SOME DOUBLE LACUNARY SEQUENCE SPACES

Kuldip Raj
Sunil K. Sharma
Seema Jamwal

School of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J & K
India
e-mail: kuldipraj68@gmail.com
sunilksharma42@yahoo.co.in

Abstract. The purpose of this paper is to introduce some double lacunary sequence spaces defined by a sequence of Orlicz functions. We also make an effort to study some topological properties and inclusion relations between these sequence spaces.

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1. Introduction and preliminaries

The initial work on double sequences is found in Bromwich [4]. Later on, it was studied by Hardy [7], Moricz [16], Moricz and Rhoades [17], Tripathy ([30], [31]), Başarir and Sonalcan [2] and many others. Hardy [7] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [33] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [21] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [18] and Mursaleen and Edely [22] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{k,l})$ into one whose core is a subset of the M-core of x. More recently, Altay and Başar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$,

$\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Recently, Başar and Sever [3] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. By the convergence of a double sequence we mean the convergence in the Pringsheim sense, i.e., a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [25]. We shall write more briefly as P-convergent. The double sequence $x = (x_{k,l})$ is bounded if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l.

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{x \in w : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty\Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$||x|| = \inf\Big\{\rho > 0 : \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1\Big\}.$$

It is shown in [10] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. The notion of difference sequence spaces was introduced by Kızmaz [8], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [5] by introducing the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$.

Let m, n be non-negative integers, then for $Z = c, c_0$ and l_∞, we have sequence spaces

$$Z(\Delta_n^m) = \{x = (x_k) \in w : (\Delta_n^m x_k) \in Z\},$$

where $\Delta_n^m x = (\Delta_n^m x_k) = (\Delta_n^{m-1} x_k - \Delta_n^{m-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_n^m x_k = \sum_{v=0}^{m} (-1)^v \begin{pmatrix} m \\ v \end{pmatrix} x_{k+nv}.$$

Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ studied by Kızmaz [8]. Taking $n = 1$, we get the spaces $l_\infty(\Delta^m)$, $c(\Delta^m)$ and $c_0(\Delta^m)$ studied by Et and Colak [5]. Similarly, we can define difference operators on double sequence spaces as:

$$\begin{aligned} \Delta x_{k,l} &= (x_{k,l} - x_{k,l+1}) - (x_{k+1,l} - x_{k+1,l+1}) \\ &= x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}, \end{aligned}$$

$$\Delta^m x_{k,l} = \Delta^{m-1} x_{k,l} - \Delta^{m-1} x_{k,l+1} - \Delta^{m-1} x_{k+1,l} + \Delta^{m-1} x_{k+1,l+1}$$

and

$$\Delta_n^m x_{k,l} = \Delta_n^{m-1} x_{k,l} - \Delta_n^{m-1} x_{k,l+1} - \Delta_n^{m-1} x_{k+1,l} + \Delta_n^{m-1} x_{k+1,l+1}.$$

A double sequence $x = (x_{k,l})$ of real numbers is called almost P-convergent to a limit L if

$$P - \lim_{p,q\to\infty} \sup_{m,n\geq 0} \frac{1}{pq} \sum_{k=m}^{m+p-1} \sum_{l=n}^{n+q-1} |x_{k,l} - L| = 0$$

i.e., the average value of $(x_{k,l})$ taken over any rectangle

$$\{(k,l) : m \leq k \leq m+p-1,\ n \leq l \leq n+q-1\}$$

tends to L as both p and q tends to ∞, and this P-convergence is uniform in m and n.

By a lacunary sequence $\theta = (i_r)$, $r = 0, 1, 2, ...$, where $i_0 = 0$, we shall mean an increasing sequence of non-negative integers $h_r = (i_r - i_{r-1}) \to \infty\ \ (r \to \infty)$. The intervals determined by θ are denoted by $I_r = (i_{r-1}, i_r]$ and the ratio i_r/i_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [6] as follows:

$$N_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{g_r} \sum_{k\in I_r} |x_k - L| = 0 \text{ for some } L \right\}.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary if there exist two increasing sequences of integers such that

$$k_0 = 0,\ h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0,\ \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Let $k_{r,s} = k_r l_s$, $h_{r,s} = h_r \bar{h}_s$ and $\theta_{r,s}$ is determined by $I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r$ and $l_{s-1} < l \leq l_s\}$, $q_r = \dfrac{k_r}{k_{r-1}}$, $\bar{q}_s = \dfrac{l_s}{l_{s-1}}$ and $q_{r,s} = q_r \bar{q}_s$.

A sequence space E is said to be solid if $(\alpha_{k,l} x_{k,l}) \in E$, whenever $(x_{k,l}) \in E$ and for all sequence $(\alpha_{k,l})$ of scalars with $|\alpha_{k,l}| \leq 1$, for all $k, l \in \mathbb{N}$.

A sequence space E is said to be symmetric if $(x_{k,l}) \in E$ implies $(x_{\pi(k,l)}) \in E$, where π is a permutation of $\mathbb{N}$.

A sequence space E is said to be convergence free if $(y_{k,l}) \in E$ whenever $(x_{k,l}) \in E$ and $x_{k,l} = 0$ implies $y_{k,l} = 0$.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

1. $p(x) \geq 0$, for all $x \in X$,
2. $p(-x) = p(x)$, for all $x \in X$,

3. $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$,

4. if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ *as* $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ *as* $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ *as* $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [32], Theorem 10.4.2, P-183). For more details about sequence spaces see ([9], [11], [12], [13], [14], [19], [23], [24], [26], [27], [28], [29]).

Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of positive real numbers. Let X be a seminormed space over the field of complex numbers with the seminorm q and $w(X)$ denotes the space of all sequences $x = (x_{k,l})$, where $x_{k,l} \in X$. Now, we define the following sequence spaces in this paper:

$$[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1 = \Big\{ x = (x_{k,l}) \in w(X) : \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l} - L)}{\rho} \Big) \Big]^{p_{k,l}} = 0, \text{ for some } L \text{ and } \rho > 0 \Big\},$$

$$[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0 = \Big\{ x = (x_{k,l}) \in w(X) : \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l})}{\rho} \Big) \Big]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \Big\}$$

$$[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty = \Big\{ x = (x_{k,l}) \in w(X) : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l})}{\rho} \Big) \Big]^{p_{k,l}} < \infty, \text{ for some } \rho > 0 \Big\}.$$

If we take $\mathcal{M}(x) = x$, we get the following spaces:

$$[N_\theta, \Delta_n^m, p, q, u]_1 = \Big\{ x = (x_{k,l}) \in w(X) : \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l} - L)}{\rho} \Big)^{p_{k,l}} = 0, \text{ for some } L \text{ and } \rho > 0 \Big\},$$

$$[N_\theta, \Delta_n^m, p, q, u]_0 = \Big\{ x = (x_{k,l}) \in w(X) : \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l})}{\rho} \Big)^{p_{k,l}} = 0, \text{ for some } \rho > 0 \Big\}$$

$$[N_\theta, \Delta_n^m, p, q, u]_\infty = \Big\{x = (x_{k,l}) \in w(X) : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l})}{\rho}\Big)^{p_{k,l}} < \infty, \text{ for some } \rho > 0\Big\}.$$

If we take $p = (p_{k,l}) = 1$, we get the spaces like:

$$[N_\theta, \mathcal{M}, \Delta_n^m, q, u]_1 = \Big\{x = (x_{k,l}) \in w(X) : \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l} - L)}{\rho}\Big)\Big] = 0, \text{ for some } L \text{ and } \rho > 0\Big\},$$

$$[N_\theta, \mathcal{M}, \Delta_n^m, q, u]_0 = \Big\{x = (x_{k,l}) \in w(X) : \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l})}{\rho}\Big)\Big] = 0, \text{ for some } \rho > 0\Big\}$$

$$[N_\theta, \mathcal{M}, \Delta_n^m, q, u]_\infty = \Big\{x = (x_{k,l}) \in w(X) : \sup_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l})}{\rho}\Big)\Big] < \infty, \text{ for some } \rho > 0\Big\}.$$

The following inequality will be used throughout the paper.

If $0 \le p_{k,l} \le \sup p_{k,l} = H$, $K = \max(1, 2^{H-1})$, then

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \le K\{|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}\} \tag{1.1}$$

for all k, l and $a_{k,l}, b_{k,l} \in \mathbb{C}$. Also $|a|^{p_{k,l}} \le \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main aim of the present paper is to study some topological properties and inclusion relations between the above defined sequence spaces.

2. Main results

Theorem 2.1. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers. Then, $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$, $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$ are linear spaces over the set of complex numbers $\mathbb{C}$.*

Proof. Suppose $x = (x_{k,l})$ and $y = (y_{k,l}) \in [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$. Then, there exist positive numbers ρ_1, ρ_2 such that

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m (x_{k,l}))}{\rho_1}\Big)\Big]^{p_{k,l}} = 0$$

and

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m(y_{k,l}))}{\rho_2}\Big)\Big]^{p_{k,l}} = 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_{k,l})$ is a non-decreasing and convex so by using inequality (1.1), we have

$$\begin{aligned}
&\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m(\alpha x_{k,l}+\beta y_{k,l}))}{\rho_3}\Big]^{p_{k,l}} \\
&\le K \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \frac{1}{2^{p_{k,l}}}\Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m(x_{k,l}))}{\rho_1}\Big)\Big]^{p_{k,l}} \\
&+K \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \frac{1}{2^{p_{k,l}}}\Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m(y_{k,l}))}{\rho_2}\Big)\Big]^{p_{k,l}} \\
&\le K \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m(x_{k,l}))}{\rho_1}\Big)\Big]^{p_{k,l}} \\
&+K \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m(y_{k,l}))}{\rho_2}\Big)\Big]^{p_{k,l}} \\
&= 0.
\end{aligned}$$

Thus, $\alpha x + \beta y \in [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$. This prove that $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ is a linear space. Similarly, we can prove other cases.

Theorem 2.2. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers and $\theta = (k_{r,s})$ be a lacunary sequence. Then $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ is a paranormed spaces with the paranorm*

$$g(x) = \sum_{i,j=1}^{m,n} |x_{i,j}| + \inf\Big\{\rho^{p_{k,l}/H} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l}}{\rho}\Big)\Big] \le 1, \text{ for some } \rho > 0 \text{ and } r = 1,2,3,...\Big\},$$

where $H = \max(1, \sup_{k,l} p_{k,l})$.

Proof. Clearly $g(x) = g(-x)$. Since $M_{k,l}(0) = 0$, for all $k,l \in \mathbb{N}$, we get $g(0) = 0$. Let $x = (x_{k,l})$ and $y = (y_{k,l}) \in [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l})}{\rho_1}\Big)\Big] \le 1, \quad r = 1,2,3,...,$$

and

$$\sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m y_{k,l}}{\rho_2}\Big)\Big] \le 1, \quad r = 1,2,3,....$$

Let $\rho = \rho_1 + \rho_2$. We have

$$\begin{aligned}
&\sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{kl}\Delta_n^m (x_{k,l} + y_{k,l}))}{\rho} \Big) \Big] \\
&\leq \Big(\frac{\rho_1}{\rho_1 + \rho_2} \Big) \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m x_{k,l})}{\rho_1} \Big) \Big] \\
&+ \Big(\frac{\rho_2}{\rho_1 + \rho_2} \Big) \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m y_{k,l})}{\rho_2} \Big) \Big] \\
&\leq 1.
\end{aligned}$$

Since $\rho > 1$, we have

$$\begin{aligned}
&g(x+y) \\
&= \sum_{i,j=1}^{m,n} |x_{i,j} + y_{i,j}| + \inf \Big\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m (x_{k,l} + y_{k,l}))}{\rho} \Big) \Big] \Big\} \\
&\leq \sum_{i,j=1}^{m,n} |x_{i,j}| + \inf \Big\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m (x_{k,l}))}{\rho} \Big) \Big] \leq 1,\ r=1,2,3,... \Big\} \\
&+ \sum_{i,j=1}^{m,n} |y_{i,j}| + \inf \Big\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m (y_{k,l}))}{\rho} \Big) \Big] \leq 1,\ r=1,2,3,... \Big\} \\
&= g(x) + g(y).
\end{aligned}$$

Finally, let λ be a given non-zero scalar in $\mathbb{C}$. Then the continuity of the product follows from the following expression

$$\begin{aligned}
g(\lambda x) &= \sum_{i,j=1}^{m,n} |\lambda x_{i,j}| + \inf \Big\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m (\lambda x_{k,l}))}{\rho} \Big) \Big] \leq 1, \\
&\qquad \text{for some } \rho > 0 \text{ and } r = 1,2,3,... \Big\} \\
&= \lambda \sum_{i,j=1}^{m,n} |x_{i,j}| + \inf \Big\{ (|\lambda| \eta)^{\frac{p_{k,l}}{H}} : \sup_{r,s} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m (x_{k,l}))}{\eta} \Big) \Big] \leq 1, \\
&\qquad \text{for some } \rho > 0 \text{ and } r = 1,2,3,... \Big\},
\end{aligned}$$

where $\eta = \dfrac{\rho}{|\lambda|} > 0$. This completes the proof of the theorem. ∎

Theorem 2.3. *Let $\mathcal{M} = (M_{k,l})$ and $\mathcal{M}' = (M'_{k,l})$ be two sequences of Orlicz functions and $p = (p_{k,l})$ be a bounded sequence of positive real numbers. Then*

(i) $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z \subseteq [N_\theta, \mathcal{M} \circ \mathcal{M}', \Delta_n^m, p, q, u]_Z$

(ii) $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z \cap [N_\theta, \mathcal{M}', \Delta_n^m, p, q, u]_Z \subseteq [N_\theta, \mathcal{M} + \mathcal{M}', \Delta_n^m, p, q, u]_Z$, *where $Z = 0, 1, \infty$.*

Proof. The proof of the theorem is easy, so we omit it. ∎

Theorem 2.4. *The inclusion* $[N_\theta, \mathcal{M}, \Delta_n^{m-1}, p, q, u]_Z \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$ *hold, for* $m \geq 1$. *In general,* $[N_\theta, \mathcal{M}, \Delta_n^i, p, q, u]_Z \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$, *for* $i = 1, 2, 3, ..., m-1$ *and the inclusions are strict, where* $Z = 0, 1, \infty$.

Proof. Let $x = (x_{k,l}) \in [N_\theta, \mathcal{M}, \Delta_n^{m-1}, p, q, u]_0$. Then, there exists $\rho > 0$ such that

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^{m-1}(x_{k,l}))}{\rho}\Big)\Big]^{p_{k,l}} = 0.$$

Since $\mathcal{M}$ is non-decreasing and convex, we have

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^{m}(x_{k,l}))}{2\rho}\Big)\Big]^{p_{k,l}}$$

$$= \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^{m-1}(x_{k,l}) - u_{k,l}\Delta_n^{m-1}(x_{k+1,l+1}))}{2\rho}\Big)\Big]^{p_{k,l}}$$

$$\leq \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^{m-1}(x_{k,l})}{2\rho}\Big)\Big]^{p_{k,l}}$$

$$+ \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^{m-1}(x_{k+1,l+1})}{2\rho}\Big)\Big]^{p_{k,l}}$$

$$\leq \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^{m-1}(x_{k,l})}{\rho}\Big)\Big]^{p_{k,l}}$$

$$+ \lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^{m-1}(x_{k+1,l+1})}{\rho}\Big)\Big]^{p_{k,l}} \longrightarrow 0 \text{ as } r,s\to\infty,$$

i.e., $x = (x_{k,l}) \in [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$. The other cases can be proved in the similar way. ∎

Theorem 2.5. *Let* $\theta = (k_r, l_s)$ *be a lacunary sequence and let* $\mathcal{M} = (M_{k,l})$ *be a sequence of Orlicz functions. Then*

(i) $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0 \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1 \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$, *and the inclusion is strict.*

(ii) *If* $|u_{k,l}| \leq 1$, *then* $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q,]_Z$ *for* $Z = 0, 1, \infty$.

Proof. (i) The inclusion $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0 \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ is obvious. Let $(x_{k,l})$ be an element of $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$. Then, there exists $\rho > 0$ such that

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l\in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^{m}(x_{k,l} - L))}{\rho}\Big)\Big]^{p_{k,l}} = 0.$$

Since $(M_{k,l})$ is non-decreasing and convex, we have

$$\begin{aligned}
&\frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m(x_{k,l}))}{\rho}\Big)\Big]^{p_{k,l}} \\
&\quad \le K \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}\Delta_n^m(x_{k,l} - L))}{\rho}\Big)\Big]^{p_{k,l}} \\
&\quad + K \ \max\Big[1, M_{k,l}\Big(q\big(\frac{L}{\rho}\big)\Big)\Big]^{H},
\end{aligned}$$

where $H = \sup_{k,l}(p_{k,l}), K = \max(1, 2^{H-1})$. This completes the proof of the (i).

(ii) The proof of the (ii) is easy, so we omit it. ∎

Example. Let

$$(p_{k,l}) = \begin{cases} 4, & \text{if } k, l \text{ are even;} \\ 5, & \text{otherwise.} \end{cases}$$

Let $m, n \ge 0$ be given. Let $u_{k,l} = (kl), M_{k,l}(x) = x^2$, for all $k, l \in \mathbb{N}$ and $q(x) = |x|$. Let $\theta = (2^{rs})$ be a lacunary sequence. Consider a sequence $(x_{k,l})$ defined by

$$(x_{k,l}) = \{(kl)^{(mn)}, (kl)^{(mn)}, (kl)^{(mn)}, ...\}.$$

Thus, the sequence $(x_{k,l})$ belongs to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$, but $(x_{k,l})$ does not belongs to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$.

Theorem 2.6. *Let $\mathcal{M} = (M_{k,l})$ and $\mathcal{M}' = (M'_{k,l})$ be two sequences of Orlicz functions. If $\mathcal{M}$ and $\mathcal{M}'$ are equivalent for each $k, l \in \mathbb{N}$ and $\theta = (k_r, l_s)$ be a lacunary sequence. Then*

$$[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z = [N_\theta, \phi, \Delta_n^m, p, q, u]_Z,$$

where $Z = 0, 1, \infty$.

Proof. The proof of the theorem is easy, so omitted. ∎

Theorem 2.7. *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions and let q_1 and q_2 be two seminorms. Then*

(i) $[N_\theta, \mathcal{M}, \Delta_n^m, p, q_1, u]_Z \cap [N_\theta, \mathcal{M}, \Delta_n^m, p, q_2, u]_Z \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q_1 + q_2, u]_Z$;

(ii) $[N_\theta, \mathcal{M}, \Delta_n^m, p, q_1, u]_Z \subseteq [N_\theta, \mathcal{M}, \Delta_n^m, p, q_2, u]_Z$, *where $Z = 0, 1, \infty$.*

Proof. The proof is easy, so omitted.

Proposition 2.8. *The spaces $[N_\theta, \mathcal{M}, p, q, u]_0$ and $[N_\theta, \mathcal{M}, p, q, u]_\infty$ are solid as well as monotone. The spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$ are not solid in general, for $Z = 0, 1, \infty$.*

Proof. Let $(x_{k,l}) \in [N_\theta, \mathcal{M}, p, q, u]_0$. Then, there exists $\rho > 0$ such that

$$\lim_{r,s\to\infty} \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}x_{k,l})}{\rho}\Big)\Big]^{p_{k,l}} = 0.$$

Let $(\alpha_{k,l})$ be a sequence of scalers such that $|\alpha_{k,l}| \leq 1$, for all $k, l \in \mathbb{N}$. Since $|\alpha_{k,l}| \leq \max(1, |\alpha_{k,l}|^H) \leq 1$, for all $k, l \in \mathbb{N}$, where $H = \sup_{k,l} p_{k,l} < \infty$, then, for each r, s, we have

$$\frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(\alpha_{k,l}(u_{k,l}x_{k,l}))}{\rho}\Big)\Big]^{p_{k,l}} \leq \frac{1}{h_{r,s}} \sum_{k,l \in I_{r,s}} \Big[M_{k,l}\Big(\frac{q(u_{k,l}x_{k,l})}{\rho}\Big)\Big]^{p_{k,l}}.$$

Therefore, $(\alpha_{k,l}x_{k,l}) \in [N_\theta, \mathcal{M}, p, q, u]_0$. Hence $[N_\theta, \mathcal{M}, p, q, u]_0$ is solid. Therefore, the space $[N_\theta, \mathcal{M}, p, q, u]_0$ is monotone.

Hence the space $[N_\theta, \mathcal{M}, p, q, u]_\infty$ is solid as well as monotone.

In order to prove that the spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$ are not solid in general, we consider the following example.

Example. Let $M_{k,l}(x) = x^{st}$, for all $k, l \in N$ and $s, t \geq 1$. Let $(p_{k,l}) = (\frac{1}{kl})$, $(u_{k,l}) = (kl)$, for all $k, l \in N$ and $q(x) = |x|$. Let $\theta = (2^{rs})$ be a lacunary sequence, for all $k, l \in \mathbb{N}$. Consider a sequence $(x_{k,l})$ defined by

$$(x_{k,l}) = (kl)^2, \text{ for all } k, l \in \mathbb{N}.$$

Then, $(x_{k,l})$ belongs to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$ for $m, n = 1$.

Let $(\alpha_{k,l}) = (-1)^{kl}$, for all $k, l \in \mathbb{N}$. Then, $(\alpha_{k,l}x_{k,l})$ does not belong to the spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$.

Hence the spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$ are not solid. Therefore, the spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_1$ and $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_\infty$ are not monotone.

Proposition 2.9. *The spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$ are not symmetric in general, for $Z = 0, 1, \infty$.*

Proof. The proof of the result follows from the following example. ∎

Example. Let $M_{k,l}(x) = x^2$, $(p_{k,l}) = (kl)$ and $(u_{k,l}) = (kl)^2$, for all $k, l \in \mathbb{N}$ and $q(x) = |x|$. Let $\theta = (2^{rs})$ be a lacunary sequence for all $k, l \in \mathbb{N}$. Consider a sequence $(x_{k,l})$ defined by $(x_{k,l}) = (kl)^3$, for all $k, l \in \mathbb{N}$. Then $(x_{k,l})$ belong to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$, for $m, n = 1$. Consider the sequence $(y_{k,l})$ which is the rearrangement of the sequence $(x_{k,l})$ defined by

$$(y_{k,l}) = (x_{1,1}, x_{2,2}, x_{4,4}, x_{3,3}, x_{9,9}, x_{5,5}, x_{16,16}, x_{6,6}, x_{25,25}, x_{7,7}, ...).$$

Then, $(y_{k,l})$ does not belongs to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$.

Hence the spaces $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_Z$ are not symmetric in general.

Proposition 2.10. *The space $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ is not convergence free.*

Proof. The proof of the result follows from the following example.

Example. Let $M_{k,l}(x) = x$, $(p_{k,l}) = (kl)$, $(u_{k,l}) = (kl)$, for all $k, l \in \mathbb{N}$ and $q(x) = |x|$. Let $\theta = (2^{rs})$ be a lacunary sequence for all $k, l \in \mathbb{N}$. Consider a sequence $(x_{k,l})$ defined by

$$(x_{k,l}) = \begin{cases} 2, & \text{if } k, l \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

Then $(x_{k,l})$ belongs to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$, for $m, n = 2$.

Consider the sequence $(y_{k,l})$ defined by

$$(y_{k,l}) = \begin{cases} (kl)^2, & \text{if } k, l \text{ are even;} \\ 0, & \text{otherwise.} \end{cases}$$

Then, $(y_{k,l})$ does not belong to $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$.

Hence the space $[N_\theta, \mathcal{M}, \Delta_n^m, p, q, u]_0$ is not convergence free.

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