EXP-FUNCTION METHOD USING MODIFIED RIEMANN–LIOUVILLE DERIVATIVE FOR SINGULARLY PERTURBED BOUSSINESQ EQUATIONS OF FRACTIONAL-ORDER

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Abstract. This paper witnesses the combination of an efficient transformation and Exp-function method to construct generalized solitary wave solutions of the nonlinear singularly perturbed sixth-order Boussinesq equations of fractional-order. Computational work and subsequent numerical results re-confirm the efficiency of proposed algorithm. It is observed that suggested scheme is highly reliable and may be extended to other nonlinear differential equations of fractional order.

Keywords: perturbed sixth-order Boussinesq equations, fractional calculus, exp-function method, modified Riemann-Liouville derivative.

1. Introduction

The subject of factional calculus [1], [2] is a rapidly growing field of research, at the interface between chaos, probability, differential equations, and mathematical physics. In recent years, nonlinear fractional differential equations (NFDEs) have gained much interest due to exact description of nonlinear phenomena of many real-time problems. The fractional calculus is also considered as a novel topic [3], [4]; has gained considerable popularity and importance during the recent past. It has been the subject of specialized conferences, workshops and treatises or so, mainly due to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. Some of the areas of present-day applications of fractional models [5]–[8] include fluid flow, solute transport or dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, material viscoelastic theory, electromagnetic theory, dynamics of earthquakes, control theory of dynamical systems, optics and signal processing, bio-sciences, economics, geology, astrophysics, probability and statistics, chemical

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physics, and so on. As a consequence, there has been an intensive development of the theory of fractional differential equations, see [1]–[8] and the references therein. Recently, He and Wu [9] developed a very efficient technique which is called exp-function method for solving various nonlinear physical problems. The through study of literature reveals that Exp-function method has been applied on a wide range of differential equations and is highly reliable. The exp-function method has been extremely useful for diversified nonlinear problems of physical nature and has the potential to cope with the versatility of the complex nonlinearities of the problems. The subsequent works have shown the complete reliability and efficiency of this algorithm. He et.al. [10]–[11] used this scheme to find periodic solutions of evolution equations; Mohyud-Din [12-13] extended the same for nonlinear physical problems including higher-order BVPs; Oziz [14] tried this novel approach for Fisher's equation; Wu et.al. [15], [16] for the extension of solitary, periodic and compacton-like solutions; Yusufoglu [17] for MBBN equations, Zhang [18] for high-dimensional nonlinear evolutions; Zhu [19], [20] for the Hybrid-Lattice system and discrete m KdV lattice; Kudryashov [21] for exact soliton solutions of the generalized evolution equation of wave dynamics; Momani [22] for an explicit and numerical solutions of the fractional KdV equation; The basic motivation of this paper is the development of an efficient combination comprising an efficient transformation, exp-function method using Jumarie's derivative approach [23]–[26] and its subsequent application to construct generalized solitary wave solutions of the nonlinear singularly perturbed sixth-order Boussinesq equations of fractional-order [27]–[28]. It is to be highlighted that Ebaid [29] proved that $c = d$ and $p = q$ are the only relations that can be obtained by applying exp-function method to any nonlinear ordinary differential equation.

Theorem 1 [29] *Suppose that $u^{(r)}$ and $\left(u^{(\gamma)}\right)^{\lambda}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and γ are both positive integers. Then the balancing procedure using the Exp-function ansatz; $U(\eta) = \dfrac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}$, leads to $c = d$ and $p = q, \forall r, s, \lambda \geq 1$.*

Theorem 2 [29] *Suppose that $u^{(r)}$ and $u^{(s)}u^{k}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and$p = q, \forall r, s, k \geq 1$.*

Theorem 3 [29] *Suppose that $u^{(r)}$ and$(u^{(s)})^{\Omega}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where $r, s$and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and$p = q, \forall r, s \geq 1, \forall \Omega \geq 2$.*

Theorem 4 [29] *Suppose that $u^{(r)}$ and $(u^{(s)})^{\Omega}u^{\lambda}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r, s, Ω and λ are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c = d$ and$p = q$, $\forall r, s, \Omega, \lambda \geq 1$.*

2. Jumarie's fractional derivative

Jumarie's fractional derivative is a mo1dified Riemann-Liouville derivative defined as [27]–[30]

$$(1) \qquad D_t^{\alpha} f(x) = \left\{ \begin{array}{l} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (x-t)^{-\alpha-1} (f(t) - f(0))\, dt, \alpha \leq 0, \\ \dfrac{1}{\Gamma(-\alpha)} \dfrac{d}{dx} \displaystyle\int_0^x (x-t)^{-\alpha} (f(t) - f(0))\, dt, 0 \leq \alpha \leq 1 \\ \left[f^{\alpha-n}(x)^n\right]^n, n \leq \alpha \leq n+1, n \geq 1 \end{array} \right\}$$

where f : R → R, x → f(x) denotes a continuous (but not necessarily differentiable) function.Some useful formulas and results of Jumarie's modified Riemann–Liouville derivative were summarized in [27]–[28].

$$(2) \qquad D_x^{\alpha} c = 0, \ \alpha \geq 0, \ c = \text{constant}$$

$$(3) \qquad D_x^{\alpha} [cf(x)] = cD_x^{\alpha} f(x) \, \alpha \geq 0, \ c = \text{constant}$$

$$(4) \qquad D_x^{\alpha} x^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}, \ \beta \geq \alpha \geq 0,$$

$$(5) \qquad D_x^{\alpha} [f(x)\, g(x)] = [D_x^{\alpha} f(x)\, g(x) + f(x) [D_x^{\alpha} g(x)] \, .$$

$$(6) \qquad D_x^{\alpha} f(x(t)) = f_x'(x) \, .x^{\alpha}(t) \, .$$

3. Exp-function method [28]–[31]

We consider the general nonlinear FPDE of the type

$$(7) \qquad P(u, \, u_t, \, u_x, u_{xx}\, u_{xxx}, ..., D_t^{\alpha} u, D_x^{\alpha} u, D_{xx}^{\alpha} u, ...) = 0, \quad 0 < \alpha \leq 1,$$

where $D_t^{\alpha} u, D_x^{\alpha} u, D_{xx}^{\alpha} u$ are the modified Riemann-Liouville derivative of u with respect to t, x, xx, respectively.

Using a transformation [32]

$$(8) \qquad \eta = kx + \frac{\omega t^{\alpha}}{\Gamma(1+\alpha)} + \eta_0, k, \omega, \eta_0 \text{ are all constants with } k, \omega \neq 0$$

we can rewrite equation (7) in the following nonlinear ODE;

$$(9) \qquad Q(u, \, u', u'', u''', u^{\text{iv}}) = 0,$$

where the prime denotes derivative with respect to η.

According to the Exp-function method, we assume that the wave solution can be expressed in the following form

$$(10) \qquad u(\eta) = \frac{\sum_{n-c}^{d} a_n \exp[n\eta]}{\sum_{m-p}^{q} b_m \exp[m\eta]}$$

where p, q, c and d are positive integers which are known to be further determined ,a_n and b_m are unknown constants. We can rewrite equation (4) in the following equivalent form

$$u(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \tag{11}$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems. To determine the value of q and p by using [25],

$$p = c, \ q = d \tag{12}$$

4. Solution procedure

In this section, we apply the exp-function method to construct generalized solitary solutions for Burger's Equations of fractional-order. Numerical results are very encouraging. In this section, we apply exp-function method to construct generalized solitary of the singularly perturbed sixth-order Boussinesq equations. Numerical results are very encouraging.

Example 4.1 Consider the singularly perturbed sixth-order Boussineqes equation where a and k are arbitrary constants.

$$u_{tt} = u_{xx} + (p(u))_{xx} + \beta u_{xxxx} + \delta u_{xxxxxx}$$

Taking $\beta = 1, \ \delta = 1$ and $p(u) = 3u^2$, the model equation is given as

$$u_{tt} = u_{xx} + 3(u^2)_{xx} + u_{xxxx} \tag{13}$$

with the initial conditions

$$u(x,0) = \frac{2ak^2 e^{kx}}{(1 + ae^{kx})}, u_t(x,0) = \frac{2ak^3\sqrt{1+k^2}\left(1 - ae^{kx}\right)e^{kx}}{(1 + ae^{kx})^3}$$

Using (8), equation (13) can be converted to an ordinary differential equation

$$(\omega^2 + k^2)\, u'' + 6k^2(u'^2 + u''u)uu''' + k^4 u^{(\text{iv})} = 0, \tag{14}$$

where the prime denotes the derivative with respect to η. The solution of the equation (13) can be expressed in the form, equation (11). To determine the value of c and$p$, by using [25],

$$p = c, q = d \tag{15}$$

Case 4.1.1. We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of $c$and$d$. For simplicity, we set $p = c = 1$ and $q = d = 1$ equation (11) reduces to

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}. \tag{16}$$

Substituting equation (16) into equation (14), we have

$$\frac{1}{A}\left[\begin{array}{l} c_5\exp(5\eta)+c_4\exp(4\eta)+c_3\exp(3\eta)+c_2\exp(2\eta)+ \\ c_1\exp(\eta)+c_0+c_{-1}\exp(-\eta)+c_{-2}\exp(-2\eta)+ \\ c_{-3}\exp(-3\eta)+c_{-4}\exp(-4\eta)+c_{-5}\exp(-5\eta) \end{array}\right]=0 \tag{17}$$

$A=5\left(b_1\exp(\eta)+b_0+b_{-1}\exp(-\eta)\right)^6$ where $c_i\ (i=-5,-4,...,4,5)$ are constants obtained by Maple 16.

Equating the coefficients of $\exp(n\eta)$ to be zero, we obtain

$$\left(\begin{array}{c} c_{-5}=0, c_{-4}=0, c_{-3}=0, c_{-2}=0, c_{-1}=0, c_0=0, \\ c_1=0, c_2=0, c_3=0, c_4=0, c_5=0 \end{array}\right) \tag{18}$$

By solving (18), we get different solutions of (13).

1^{st} Solution set:

$$\begin{aligned} a_{-1} &= \frac{-1}{24}\frac{b_0^2\left(K^4-\omega^2+K^2\right)}{K^2 b_1}, a_0=\frac{1}{6}\frac{b_0\left(5K^4+\omega^2-K^2\right)}{K^2}, \\ a_1 &= -\frac{1}{6}\frac{b_1\left(K^4-\omega^2+K^2\right)}{K^2}, b_{-1}=\frac{1}{4}\frac{b_0^2}{b_1}, b_0=b_0, b_1=b_1 \end{aligned} \tag{19}$$

Therefore, we obtained the following generalized solitary solution $u(x,t)$ of equation (13)

(20)

$$u(x,t)=\frac{\left(\frac{-1}{24}\frac{b_0^2\left(K^4-\omega^2+K^2\right)e^{-\frac{xK+\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{K^2 b_1}+\frac{1}{6}\frac{b_0\left(5K^4+\omega^2-K^2\right)}{K^2}-\frac{1}{6}\frac{b_1\left(K^4-\omega^2+K^2\right)e^{\frac{xK+\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{K^2}\right)}{\frac{1}{4}\frac{b_0^2 e^{-\frac{xK+\omega t^{\alpha}}{\Gamma(1+\alpha)}}}{b_1}+b_0+b_1 e^{\frac{xK+\omega t^{\beta}}{\Gamma(1+\alpha)}}}$$

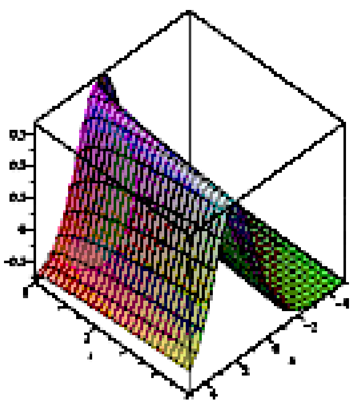

Figure 4.1(a) $\alpha=0.25$

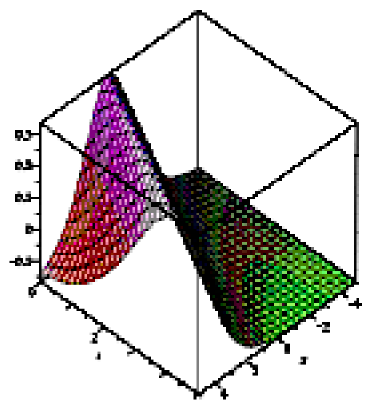

Figure 4.1(b) $\alpha=1$

Case 4.1.2. If $p=c=2$ and $q=d=1$, then equation (11) reduces to

$$u(\eta)=\frac{a_2\exp[2\eta]+a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_2\exp[2\eta]+b_1\exp[\eta]+a_0+b_{-1}\exp[-\eta]}. \tag{21}$$

Proceeding as before, we obtain

1^{st} Solution set:

$$\left\{ \begin{aligned} a_{-1} &= \frac{-1}{24}\frac{b_0^2\left(K^4-\omega^2+K^2\right)}{K^2 b_1}, a_0 = \frac{1}{6}\frac{b_0\left(5K^4+\omega^2-K^2\right)}{K^2}, \\ a_1 &= -\frac{1}{6}\frac{b_1\left(K^4-\omega^2+K^2\right)}{K^2}, b_{-1} = \frac{1}{4}\frac{b_0^2}{b_1}, b_0 = b_0, b_1 = b_1 \end{aligned} \right\}, \tag{22}$$

Hence we get the generalized solitary wave solution of equation (13) as follows (23)

$$u\left(x,t\right) = \frac{\left(\frac{-1}{24}\frac{b_0^2\left(K^4-\omega^2+K^2\right)e^{-\left(xK+\omega t^{\alpha}\right)}}{K^2 b_1} + \frac{1}{6}\frac{b_0\left(5K^4+\omega^2-K^2\right)}{K^2} - \frac{1}{6}\frac{b_1\left(K^4-\omega^2+K^2\right)e^{xK+\omega t^{\alpha}}}{K^2}\right)}{\frac{1}{4}\frac{b_0^2 e^{-\left(xK+\omega t^{\alpha}\right)}}{b_1} + b_0 + b_1 e^{xK+\omega t^{\alpha}}}$$

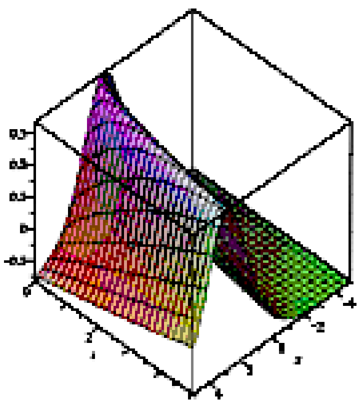

Figure 4.1(c) $\alpha = 0.25$

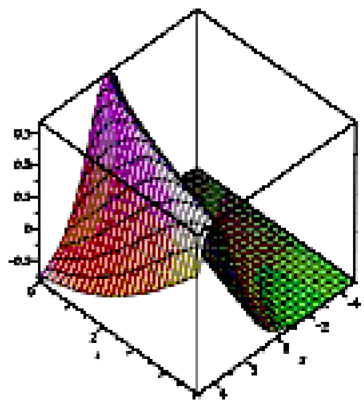

Figure 4.1(d) $\alpha = 1$

We get the same soliton solutions which clearly illustrate that final solution does not strongly depends upon these parameters.

5. Conclusion

In this paper, we applied exp-function method to construct generalized solitary solutions of the nonlinear fractional order singularly perturbed sixth-order Boussinesq equations. It is observed that the Exp-function method is very convenient to apply and is very useful for finding solutions of a wide class of nonlinear problems.

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