FROM NEWTON TO KEPLER.
One simple derivation of Kepler's laws from Newton's ones.

František Mošna

Department of Mathematics
Technical Faculty
Czech University of Life Sciences, Prague
Czech Republic
e-mail: mosna@tf.czu.cz

Abstract. There is plenty of ways how to deduce Kepler's laws of planetary motion from Newton's ones (law of universal gravitation and law of motion). We offer one of them which is very simple and direct. It uses only mathematical tools and is suitable for teaching purposes.

1. Kepler's and Newton's laws

Johannes Kepler (1571–1630) deduced his laws of planetary motion due to the exact measurement of planets performed by astronomer Tycho Brahe (1546–1601). The first and the second law were derived in Prague and published in work Astronomia Nova in 1609. The third one was formulated in Linz and published Harmoniae Mundi in 1619.

Sixty five years later (in 1684), Isaac Newton (1643–1727) deduced his law of universal gravitation from Kepler's laws.

Plenty of literature deals with the relation between Kepler's and Newton's laws. The most of them use physical concepts, e.g. [1] and [2]. The geometrical way of derivation was discussed in Richard Feynmann lecture presented on 13th March 1964 at Caltech (its manuscript had been lost and it was found again in 1992 at the office of author's colleague Leighton) and it is very interesting [3].

In this text, we would like to present one way how to derive Kepler's laws from the Newton's law of universal gravitation and motion. The aim and advantage of our deduction of especially the first one is its simplicity, directness, straightness and using only mathematical tools without introducing other physical variables. It can be presented maybe even in secondary schools.

2. Formulation of the laws

Let us formulate this laws at first.

- Kepler's laws of planetary motion:

(K1) The orbit of every planet is an elipse with the Sun at one of the two foci.

(K2) A line joining a planet and the Sun sweeps out equal areas during equal intervals of time, i.e. area velocity is constant.

(K3) Quotient $\frac{T^3}{a^2}$ is constant, where T is orbital period of any planet and a is the major semi-axis of its orbit.

- Newton's laws of universal gravitation and the second law of motion:

(NG) Any point mass M attracts every single other point mass m situated in the distance r from M by a force

$$F = \kappa \frac{mM}{r^2} \,, \qquad \text{where} \;\; \kappa = 6.672 \cdot 10^{-11} Nm^2/kg^2$$

(acting in direction of radius-vector of both points).

(N2) The acceleration a of a body is parallel and directly proportional to the net force F acting on the body,

$$F = am \,, \qquad \text{where} \;\; m \text{ is mass of a body.}$$

3. Derivation of Kepler's laws

At first, we will describe the situation in Figure 1. In the beginning, the point mass m (planet) in the distance r_0 from the point mass M (the Sun) has the velocity v_0 in the direction perpendicular to the radius-vector of both points (M and m).

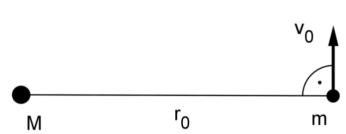

Figure 1: Situation and start condition

Let the point mass M be situated to origin of axes and let point mass m be described by the position vector $\vec{r}(t) = (x(t), y(t), z(t))$. According to (NG), the force by which the point mass m is attracted to the point mass M can be expressed by the formula

$$\vec{F} = -\kappa \cdot \frac{mM}{|\vec{r}|^3} \cdot \vec{r} = -k \cdot \frac{m\vec{r}}{|\vec{r}|^3}\,, \quad \text{where } k = \kappa M. \tag{1}$$

By (N2) it holds

$$\vec{F} = m \cdot \ddot{\vec{r}}. \tag{2}$$

From these two relations (1) and (2) we obtain equation

$$\ddot{\vec{r}} + k \cdot \frac{\vec{r}}{|\vec{r}|^3} = 0. \tag{3}$$

3.1. Trajectory is a plane curve

At first, we derive that the trajectory of point mass m is situated in some plane containing the point mass M.

This fact will be proved if we show that the velocity $\dot{\vec{r}}$ of the point m is situated in the same plane as radius-vector $\vec{r}$ of that point, i.e., when $\dot{\vec{r}} \times \vec{r}$ is a constant vector. Let us calculate the derivation of this cross product using relation (3)

$$\frac{d}{dt}(\dot{\vec{r}} \times \vec{r}) = \ddot{\vec{r}} \times \vec{r} + \underbrace{\dot{\vec{r}} \times \dot{\vec{r}}}_{=0} = -\frac{k}{|\vec{r}|^3}(\vec{r} \times \vec{r}) = 0.$$

That is why the cross product $\dot{\vec{r}} \times \vec{r}$ is really constant. So, it is proved that the motion of a point in so called central-force field is a plane curve.

3.2. Trajectory is an ellipse (1. Kepler law)

We can assume for further calculation that the motion is situated in the plane xy and we shall describe this motion by position vector $\vec{r}(t) = (x(t), y(t))$ and $\vec{r}(0) = (r_0, 0)$ and $\dot{\vec{r}}(0) = (0, v_0)$ in consistency with initial conditions.

We transform the position vector to the polar coordinates r and φ

$$\vec{r} = (x, y) = (r\cos\varphi, r\sin\varphi).$$

We can differentiate twice

$$\dot{\vec{r}} = (\dot{x}, \dot{y}) = \dot{r}(\cos\varphi, \sin\varphi) + r\dot{\varphi}(-\sin\varphi, \cos\varphi),$$

$$\ddot{\vec{r}} = (\ddot{x}, \ddot{y}) = (\ddot{r} - r\dot{\varphi}^2)(\cos\varphi, \sin\varphi) + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})(-\sin\varphi, \cos\varphi)$$

and rewrite the initial conditions $\dot{r}(0) = 0$ and $\dot{\varphi}(0) = \dfrac{v_0}{r_0}$.

This derivatives can be put into (3) and we obtain

$$(\ddot{r} - r\dot{\varphi}^2 + \frac{k}{r^2})(\cos\varphi, r\sin\varphi) + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})(-\sin\varphi, \cos\varphi) = 0.$$

From here, we have the equations

$$\ddot{r} - r\dot{\varphi}^2 + \frac{k}{r^2} = 0 \quad \text{and} \quad 2\dot{r}\dot{\varphi} + r\ddot{\varphi} = 0. \tag{4}$$

We multiply both sides of the second part of (4) by r and get

$$0 = 2r\dot{r}\dot{\varphi} + r^2\ddot{\varphi} = \frac{d}{dt}(r^2\dot{\varphi}),$$

and so, $r^2\dot{\varphi}$ is a constant. The initial conditions give us

$$\dot{\varphi} = \frac{r_0 v_0}{r^2}. \tag{5}$$

We put this result into the first part of (4)

$$\ddot{r} - \frac{r_0^2 v_0^2}{r^3} + \frac{k}{r^2} = 0 \tag{6}$$

and we deal similarly with this equation (6). We multiply both sides of it by $\dot{r}$ and we obtain

$$0 = \dot{r}\ddot{r} - r_0^2 v_0^2 \cdot \frac{\dot{r}}{r^3} + k \cdot \frac{\dot{r}}{r^2} = \frac{d}{dt}\left(\frac{1}{2}\dot{r}^2 + \frac{1}{2}r_0^2 v_0^2 \cdot \frac{1}{r^2} - k \cdot \frac{1}{r}\right).$$

This derivative is equal to zero and it again proves that

$$\dot{r}^2 + r_0^2 v_0^2 \cdot \frac{1}{r^2} - 2k \cdot \frac{1}{r}$$

is a constant. With respect to the initial conditions we get

$$\dot{r}^2 + r_0^2 v_0^2 \cdot \frac{1}{r^2} - 2k \cdot \frac{1}{r} = v_0^2 - 2k \cdot \frac{1}{r_0}. \tag{7}$$

We can express the derivative $\dot{r}$ from (7)

$$\dot{r} = \sqrt{v_0^2 - 2k \cdot \frac{1}{r_0} - \frac{r_0^2 v_0^2}{r^2} + 2k \cdot \frac{1}{r}}. \tag{8}$$

Now, we would like to express φ as a function of r, so we are looking for the function g such that $\varphi = g(r)$ (see Figure 2).

The chain rule for derivatives gives us

$$\frac{d\varphi}{dt}(t) = \frac{dg}{dr}(r(t)) \cdot \frac{dr}{dt}(t) \quad \text{or} \quad \dot{\varphi}(t) = \frac{d\varphi}{dr}(r(t)) \cdot \dot{r}(t).$$

We put (5) and (8) into this relation and we obtain

$$\frac{r_0 v_0}{r^2} = \frac{d\varphi}{dr} \cdot \sqrt{v_0^2 - 2k \cdot \frac{1}{r_0} - \frac{r_0^2 v_0^2}{r^2} + 2k \cdot \frac{1}{r}}. \tag{9}$$

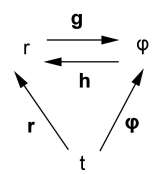

Figure 2: Functions

We can rewrite the equation (9) into form

$$\frac{d\varphi}{dr} = \frac{r_0 v_0}{r\sqrt{Ar^2 + Br + C}}, \tag{10}$$

where

$$A = v_0^2 - 2k \cdot \frac{1}{r_0}, \quad B = 2k \ , \quad C = -r_0^2 v_0^2 < 0$$

and the discriminant

$$D = 4\left(k - r_0 v_0^2\right)^2 > 0.$$

We shall solve (10) by integration

$$\varphi = \frac{r_0 v_0}{\sqrt{-C}} \arcsin \frac{Br + 2C}{r\sqrt{D}} + c_1 = \arcsin \frac{2kr - 2r_0^2 v_0^2}{2r\left(k - r_0 v_0^2\right)} + c_1.$$

The constant $c_1 = -\frac{\pi}{2}$ can be got from initial condition, hence

$$\varphi = \arcsin \frac{r - \frac{1}{k} r_0^2 v_0^2}{r\left(1 - \frac{1}{k} r_0 v_0^2\right)} - \frac{\pi}{2}.$$

We continue the calculation

$$\cos\varphi = \sin(\varphi + \frac{\pi}{2}) = \frac{r - \frac{1}{k} r_0^2 v_0^2}{r\left(1 - \frac{1}{k} r_0 v_0^2\right)}$$

and

$$r = \frac{\frac{1}{k} r_0^2 v_0^2}{1 - \left(1 - \frac{1}{k} r_0 v_0^2\right)\cos\varphi}. \tag{11}$$

This is an equation of conic section

$$r = \frac{p}{1 + \epsilon\cos\varphi} \ ,$$

where

$$p = \frac{1}{k} r_0^2 v_0^2 \geq 0 \quad \text{and} \quad \epsilon = \frac{1}{k} r_0 v_0^2 - 1 \geq -1.$$

Parameter $\epsilon = 0$ corresponds to the circle (and to the formula about the first cosmic speed $v_1 = \sqrt{\frac{\kappa M}{r_0}}$), parameter $\epsilon = 1$ corresponds to the parabola (and to the relations concerning the second cosmic speed $v_2 = \sqrt{\frac{2\kappa M}{r_0}}$). The elliptic trajectory remains for $|\epsilon| < 1$, i.e., $v_0 < v_2$. So we got the 1. Kepler law from here.

3.3. Area velocity is constant (2. Kepler's law)

In order to derive the 2. Kepler law, we must express the area, which is swept out by radius-vector of point mass m between time t_1 and t_2 in polar coordinates, $\Delta t = t_2 - t_1$. We use again the expression of r by φ, i.e., $r = h(\varphi)$, function h is inverse to the function g (see Figure 2).

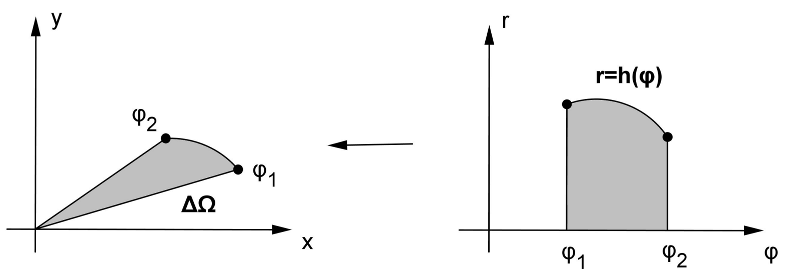

Figure 3: Area of radius-vector

Let us calculate the area of $\Delta\Omega$. We use step by step the translation to the polar coordinates (see Figure 3), the Fubini theorem and the substitution $\varphi = \varphi(t)$

$$\begin{aligned}
|\Delta\Omega| &= \iint_{\Delta\Omega} 1\,dxdy = \iint_{D} r\,drd\varphi \\
&= \int_{\varphi_1}^{\varphi_2} \left(\int_0^{h(\varphi)} r\,dr \right) d\varphi = \int_{\varphi_1}^{\varphi_2} \frac{1}{2} h^2(\varphi)\,d\varphi \\
&= \int_{t_1}^{t_2} \frac{1}{2} h^2(\varphi(t)) \cdot \dot{\varphi}(t)\,dt = \int_{t_1}^{t_2} \frac{1}{2} r^2(t) \cdot \dot{\varphi}(t)\,dt.
\end{aligned}$$

Now, we use relation (5) and get

$$|\Delta\Omega| = \int_{t_1}^{t_2} \frac{1}{2} v_0 r_0\,dt = \frac{1}{2} v_0 r_0 (t_2 - t_1) = \frac{1}{2} v_0 r_0 \Delta t, \tag{12}$$

which gives us the 2. Kepler law about the constant area speed of radius-vector of motion.

3.4. Relation between period and semi-major axes (3. Kepler's law)

The 3. Kepler's law is a simple consequence of obtained results. We shall use (12) for $t_1 = 0$ and $t_2 = T$, where T is a period of motion of point mass m, and we can compare it with formula of ellipse area with semi-axes a and b

$$\pi ab = \frac{1}{2} r_0 v_0 T. \tag{13}$$

The relations

$$a = \frac{p}{1-\epsilon^2} \qquad \text{and} \qquad b = a\sqrt{1-\epsilon^2} \tag{14}$$

hold for parameters p, ϵ used for expression of ellipse in polar coordinates and semi-axes a, b used in cartesian coordinates. From (14) we have $b = \sqrt{ap}$ and (13) gives us $\pi a\sqrt{ap} = \frac{1}{2} r_0 v_0 T$. Because $p = \frac{1}{k} v_0^2 r_0^2$, we have

$$\pi a \sqrt{\frac{a}{k}} r_0 v_0 = \frac{1}{2} r_0 v_0 T$$

and from here we obtain

$$\frac{T^2}{a^3} = \frac{4\pi^2}{k} \ ,$$

which gives the 3. Kepler's law.

3.5. Mathematical apparatus

Finally, we summarize the mathematical apparatus used in our deductions:

1. properties of vectors, the dot (scalar, inner) product and cross (vector) product;
2. equation of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and formula of ellipse area $P = ab\pi$, where a and b are semi-axes;
3. polar coordinates $x = r\cos\varphi$, $y = r\sin\varphi$;
4. equation of conic section in polar coordinates $r(1 + \epsilon\cos\varphi) = p$, where p is parameter and ϵ is angular eccentricity, conic section is a hyperbola for $|\epsilon| > 1$, parabola for $|\epsilon| = 1$, ellipse for $|\epsilon| < 1$, (circle for $|\epsilon| = 0$); relations $b^2 = a^2(1-\epsilon^2)$, $p = a(1-\epsilon^2)$, where a, b are semi-axes of ellipse;
5. chain rule for derivative, derivative of cross product;
6. integral

   $$\int \frac{dx}{x\sqrt{Ax^2 + Bx + C}} = \frac{1}{\sqrt{-C}} \arcsin \frac{Bx + 2C}{x\sqrt{D}} + c_1 \ ,$$

   where $C < 0$, $D = B^2 - 4AC > 0$;

7. double integral, its property, Fubini theorem and substitution theorem;
8. formula of ellipse area $P = ab\pi$.

References

[1] Arnold, V.I., *Mathematical Methods of Classical Mechanics*, Springer, New York, 1989.

[2] Sommerfeld, A., *Mechanics*, Academic Press, New York, 1952.

[3] Goodstein, D.L., Goodstein, J.R., *Feynman's lost lecture*, Norton, New York, 1996, 1999.

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