarrow fh) = (l_1 \longrightarrow f)(l_2 \longrightarrow h)$ for $l \in L,\ h \in H$, $f \in H^*$, since for $x \in H$

$$\begin{aligned}
((l_1 \longrightarrow f)(l_2 \longrightarrow h))(x) &= (l_1 \longrightarrow f)((l_2 \longrightarrow h)x),\\
&= f(S_L(l_1) \longrightarrow ((l_2 \longrightarrow h)x)),\\
&= f((S_L(l_2)l_3 \longrightarrow h)(S_L(l_1) \longrightarrow x)),\\
&= f((\varepsilon_s(l_2) \longrightarrow h)(S_L(l_1) \longrightarrow x)),\\
&= f((S_L(1_{(2)}) \longrightarrow h)(S_L(l1_{(1)}) \longrightarrow x)),\\
&= f((1_{(1)} \longrightarrow h)(1_{(2)} \longrightarrow (S_L(l) \longrightarrow x))),\\
&= f(h(S_L(l) \longrightarrow x)),\\
&= (fh)(S_L(l) \longrightarrow x),\\
&= (l \longrightarrow fh)(x).
\end{aligned}$$

We have $f^{-1} \otimes (f^0)_0 \otimes (f^0)_1 = f_0^{-1}f_1^{-1} \otimes f_0^0 \otimes f_1^0$ for $f \in H^*$, since for $x \in H$

$$\begin{aligned}
&f_0^{-1}\ \ f_1^{-1} \otimes \theta_H(f_0^0 \otimes f_1^0)(x)\\
&= f_0^{-1}f_1^{-2} \otimes f_0^0(f_1^{-1} \longrightarrow x)f_1^0,\\
&= f_0((f_1^{-1} \longrightarrow x)^0)S_L^{-1}((f_1^{-1} \longrightarrow x)^{-1})f_1^{-2} \otimes f_1^0,\\
&= f_0(f_1^{-2} \longrightarrow x^0)S_L^{-1}(f_1^{-3}x^{-1}S(f_1^{-1}))f_1^{-4} \otimes f_1^0,\\
&= f_0(f_1^{-2} \longrightarrow x^0)f_1^{-1}S_L^{-1}(x^{-1})\varepsilon_t(f_1^{-3}) \otimes f_1^0,\\
&= f_0((f_1^{-2}1_{(2)} \longrightarrow x^0)f_1^{-1}S_L^{-1}(1_{(1)}x^{-1}) \otimes f_1^0,\\
&= f_0(f_1^{-2} \longrightarrow x^0)f_1^{-1}S_L^{-1}(x^{-1}) \otimes f_1^0,
\end{aligned}$$

$$
\begin{aligned}
&= f_1^{-1}S_L^{-1}(x^{-1}) \otimes f_1^0 f_0(f_1^{-2} \longrightarrow x^0),\\
&= (x^0)_2^{-1}S_L^{-1}(x^{-1}) \otimes f((x^0)_1)S(((x^0)_2)^0),\\
&= x_2^{-1}S_L^{-1}(x_1^{-1}x_2^{-2}) \otimes f(x_1^0)S(x_2^0),\\
&= S_L^{-1}(x_1^{-1}\varepsilon_t(x_2^{-1})) \otimes f(x_1^0)S(x_2^0),\\
&= S_L^{-1}(x_1^{-1}S_L(1_{(1)})) \otimes f(x_1^0)S(1_{(2)} \longrightarrow x_2),\\
&= S_L^{-1}((1_{(1)} \longrightarrow x_1)^{-1}) \otimes f((1_{(1)} \longrightarrow x_1)^0)S(1_{(2)} \longrightarrow x_2),\\
&= S_L^{-1}(x_1^{-1})f(x_1^0) \otimes S(x_2),\\
&= f^0(x_1)f^{-1} \otimes S(x_2),\\
&= f^{-1} \otimes f^0(x_1)S(x_2),\\
&= f^{-1} \otimes (f^0)_0((f^0)_1^{-1} \longrightarrow x)(f_1^0)_1^0,\\
&= f^{-1} \otimes \theta_H((f^0)_0 \otimes (f^0)_1)(x).
\end{aligned}
$$

Finally, we show that $\rho_{H^*}(l \longrightarrow f) = l_1 \longrightarrow f_0 \otimes l_2 \longrightarrow f_1$. Since for $l \in L,\ f \in H^*,\ x \in H$

$$
\begin{aligned}
\theta_H\ \ &((l_1 \longrightarrow f_0) \otimes (l_2 \longrightarrow f_1))(x)\\
&= (l_1 \longrightarrow f_0)((l_2 \longrightarrow f_1)^{-1} \longrightarrow x)(l_2 \longrightarrow f_1)^0,\\
&= (l_1 \longrightarrow f_0)(l_2 f_1^{-1}S_L(l_4) \longrightarrow x)(l_3 \longrightarrow f_1^0),\\
&= f_0(S_L(l_1)l_2 f_1^{-1}S_L(l_4) \longrightarrow x)(l_3 \longrightarrow f_1^0),\\
&= f_0(\varepsilon_s(l_1)f_1^{-1}S_L(l_3) \longrightarrow x)(l_2 \longrightarrow f_1^0),\\
&= f_0(1_{(1)}f_1^{-1}S_L(l_2) \longrightarrow x)(l_1 1_{(2)} \longrightarrow f_1^0),\\
&= f_0(f_1^{-1} \longrightarrow (S_L(l_2) \longrightarrow x))(l_1 \longrightarrow f_1^0),\\
&= f((S_L(l_2) \longrightarrow x)_1)(l_1 \longrightarrow S((S_L(l_2) \longrightarrow x)_2)),\\
&= f(S_L(l_3) \longrightarrow x_1)(l_1 \longrightarrow S(S_L(l_2) \longrightarrow x_2)),\\
&= f(S_L(l_3) \longrightarrow x_1)(l_1 S_L(l_2) \longrightarrow S(x_2)),\\
&= f(S_L(l_2) \longrightarrow x_1)(\varepsilon_t(l_1) \longrightarrow S(x_2)),\\
&= F(S_L(1_{(2)}l) \longrightarrow x_1)(S_L(1_{(1)}) \longrightarrow S(x_2)),\\
&= f(S_L(l) \longrightarrow (1_{(1)} \longrightarrow x_1))S(1_{(2)} \longrightarrow x_2),\\
&= f(S_L(l) \longrightarrow x_1)S(x_2),\\
&= (l \longrightarrow f)(x_1)S(x_2),\\
&= \theta_H((l \longrightarrow f)_0 \otimes (l \longrightarrow f)_1)(x).
\end{aligned}
$$

From all above, H^* is a right H-Hopf module in ${}_L^L\mathcal{YD}$. ■

Applying Theorem 4.3, we can obtain the following result.

Corollary 5.3. *H^* is defined a right H-Hopf module in ${}_L^L\mathcal{YD}$ as above, then $H^{*CoH} \otimes_t H \cong H^*$.*

Corollary 5.4. *Let L be a finite-dimensional Hopf algebra. Assume that H is a finite-dimensional Hopf algebra in the Yetter-Drinfeld category in ${}_L^L\mathcal{YD}$. H^* is defined a right H-Hopf module in ${}_L^L\mathcal{YD}$ as above, then $H^{*CoH} = Hom(H,k) = \{f \in H^* | f(x_1)x_2 = f(x)1_H,\ \ x \in H\}$.*

Example. Recall that a groupoid G is a category in which every morphism is an isomorphism. In this section, we consider finite groupoids, i.e., groupois which a finite number of objects. The set of objects of G will be denoted by G_0, and the set of morphisms by G_1. The identity morphism on $x \in G_0$ will also be denoted by x. For $\sigma : x \longrightarrow y$ in G_1, we write $s(\sigma) = x$ and $t(\sigma) = y$, respectively for the source and the target of σ. For every $x \in G$, $G_x = \{\sigma \in G_1 | s(\sigma) = t(\sigma) = x\}$ is a group.

Let G be a groupoid, and k a commutative ring. The groupoid algebra is the direct product $kG = \bigoplus\limits_{\sigma \in G_1} ku_\sigma$, with multiplication defined by the formula $u_\sigma u_\tau = u_{\sigma\tau}$, if $t(\tau) = s(\sigma)$ (0 if $t(\tau) \neq s(\sigma)$). The unit element is $1 = \sum\limits_{x \in G_0} u_x$. kG is a weak Hopf algebra, with comultiplication, counit and antipode given by the formulas

$$\Delta(u_\sigma = u_\sigma \otimes u_\sigma, \quad \varepsilon(u_\sigma) = 1, \quad S(u_\sigma) = u_{\sigma^{-1}}$$

Using the formula

$$\Delta(1) = \sum_{x \in G_0} u_x \otimes u_x,$$

suppose kG is free of finite rank as a k-module, hence $(kG)^*$ is also a weak Hopf algebra. As a k-module, $(kG)^* = \bigoplus\limits_{\sigma \in G_1} kv_\tau$ with $\langle v_\sigma,\ u_\tau \rangle = \delta_{\sigma,\ \tau}$. Suppose $(kG)^*$ is defined a right H-Hopf module in ${}_L^L\mathcal{YD}$ as above. Then

$$\begin{aligned}((kG)^*)^{CoH} =& \{f \in (kG)^* | f(u_\sigma)u_\sigma = \sum_{x \in G_0} f(u_x u_\sigma)u_x\},\\ =& \{f \in (kG)^* | f(u_\sigma)u_\sigma = \sum_{x \in G_0} f(u_\sigma)u_{t(\sigma)}\}.\end{aligned}$$

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References

[1] Kadison, L., Nikshych, D., *Frobenius Extensions and Weak Hopf Algebras*, J. Algebra, 244 (2001), 312-342.

[2] Etingof, P., Nikshych, D., *Dynamical Quantum Groups at Roots of* 1, Duke Math. J., 108 (2001), 135-168.

[3] Bohm, G., Nill, F., Szlachanyi, K., *Weak Hopf Algebras* I. *Integral Theory and C^*-structure*, J. Algebra, 221 (1999), 385-438.

[4] Caenpeel, S., Wang Ding-duo, Yin Yan-min, *Yetter-Drinfeld modules over Weak Bialgebras*, Ann. Univ. Ferrara, Sez. VII-Sc. Mat., 51 (2005), 69-98.

[5] Shen Bing-liang, Wang Shuan-hong,, *Weak Hopf Algebra Duality in Weak Yetter-Drinfeld Categories and Applications*, Internation Electronic Journal of Algebra, 6 (2009), 74-94.

[6] Takeuchi, M., *Hopf Modules in Yetter-Drinfeld-Drinfeld Categories*, Comm. Algebra, 26 (9) (1998), 3057-3070.

[7] Bohm, G., *Doi-Hopf Modules over Weak Hopf Algebras*, Comm. Algebra, 28 (2000), 4687-4689.

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