φJ-MULTIPLIERS AND φJ-MULTIPLIERS QUADRATIC ON JORDAN BANACH ALGEBRAS

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Abstract. In this work, we define the notion of φJ-multipliers on Jordan-Banach algebras without order and investigate some of their properties. We show that a φJ-multiplier satisfies the condition $T[U_x(y)] = U_{\varphi(x)}[T(y)]$. This suggests that we define a new concept which is the φJ-multiplier quadratic. We show several algebraic or topological properties for both concepts. In particular, we extend some known results for φ-multipliers to φJ-multipliers and φJ-multipliers quadratic.

Keywords: Multiplier, J-Multiplier, J-multiplier quadratic, φ-Multiplier, φJ-Multiplier, φJ-multiplier quadratic, Idempotent homomorphism, Quadratic operator, Spectrum of an element, Product of Jordan, Jordan Banach Algebras, Algebra without order, Special Jordan algebra, Full algebra. Strong topology.

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1. Introduction

The aim of this work is to study the φJ-Multipliers and φJ-Multipliers Quadratic in the case of Jordan-Banach algebras.

In the first part, we are going to resume the following definition of a φ-Multiplier T which is due to M. Adib and A. Riazi [1]:

$$T(x \cdot y) = \varphi(x) \cdot Ty = T(x) \cdot \varphi(y).$$

We will show that some results that are true in the case of Banach algebras remain true in the case of Jordan-Banach. We will demonstrate that every φJ-multiplier of Jordan-Banach algebras verifies the relation:

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \ \forall x, y \in A,$$

and that this condition is not sufficient for a linear operator to be a φJ-multiplier. This suggests to us to define a new notion which is that of φJ-multiplier quadratic.

In the second part, we will define φJ-multipliers quadratic of Jordan-Banach algebras. We will show that this generalized definition is the one given in the case of Banach. We will compare the two notions just defined and show some theorems of the type Wang. More particularly, we will establish some algebraic and topological relations between all φJ-multipliers quadratic $M_{\varphi JQ}(A)$ of Jordan-Banach and the algebra of continuous linear operators of A, noted $\mathcal{L}(A)$. We will also show some algebraic properties of $M_{\varphi JQ}(A)$.

2. Preliminaries

Let $\mathbb{K}$ be a commutative field of characteristic zero.

We call $\mathbb{K}$-algebra every $\mathbb{K}$-vector space A provided with a bilinear product $(x, y) \to x.y$ of $A \times A$ in A. If the product is associative (resp. commutative), we say that the algebra is associative (resp. commutative).

If A is a non-commutative algebra, we define the following operators: $R_x(y) = xy$ and $L_x(y) = yx$. In this case, we note A^+ the algebra of the same vector space structure as A, provided with product $\circ$ Defined by

$$x \circ y = \frac{1}{2}(xy + yx), \forall x, y \in A$$

$\circ$ is called Jordan product.

If A is an associative algebra not commutative, the subalgebras of A^+ are called special Jordan algebras.

Among the remarkable identities of these algebras, we note the two following identities:

$$\begin{array}{lcll} x \circ y & = & y \circ x & (C) \\ (x^2 \circ y) \circ x & = & x^2 \circ (y \circ x) & (J) \end{array}$$

More generally, an algebra $(A, \cdot)$ that verifies the two previous identities is called Jordan algebra.

In a Jordan algebra A, we define the following applications:

$$U_x(y) = 2x(xy) - x^2y$$

$$U_{x,y} = \frac{1}{2}\left(U_{x+y} - U_x - U_y\right)$$

The application $x \to U_x$ is quadratic, thus we have: $U_{ax} = a^2U_x$ for every $a \in \mathbb{K}$ and the application $(x, y) \to U_{x,y}$ is bilinear.

If A has a unity, then $R_x = U_{x,e} = U_{e,x}$.

In the case of special Jordan algebra, we have

$$U_x(y) = xyx, \ (\forall x, y \in A).$$

$a \in A$ is invertible if and only if U_a is invertible in $\mathcal{L}(A)$ and we have

$$U_{a^{-1}} = (U_a)^{-1}.$$

A normed Jordan algebra A is a Jordan algebra provided with a vector space norm $\|.\|$ verifying

$$\|xy\| \leq \|x\|\|y\|, \ (\forall x, y \in A).$$

A Jordan Banach algebra $(A, \|.\|)$ is a normed Jordan algebra and complete for its norm.

The spectrum of an element x of Jordan algebra A is a set of scalars λ verifying that $x - \lambda e$ is not invertible in A.

The spectrum of an element of Jordan-Banach algebra is a nonempty compact.

3. The φJ-multipliers of Jordan Banach Algebra

In [2], Helgason has defined a multiplier T of Banach algebra by posing for x et y elements of A:

$$(Tx) \cdot y = x \cdot (Ty)$$

Then, he has showed that, if A is without order, then this definition is equivalent to

$$T(x \cdot y) = x \cdot Ty \tag{3.1}$$

This remark has allowed Wang to establish a number of results on multipliers [9].

As the relation (3.1) arises out quite simply of the associativity of A, we have had the idea of taking it as a definition of a J-multiplier of Jordan Banach algebra [8], in order to find in the case Jordan some results of the type Wang.

In [1], the authors defined the concept of φ-multipliers and proved a some useful results.

We propose in this section to extend the concept of φ-multiplier on Jordan Banach algebras by asking the Definition 3.2.

Definition 3.1 Let A be a Jordan-Banach algebra. A is without order if for all x element of A, $xA = 0$ implies $x = 0$, or, for all x element of A, $Ax = 0$ implies $x = 0$.

Obviously, if A has a unit or if A is semi-simple, then it is without order.

Definition 3.2 Let A be a Jordan-Banach algebra and φ an idempotent homomorphism A. A left (resp.right) φJ-multiplier on A is a bounded linear mapping T of A in A such that $T(x \cdot y) = T(x)\varphi(y)$ (resp. $T(x \cdot y) = \varphi(x)T(y)$) for all $x, y \in A$. We say T is a φJ-multiplier on A if it is both a left φJ-multiplier and right φJ-multiplier. We denote $M_{\varphi J}(A)$ the collection of all φJ-multipliers of A.

Remark 3.1 It is clear that the condition "idempotent" is not necessary in the definition of a φJ-multiplier. But it plays an important role in the following theorems.

Remark 3.2 It is obvious that φ is a φJ-multiplier of A.

To illustrate this theory, we give two important examples of idempotent homomorphisms algebra. It is therefore an example of φ-multiplier and another φJ-multiplier:

Exemple 3.1 Let $\omega = (\omega_n)_{n\geq 0}$ be a sequence of real numbers such that $\omega_n \geq 1$. Consider the space $c_0(\omega)$ of all sequences $(x_n)_{n\geq 0}$ of complex number for which $\lim\limits_{n\longrightarrow\infty} x_n\omega_n = 0$ (see [6]). We equip this space with the weighted supremum norm $\|\cdot\|_w$ given by

$$\|x\|_w := \sup_{n\geq 0} |x_n\omega_n|, \text{ for all } x = (x_n)_{n\geq 0}$$

Moreover, with respect to coordinatewise operations $c_0(\omega)$ is a commutative Banach algebra, consider the standard basis $(e_n)_{n\geq 0}$ where $e_n = (\delta_{nj})_{j\geq 0}$, then e_n is idempotent for all $n \geq 0$.

Now, let the multiplication operator by the element e_k

$$\begin{array}{rcccl} L_{e_k} & : & c_0(\omega) & \longrightarrow & c_0(\omega) \\ & & x & \longrightarrow & e_k x \end{array}$$

Since e_k is idempotent then L_{e_k} is idempotent homomorphism of algebra $c_0(\omega)$, indeed:

$(L_{e_k}(xy) = e_k xy = e_k^2 xy = e_k x e_k y = L_{e_k}(x)L_{e_k}(y)$ because $c_0(\omega)$ is commutative)

Therefore, L_{e_k} is a φ-multiplier of $\mathbf{A}$.

Exemple 3.2 Let A be a non-commutative Banach algebra, E=$\{e \in \mathbf{A} : e^2 = e\}$ denote the set of idempotents in $\mathbf{A}$ and $C(\mathbf{A}) = \{x \in \mathbf{A} : ax = xa$, for all a in $\mathbf{A}\}$ the centre of $\mathbf{A}$.

The characterization of Banach algebras itempotents in $C(\mathbf{A})$ has been obtained by Zemanek ([10], Theorem 5.2), he showed that an idempotent $e \in E$ belongs to the centre $C(\mathbf{A})$ if and only if e is an isolated point in the set E. In other words, the centre of a Banach algebra meets the set of idempotents just in its isolated points.

Let $e \in E$ belongs in $C(\mathbf{A})$, and consider the multiplication operator by e. Then L_e is an idempotent homomorphism algebra for $\mathbf{A}^+$ (special Jordan algebra). Therefore, L_e is a φJ-multiplier of $\mathbf{A}^+$.

Proposition 3.1 *Let A be a non-commutative Banach algebra. If T is a φ-multiplier of A, then T is a φJ-multiplier in special Jordan algebra A^+.*

Proof. Let A be a non-commutative Banach algebra and T a φ-multiplier of A. Then we have:

$$\begin{aligned} T(x \circ y) &= T\left[\frac{1}{2}(x \cdot y + y \cdot x)\right] \\ &= \frac{1}{2}[T(x \cdot y) + T(y \cdot x)] \\ &= \frac{1}{2}[\varphi(x)T(y) + T(y)\varphi(x)] \\ &= \varphi(x) \circ T(y). \end{aligned}$$

$$\begin{aligned} T(x \circ y) &= T\left[\frac{1}{2}(x \cdot y + y.x)\right] \\ &= \frac{1}{2}[T(x \cdot y) + T(y \cdot x)] \\ &= \frac{1}{2}[T(x)\varphi(y) + \varphi(y)T(x)] \\ &= T(x) \circ \varphi(y). \end{aligned}$$

$$T(x \circ y) = \varphi(x) \circ T(y) = T(x) \circ \varphi(y).$$

Therefore, T is a φJ-multiplier in special Jordan algebra A^+. ∎

Remark 3.3 The definition of a φJ-multiplier of Jordan Banach algebra generalizes well that given in the case of Banach algebra.

For the adopted definition of a φJ-multiplier of Jordan Banach algebra, we have the following theorems the demonstration of which is exactly the same in the case of Banach algebras.

Theorem 3.1 *Let A be a Jordan-Banach algebra without order and φ be an idempotent homomorphism. Then the set $M_{\varphi J}(A)$ is a closed and commutative subalgebra of Banach algebra of continuous linear operators $\mathcal{L}(A)$, for the topology of simple convergence. Moreover if $A^2 = A$ and φ commutes with any φJ-multiplier of A, then $M_{\varphi J}(A)$ is commutative and without order.*

Theorem 3.2 *Let A be a Jordan-Banach algebra without order. Then $M_{\varphi J}(A)$ is complete in the strong operator topology.*

Proof. Suppose (T_λ) is a Cauchy net in the strong operator. Then, for each x in A, $(T_\lambda(x))$ is a Cauchy net in A and hence there exists $T(x)$ in A such that $\lim\limits_{\lambda \to +\infty} \|T_\lambda(z) - T(z)\| = 0$.

If x,y in A, then:

$$\begin{aligned} \|T(xy) - \varphi(x)T(y)\| &\leq \|T(xy) - T_\lambda(xy)\| + \|T_\lambda(xy) - \varphi(x)T(y)\| \\ \|T(xy) - \varphi(x)T(y)\| &\leq \|T(xy) - T_\lambda(xy)\| + \|\varphi(x)T_\lambda(y) - \varphi(x)T(y)\| \\ \|T(xy) - \varphi(x)T(y)\| &\leq \|T - T_\lambda\|\|xy\| + \|\varphi(x)\|\|T_\lambda - T\|\|y\| \end{aligned}$$

and so $T(xy) = \varphi(x)T(y)$.

The same way, we show that $T(xy) = T(x)\varphi(y)$, for all x, y in A. Therefore, T in $M_{\varphi J}(A)$ and $M_{\varphi J}(A)$ is complete in the strong operator topology. ∎

Theorem 3.3 *Let A be a Jordan-Banach algebra without order and φ be an isomorphism from A into A. If T is a φJ-multiplier of A, then the following statements are equivalent:*

i) *T is bijective.*

ii) *T^{-1} exists and $T^{-1} \in M_{\varphi^{-1}J}(A)$.*

Corollary 3.1 *Let A be a Jordan-Banach unitary algebra. Then, for every multiplier T, we have:*

$$Sp_{M_{\varphi J}(A)}(A) = Sp_{\mathcal{L}(A)}(T).$$

$M_{\varphi J}(A)$ is a full subalgebra de $\mathcal{L}(A)$.

Remark 3.4 Like all Jordan Banach algebra are commutative, we have the following theorem whose proof is exactly the same as in the case of Banach algebras.

Theorem 3.4 *Let A be a Jordan Banach algebra without order and φ be a homomorphism from A to A with dense range. Then T is a φJ-multiplier if, and only if, $Tx^2 = \varphi(x)T(x)$ for all x in A.*

Corollary 3.2 *Let A be a Jordan Banach Algebra without order and T a linear map from A to A. Then T is a φJ-multiplier of A if, and only if, $Tx^2 = xT(x)$ for all x in A.*

Proof. Just take $\varphi(x) = x$. ∎

Proposition 3.2 *Let A be a Banach algebra and T a φ-multiplier of A. Then:*

i) *We have*

$$T \circ U_x \circ T = U_{\varphi(x)} \circ T^2, \quad \forall x \in A. \tag{3.2}$$

ii) (3.2) *remains true in special algebra A^+.*

Proof. Let A be a Banach algebra and $T \in M_{\varphi J}(A)$.

i) We have :

$$U_x(T(y)) = x^2 \cdot T(y), \quad \forall x, y \in A$$

Therefore:

$$\begin{aligned} T[U_x(T(y))] &= T[x^2 \cdot T(y)] \\ &= \varphi(x^2) \cdot T[T(y)] \\ &= [\varphi(x)]^2 \cdot T[T(y)] \\ &= U_{\varphi(x)}[T^2(y)] \end{aligned}$$

Therefore we have :

$$T[U_x(T(y))] = U_{\varphi(x)}[T^2(y)], \quad \forall x, y \in A$$

ii) In every special Jordan algebra, we have:

$$U_x(y) = x \cdot y \cdot x, \quad \forall x, y \in A$$

Therefore, we have:

$$U_x(T(y)) = x \cdot T(y) \cdot x.$$

Thus,

$$\begin{aligned} T[U_x(T(y))] &= T[x \cdot T(y) \cdot x] \\ &= \varphi(x)T[T(y) \cdot x] \\ &= \varphi(x) \cdot T[T(y)] \cdot \varphi(x) \\ &= U_{\varphi(x)}[T^2(y)] \end{aligned}$$

whence

$$U_{\varphi(x)}[T^2(y)] = T \circ U_x \circ T(y), \quad \forall x, y \in A.$$

Taking this into account, it is normal to ask ourselves the following questions: Given a Jordan-Banach algebra A and a φJ-multiplier T, is the (3.2) relation verified? This is the purpose of the following proposition:

Proposition 3.3 *Let A be a Jordan-Banach algebra. If T is φJ-multiplier of A, we have:*

$$T \circ U_x \circ T = U_{\varphi(x)} \circ T^2, \quad \forall x \in A$$

Proof. Let $T \in M_{\varphi J}(A)$ and $x, y \in A$. Then we have:

$$\begin{aligned} TU_xT(y) &= T[U_x(T(y))] \\ &= T[2x \cdot (x \cdot T(y)) - x^2 \cdot T(y)] \\ &= 2\varphi(x) \cdot T[x \cdot T(y)] - \varphi(x^2) \cdot T[T(y)] \\ &= 2\varphi(x) \cdot [\varphi(x) \cdot T[T(y)] - [\varphi(x)]^2 \cdot T^2(y) \\ &= 2\varphi(x) \cdot [\varphi(x) \cdot T^2(y) - [\varphi(x)]^2 \cdot T^2(y)) \\ &= U_{\varphi(x)} \circ T^2(y), \quad \forall x, y \in A \end{aligned}$$

Consequently, $U_{\varphi(x)} \circ T^2 = T \circ U_x \circ T$, for all element x in A. ∎

Proposition 3.4 *Let A be a Jordan-Banach algebra. If T is a φJ-multiplier of A, then for x and y elements of A, we have:*

$$T[U_x(y)] = U_{\varphi(x)}[T(y)] \tag{3.3}$$

Proof. Let T be a φJ-multiplier of A, x and y elements of A. Then we have:

$$\begin{aligned} T[U_x(y)] &= T[2x \cdot (x \cdot y) - x^2 \cdot y] \\ &= \varphi(2x) \cdot T(x \cdot y) - \varphi(x^2) \cdot T(y) \\ &= 2\varphi(x) \cdot [\varphi(x) \cdot T(y)] - \varphi(x)^2 \cdot T(y) \\ &= U_{\varphi(x)}[T(y)]. \end{aligned}$$

∎

Remark 3.5 Let A be a Jordan-Banach algebra. If T is a φJ-multiplier of A, then for x and y elements of A, we have:

$$T[U_{\varphi(x)}(y)] = U_{\varphi(x)}[T(y)] \tag{3.4}$$

Proof. Let T be a φJ-multiplier of A, x and y elements of A. Then we have:

$$\begin{aligned} T[U_{\varphi(x)}(y)] &= T[2\varphi(x)\cdot(\varphi(x)\cdot y) - (\varphi(x))^2\cdot y] \\ &= 2\varphi(\varphi(x))T(\varphi(x)\cdot y) - \varphi((\varphi(x))^2)T(y) \\ &= 2\varphi(x)[\varphi(x)T(y)] - [\varphi(x)]^2 T(y) \\ &= U_{\varphi(x)}[T(y)] \end{aligned}$$ ∎

Remark 3.6 The previous Proposition 3.3 and Remark 3.4. suggest to define a new notion which is that of φJ-multiplier quadratic; this is what we are doing in the next section.

4. φJ-Multipliers quadratic of Jordan-Banach algebra

Definition 4.3 Let A be Jordan-Banach algebra, φ an idempotent homomorphism A and $T: A \longrightarrow A$ a continuous linear application. We say that T is a φJ-multiplier quadratic of A if, we have:

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \forall x, y \in A$$

The set of a φJ-multiplier quadratic of A is noted $M_{\varphi JQ}(A)$.

Remark 4.7 φ is an element of $M_{\varphi JQ}(A)$.

Proposition 4.5 *Let A be Banach algebra. If T is a φ-multiplier of A, then we have:*

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \forall x, y \in A$$

Proof. T is a φ-multiplier of A, then we have:

$$T(x.y) = T(x).\varphi(y) = \varphi(x).T(y), \forall x, y \in A$$

Therefore,

$$\begin{aligned} T[U_x(y)] &= T(x^2\cdot y) = T[x\cdot(x\cdot y)] = \varphi(x)\cdot T(x\cdot y) \\ &= \varphi(x)\cdot[\varphi(x)\cdot T(y)] = \varphi(x)^2\cdot T(y) = U_{\varphi(x)}[T(y)]. \end{aligned}$$ ∎

Proposition 4.6 *Let A be Banach algebra. If T is a φ-multiplier of A, then T is a φJ-multiplier quadratic in the special Jordan algebra A^+.*

Proof. As in A^+, $U_x(y) = x.y.x, \forall x, y \in A$, we have:

$$T[U_x(y)] = T(x\cdot y\cdot x) = \varphi(x)\cdot T(y\cdot x) = \varphi(x)\cdot T(y)\cdot\varphi(x) = U_{\varphi(x)}[T(y)].$$ ∎

Remark 4.8 The definition of φJ-multiplier quadratic well generalizes that of φ-multiplier given in the case of Banach.

Remark 4.9 Let A be Jordan-Banach algebra. Then we have

$$M_{\varphi J}(A) \subset M_{\varphi JQ}(A).$$

In what follows, $\mathcal{L}(A)$ will denote the Banach algebra of all continuous linear operators from A to A.

Theorem 4.5 *Let A be Jordan-Banach algebra. Then, $M_{\varphi JQ}(A)$ is a closed subalgebra of $\mathcal{L}(A)$, for the topology of simple convergence, which contains φ.*

Proof. Let T and S be two elements of $M_{\varphi JQ}(A)$ and $\alpha \in \mathbb{C}$. For any two elements x and y of A, we have:

$$U_{\varphi(x)}[(\alpha T)(y)] = U_{\varphi(x)}[\alpha(T(y))] = \alpha U_{\varphi(x)})(T(y)) = \alpha[T(U_x(y)].$$

Therefore, $\alpha T \in M_{\varphi JQ}(A)$.

$$(TS)[U_x(y)] = T[S(U_x(y)] = T[U_{\varphi(x)}(S(y))] = U_{\varphi(\varphi(x))}[T(S(y))] = U_{\varphi(x))}[(TS)(y))]$$

Therefore, $TS \in M_{\varphi JQ}(A)$.

$$\begin{aligned} U_{\varphi(x)}[(T+S)(y)] = U_{\varphi(x)}[T(y)+S(y)] &= U_{\varphi(x)}[T(y)] + U_{\varphi(x)}[S(y)] \\ &= T[U_x(y)] + S[U_x(y)] \\ &= (T+S)[U_x(y)] \end{aligned}$$

Therefore, $(T+S) \in M_{\varphi JQ}(A)$. Therefore, $M_{\varphi JQ}(A)$ is a subalgebra of $\mathcal{L}(A)$ which contains the φ. Let's show that $M(A)$ is closed in $\mathcal{L}(A)$ for the topology of simple convergence. $(T_\lambda)_{\lambda\in\Lambda}$ is a suite of elements of $M_{\varphi JQ}(A)$ and T an element de $\mathcal{L}(A)$ such that, for every element z de A, we have:

$$\lim_{\lambda\to+\infty} \|T_\lambda(z) - T(z)\| = 0$$

As with any $\lambda \in \Lambda$ and $T_\lambda \in M_{\varphi JQ}(A)$, we have:

$$U_{\varphi(x)}[T_\lambda(y)] = T_\lambda[U_x(y)], \ \forall x, y \in A$$

Therefore:

$$\|T[U_x(y)] - U_{\varphi(x)}[T(y)]\| \leq \|T[U_x(y)] - T_\lambda[U_x(y)]\| + \|U_{\varphi(x)}[T_\lambda(y) - U_{\varphi(x)}[T(y)]\|$$

Since the application $z \to U_x(z)$ is continuous on A and $(T_\lambda(z))_\lambda$ converges to $T(z)$ for every z in A then: $(\forall x, y \in A), (\forall \epsilon > 0), (\exists \lambda \in \Lambda)$ such that

$$\begin{aligned} \|T[U_x(y)] - T_\lambda[U_x(y)]\| &\leq \epsilon/2 \\ \|U_{\varphi(x)}[T_\lambda(y)] - U_{\varphi(x)}[T(y)]\| &\leq \epsilon/2. \end{aligned}$$

Therefore,

$$\|T[U_x(y)] - U_{\varphi(x)}[T(y)]\| \leq \epsilon.$$

Then, we have:

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \forall x, y \in A,$$

i.e., $T \in M_{\varphi JQ}(A)$. ∎

Remark 4.10 For φJ multipliers, we have shown that if $A^2 = A$, then $M_{\varphi J}$ is without order. We will show that in the quadratic case, we only assume that A is unitary. Hence we have the following proposition.

Proposition 4.7 *Let A be Jordan-Banach algebra with identity e. Then, $M_{\varphi JQ}(A)$ is without order.*

Proof. Let T a element of $M_{\varphi JQ}(A)$ such that $ST = 0$ for each $T \in M_{\varphi JQ}(A)$. We have, for all x, y in A

$$\begin{aligned} T[U_x(y)] &= U_{\varphi(x)}[T(y)] \\ &= U_{\varphi(\varphi(x))}T(y) \\ &= \varphi[U_{\varphi(x)}(T(y)] \\ &= U_{\varphi(x)}[\varphi(T(y)] \end{aligned}$$

As φ is a φJ-multiplier quadratic and the application $z \to U_x(z)$ is linear, then $\varphi(T(y) = 0$. Therefore, $U_{\varphi(x)}[\varphi(T(y)] = 0$, where $T[U_x(y)] = 0$, $\forall x, y \in A$.

In particular, if we take $x = e$, we obtain $T(y) = 0$, for all y in A. Therefore, $T = 0$. ∎

Theorem 4.6 *Let A be Jordan-Banach algebra. Then, $M_{\varphi JQ}(A)$ is complete in the strong operator topology.*

Proof. Suppose $(T_\lambda)_{\lambda\in\Lambda}$ is a Cauchy net in the strong operator topology. Then for each element z in A, $(T_\lambda(z))_{\lambda\in\Lambda}$ is a cauchy net in A that is complete and hence there exists $T(z)$ in A such that:

$$\lim_{\lambda\to+\infty} \|T_\lambda(z) - T(z)\| = 0.$$

It is conventional to show that T is linear continuous and that

$$\lim_{\lambda\to+\infty} \|T_\lambda - T\| = 0.$$

For any two elements x and y of A, we have:

$$\|T[U_x(y)] - U_{\varphi(x)}[T(y)]\| \le \|T[U_x(y)] - T_\lambda[U_x(y)]\| + \|U_{\varphi(x)}[T_\lambda(y) - U_{\varphi(x)}[T(y)]\|$$
$$\|T[U_x(y)] - U_{\varphi(x)}[T(y)]\| \le \|(T - T_\lambda)(U_x(y))\| + \|(U_{\varphi(x)})[T_\lambda(y) - T(y)]\|$$

Since T_λ, T and $z \to U_x(z)$ are linear and continuous applications on A, then:

$$\|(T - T_\lambda)(U_x(y))\| \le \|T - T_\lambda\|\|U_x(y)\|$$

and

$$\|(U_{\varphi(x)})[T_\lambda(y) - T(y)]\| \le \|U_{\varphi(x)}\|\|T - T_\lambda\|\|y\|.$$

We have

$$\lim_{\lambda \to +\infty} \|T_\lambda - T\| = 0.$$

Then

$$\|T[U_x(y)] - U_{\varphi(x)}[T(y)]\| = 0,$$

and so

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \forall x, y \in A.$$

Therefore, $T \in M_{\varphi JQ}(A)$ and $M_{\varphi JQ}(A)$ is complete in the strong operator topology. ∎

Theorem 4.7 *Let A be a Jordan-Banach algebra without order and φ be an isomorphism from A into A. If T is a φJ-multiplier quadratic of A, then the following statements are equivalent:*

i) *T is bijective.*

ii) *T^{-1} exists and $T^{-1} \in M_{\varphi^{-1}JQ}(A)$.*

Proof.

ii) $\Rightarrow$ i) Obvious

i) $\Rightarrow$ ii) $T^{-1}$is linear. It is continuous according to the theorem of open application. On the other hand, we have by hypothesis:

$$T[U_x(y)] = U_{\varphi(x)}[T(y)], \ \forall x, y \in A$$

Let x and z are two elements of A. We pose $y = T^{-1}(z)$.

Then, we have

$$T[U_{\varphi^{-1}(x)}(T^{-1}(z))] = U_x[T(T^{-1}(z))] = U_x(z).$$

Thus, we obtain

$$U_{\varphi^{-1}(x)}(T^{-1}(z)) = T^{-1}[U_x(z)], \ \forall x, z \in A.$$

Hence $T^{-1} \in M_{\varphi^{-1}JQ}(A)$. ∎

Corollary 4.3 *Let A be Jordan-Banach algebra without order and $T \in M_{\varphi JQ}(A)$. Then, we have*

$$Sp_{M_{\varphi JQ}(A)}(T) = Sp_{\mathcal{L}(A)}(T)$$

i.e., $M_{\varphi JQ}(A)$ is a full subalgebra of $\mathcal{L}(A)$.

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