STABILITY OF NONLINEAR NABLA FRACTIONAL DIFFERENCE EQUATIONS USING FIXED POINT THEOREMS

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Abstract. Difference equations are often used to analyze sampled data systems, in which stability problems are considered to be important. This is evident from a large number of research papers dedicated to it. However, stability results for nonlinear nabla fractional difference equations are not yet reported. The present article discusses stability of nonlinear nabla fractional difference equations of Riemann–Liouville and Caputo type, using fixed point theory.

Keywords: fractional order, difference equation, zero solution, fixed point, stability.

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1. Introduction

Fractional calculus deals with the study of fractional order integrals and derivatives and their diverse applications [11]. The analogous theory for discrete fractional calculus was initiated and a series of papers continuing this research has appeared [3], [4], [5], [6], [7], [8], [9], [10], [12], [13].

Qualitative properties of fractional difference equations assume importance in the absence of closed form solutions. One such qualitative property, which has wide applications, is the stability of solutions. Motivated by the application of fixed point theory in the study of stability of differential equations, we use Krasnoselskii fixed point theorem, Schauder fixed point theorem and discrete Arzela–Ascoli theorem to discuss the stability of nonlinear nabla fractional difference equations.

The present article is organized as follows. Section 2 contains basic definitions and results concerning nabla discrete fractional calculus. In Sections 3 and 4, we establish sufficient conditions for stability of nonlinear nabla fractional difference equations of Riemann–Liouville and Caputo type. We illustrate our main results through few examples in Section 5.

2. Nabla discrete fractional calculus

Throughout the article, we consider the discrete time scale [2]

$$\mathbb{T} = \mathbb{N}_a = \{a, a+1, a+2, ...\},$$

where $a \in \mathbb{R}$ is fixed. For any function $f : \mathbb{N}_a \to \mathbb{R}$, the backward difference or nabla operator is defined as $\nabla f(t) = f(t) - f(t-1)$ for $t \in \mathbb{N}_{a+1}$.

Definition 2.1. For any real numbers α and t, the α rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} \setminus \{......, -2, -1, 0\}, \quad 0^{\overline{\alpha}} = 0. \tag{2.1}$$

Lemma 2.1. *For any real numbers a and b, the quotient expansion of two gamma functions at infinity is given by*

$$\frac{\Gamma(t+a)}{\Gamma(t+b)} = t^{a-b}\Big[1 + O\Big(\frac{1}{t}\Big)\Big], \quad |t| \to \infty.$$

Regarding the rising factorial function we observe the following properties.

Lemma 2.2. *Assume the following factorial functions are well defined. For any positive real numbers t, α and β,*

1. $\nabla t^{\overline{\alpha}} = \alpha t^{\overline{\alpha-1}}$.
2. $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.
3. $t^{\overline{\alpha+\beta}} = \sum_{j=0}^{t} C(t,j)(t-j)^{\overline{\alpha}} j^{\overline{\beta}}$.
4. *If $\alpha < t \le r$, then $t^{\overline{-\alpha}} \ge r^{\overline{-\alpha}}$.*
5. *If $\alpha \ge \beta$, then $t^{\overline{-\alpha}} \le t^{\overline{-\beta}}$.*

6. *If* $0 < \alpha < 1$, *then* $(t-\alpha\beta)^{\overline{\alpha\beta}} \geq \left[(t-\beta)^{\overline{\beta}}\right]^{\alpha}$.

7. *If* $\alpha > 1$, *then* $\left[(t+\beta)^{\overline{-\beta}}\right]^{\alpha} < \dfrac{\Gamma(1+\alpha\beta)}{[\Gamma(1+\beta)]^{\alpha}}(t+\alpha\beta)^{\overline{-\alpha\beta}}$.

Proof. The proofs of (1), (2), (3) are straightforward. The proof of (4) follows from Eulers infinite product

$$\Gamma(t) = \frac{1}{t}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^{t}}{\left(1+\frac{t}{n}\right)}.$$

Consider

$$\begin{aligned} t^{\overline{-\alpha}} = \frac{\Gamma(t-\alpha)}{\Gamma(t)} &= \frac{t}{t-\alpha}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-\alpha}\left(\frac{n+t}{n+t-\alpha}\right) \\ &\geq \frac{r}{r-\alpha}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-\alpha}\left(\frac{n+r}{n+r-\alpha}\right) \\ &= \frac{\Gamma(r-\alpha)}{\Gamma(r)} = r^{\overline{-\alpha}}. \end{aligned}$$

The proof of (5) follows from Lemma 2.1. Consider

$$\frac{t^{\overline{-\alpha}}}{t^{\overline{-\beta}}} = \frac{\Gamma(t-\alpha)}{\Gamma(t-\beta)} = t^{\beta-\alpha}\left[1+O\left(\frac{1}{t}\right)\right] = \frac{1}{t^{\alpha-\beta}}\left[1+O\left(\frac{1}{t}\right)\right] \leq 1.$$

For the proofs of (6) and (7), we refer [4], [5]. ∎

Definition 2.2. (Nabla Fractional Sum [8], [13]) Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\alpha > 0$ be given. Then the α^{th}-order nabla fractional sum of f is given by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}} \text{ for } t \in \mathbb{N}_a \tag{2.2}$$

where $\rho(s) = s-1$. Also, we define the trivial sum by $\nabla_a^{-0} f(t) = f(t)$ for $t \in \mathbb{N}_a$.

Definition 2.3. (R-L Nabla Fractional Difference [8], [13]) Let $f : \mathbb{N}_a \to \mathbb{R}$, $\alpha > 0$ be given and let $N \in \mathbb{N}$ be chosen such that $N-1 < \alpha \leq N$. Then, the α^{th}-order Riemann–Liouville nabla fractional difference of f is given by

$$\nabla_a^{\alpha} f(t) = \nabla^N \nabla_a^{-(N-\alpha)} f(t) \text{ for } t \in \mathbb{N}_{a+N}. \tag{2.3}$$

For $\alpha = 0$, we get $\nabla_a^0 f(t) = f(t)$ for $t \in \mathbb{N}_a$.

Definition 2.4. (Caputo Nabla Fractional Difference [12]) Let $f : \mathbb{N}_a \to \mathbb{R}$, $\alpha > 0$ be given and let $N \in \mathbb{N}$ be chosen such that $N-1 < \alpha \leq N$. Then, the α^{th}-order Caputo nabla fractional difference of f is given by

$$\nabla_{a*}^{\alpha} f(t) = \nabla_a^{-(N-\alpha)}\left[\nabla^N f(t)\right] \quad \text{for} \quad t \in \mathbb{N}_{a+N}. \tag{2.4}$$

For $\alpha = 0$, we set $\nabla_{a*}^0 f(t) = f(t)$ for $t \in \mathbb{N}_a$.

Theorem 2.3. [7], [13] (Power Rule) *Let $\alpha > 0$ and $\mu > -1$. Then, for $t \in \mathbb{N}_a$, we have*

1. $\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \dfrac{\Gamma(\mu+1)}{\Gamma(\alpha+\mu+1)}(t-a)^{\overline{\alpha+\mu}}.$

2. $\nabla_a^{\alpha}(t-a)^{\overline{\mu}} = \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}(t-a)^{\overline{\mu-\alpha}}.$

Let $f(t,r) : \mathbb{N}_a \times \mathbb{R} \to \mathbb{R}$, $u(t) : \mathbb{N}_a \to \mathbb{R}$ and $0 < \alpha < 1$. Consider a nonlinear nabla fractional difference equation of Riemann–Liouville type together with an initial condition of the form

$$\nabla_{a-1}^{\alpha} u(t) = f(t, u(t)), \quad t \in \mathbb{N}_{a+1}, \tag{2.5}$$

$$\nabla_{a-1}^{-(1-\alpha)} u(t)\Big|_{t=a} = u(a) = u(0). \tag{2.6}$$

From [13], $u(t)$ is a solution of the initial value problem (2.5)–(2.6) if and only if $u(t)$ has the following representation

$$u(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)), \quad t \in \mathbb{N}_a. \tag{2.7}$$

If we consider a nonlinear nabla fractional difference equation of Caputo type together with an initial condition of the form

$$\nabla_{a*}^{\alpha} u(t) = f(t, u(t)), \quad t \in \mathbb{N}_{a+1}, \tag{2.8}$$

$$u(a) = u_0. \tag{2.9}$$

Then from [13], $u(t)$ is a solution of the initial value problem (2.8)–(2.9) if and only if $u(t)$ has the following representation

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s, u(s)), \quad t \in \mathbb{N}_a. \tag{2.10}$$

Definition 2.5. The solution $u = \varphi(t)$ of the initial value problem (2.5)–(2.6) or (2.8)–(2.9) is said to be

1. stable, if given $\epsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(\epsilon, t_0)$ such that $|u_0 - \varphi(t_0)| < \delta \Rightarrow |u(t, x_0, t_0) - \varphi(t)| < \epsilon$ for all $t \geq t_0$.
2. attractive, if there exists $\eta = \eta(t_0)$ such that $|u_0 - \varphi(t_0)| < \eta$ implies $u(t, x_0, t_0) \to \varphi(t)$ as $t \to \infty$.
3. asymptotically stable if it is stable and attractive.

Definition 2.6. The space l_a^∞ is the set of real sequences defined on the set of positive integers where any individual sequence is bounded with respect to the usual supremum norm. Clearly, l_a^∞ is a Banach space under the supremum norm.

Definition 2.7. A set Ω of sequences in l_a^∞ is uniformly Cauchy (or equi–Cauchy), if for every $\epsilon > 0$, there exists a positive integer N such that $|u(i) - u(j)| < \epsilon$, whenever i, $j > N$ for any $u = u(t)$ in Ω.

Theorem 2.4. (Discrete Arzela-Ascoli's theorem) *A bounded, uniformly Cauchy subset Ω of l_a^∞ is relatively compact.*

Theorem 2.5. (Krasnoselskii's fixed point theorem) *Let S be a nonempty, closed, convex and bounded subset of the Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators such that*

1. *A is a contraction with constant $L < 1$,*
2. *B is continuous, BS resides in a compact subset of X,*
3. *$[x = Ax + By, y \in S] \Longrightarrow x \in S$.*

Then the operator equation $Ax + Bx = x$ has a solution in S.

Theorem 2.6. (Schauder fixed point theorem) *Let Ω be a closed, convex and nonempty subset of a Banach space X. Let $T : \Omega \to \Omega$ be a continuous mapping such that $T\Omega$ is a relatively compact subset of X. Then T has at least one fixed point in Ω. That is, there exists an $x \in \Omega$ such that $Tx = x$.*

3. Riemann–Liouville type fractional difference equation

Let l_a^∞ be the set of all real sequences $u = \{u(t)\}_{t=a}^\infty$ with norm $\|u\| = \sup_{t\in\mathbb{N}_a} |u(t)|$, then l_a^∞ is a Banach space. Define the operators

$$Pu(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}u_0 + \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,u(s)). \tag{3.1}$$

$$Au(t) = \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}u_0. \tag{3.2}$$

$$Bu(t) = \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,u(s)). \tag{3.3}$$

Lemma 3.1. *Let the following condition be satisfied:*

(I) *There exist constants $\beta_1 \in (\alpha, 1)$ and $L_1 \geq 0$ such that*

$$|f(t,u(t))| \leq L_1(t-a)^{\overline{-\beta_1}}, \ \textit{for } t \in \mathbb{N}_{a+1}. \tag{3.4}$$

Define

$$S_1 = \{u(t) : |u(t)| \leq (t-a)^{\overline{-\gamma_1}} \ \textit{for } t \in \mathbb{N}_{a+n_1+1}\} \tag{3.5}$$

where $\gamma_1 = (-1/2)(\alpha - \beta_1)$ *and* $n_1 \in \mathbb{N}$ *such that*

$$\frac{|u_0|}{\Gamma(\alpha)}(n_1-\gamma_1)^{\overline{(1/2)(\alpha+\beta_1)-1}} + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(n_1-\gamma_1)^{\overline{-\gamma_1}} \le 1. \tag{3.6}$$

Then the operator B *is continuous and* BS_1 *is a compact subset of* $\mathbb{R}$ *for* $t \in \mathbb{N}_{a+n_1+1}$.

Proof. Clearly, $\gamma_1 > 0$. Using Lemma 2.1, for $t \in \mathbb{N}_{a+1}$,

$$(t-a)^{\overline{-\gamma_1}} = \frac{\Gamma(t-a-\gamma_1)}{\Gamma(t-a)} = (t-a)^{-\gamma_1}\Big[1 + O\Big(\frac{1}{t-a}\Big)\Big].$$

Further, $(t-a)^{-\overline{\gamma}_1} \to 0$ as $t \to \infty$. Then, there exists $n_1 \in \mathbb{N}$ such that $(t-a)^{-\overline{\gamma}_1} \to 0$ for $(t-a) > n_1$. Clearly $\gamma_1 < 1 - (1/2)(\alpha+\beta_1)$ and $n_1 - \gamma_1 < t - a$. Using Lemma 2.2, we get $(n_1-\gamma_1)^{\overline{(1/2)(\alpha+\beta_1)-1}} \le (n_1-\gamma_1)^{\overline{-\gamma_1}} \le (t-a)^{-\overline{\gamma}_1} \to 0$ and $(n_1-\gamma_1)^{\overline{-\gamma_1}} \le (t-a)^{-\overline{\gamma}_1} \to 0$. Thus, we have the inequality (3.6) which implies that the set S_1 exists. Now, we show that B maps S_1 in S_1. Clearly, S_1 is a closed, bounded, and convex subset of $\mathbb{R}$. Applying condition (I), Theorem 2.3, Lemma 2.2 and (3.4), we have

$$\begin{aligned}
|Bu(t)| &\le \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}|f(s,u(s))| \\
&\le \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}L_1(s-a)^{-\overline{\beta}_1} \\
&= L_1\nabla_a^{-\alpha}(t-a)^{-\overline{\beta}_1} \\
&= \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(t-a)^{\overline{\alpha-\beta_1}} \\
&= \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(t-a-\gamma_1)^{-\overline{\gamma}_1}(t-a)^{-\overline{\gamma_1}} \\
&\le \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(n_1-\gamma_1)^{-\overline{\gamma_1}}(t-a)^{-\overline{\gamma}_1} \\
&\le (t-a)^{-\overline{\gamma}_1},
\end{aligned}$$

implies $BS_1 \subset S_1$ for $t \in \mathbb{N}_{a+n_1+1}$. Next, we show that B is continuous on S_1. Let $\epsilon > 0$ be given. Then, there exists $T_1 \in \mathbb{N}$ and $T_1 \ge n_1$ such that, for $t \in \mathbb{N}_{a+T_1+1}$,

$$\frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(t-a)^{\overline{\alpha-\beta_1}} < \frac{\epsilon}{2}. \tag{3.7}$$

Since $(t-a) > T_1 \ge n_1$ and $0 < \gamma_1 < \beta_1 - \alpha < 1$, we have

$$(t-a)^{\overline{\alpha-\beta_1}} \le (t-a)^{-\overline{\gamma}_1} \to 0,$$

which implies the validity of (3.7). Let $\{u_n\}$ be a sequence such that $u_n \to u$. For $t \in \{a+n_1+1, a+n_1+2, ..., a+T_1\}$, applying the continuity of f, we have

$$
\begin{aligned}
&|Bu_n(t) - Bu(t)| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} |f(s,u_n(s)) - f(s,u(s))| \\
&\quad \leq \Big[\frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}}\Big]\Big[\max_{s\in\{a+1,a+2,...,a+T_1\}} |f(s,u_n(s)) - f(s,u(s))|\Big] \\
&\quad = \frac{(t-a)^{\overline{\alpha}}}{\Gamma(\alpha+1)}\Big[\max_{s\in\{a+1,a+2,...,a+T_1\}} |f(s,u_n(s)) - f(s,u(s))|\Big] \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

For $t \in \mathbb{N}_{a+T_1+1}$,

$$
\begin{aligned}
|Bu_n(t) - Bu(t)| &\leq \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} [|f(s,u_n(s))| + |f(s,u(s))|] \\
&\leq \frac{2L_1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} (s-a)^{-\overline{\beta}_1} \\
&= 2L_1 \nabla_a^{-\alpha} (t-a)^{\overline{-\beta_1}} = \frac{2L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t-a)^{\overline{\alpha-\beta_1}} < \epsilon.
\end{aligned}
$$

Thus, for all $t \in \mathbb{N}_{a+n_1+1}$, $|Bu_n(t) - Bu(t)| \to 0$ as $n \to \infty$ implies B is continuous. Finally, we show that BS_1 is relatively compact. Let t_1, $t_2 \in \mathbb{N}_{a+T_1+1}$ such that $t_2 > t_1$. Then, we have

$$
\begin{aligned}
&|Bu(t_1) - Bu(t_2)| \\
&\quad = \Big|\frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t_1} (t_1-\rho(s))^{\overline{\alpha-1}} f(s,u(s)) - \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t_2} (t_2-\rho(s))^{\overline{\alpha-1}} f(s,u(s))\Big| \\
&\quad \leq \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t_1} (t_1-\rho(s))^{\overline{\alpha-1}} |f(s,u(s))| + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t_2} (t_2-\rho(s))^{\overline{\alpha-1}} |f(s,u(s))| \\
&\quad \leq \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t_1-a)^{\overline{\alpha-\beta_1}} + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)} (t_2-a)^{\overline{\alpha-\beta_1}} < \epsilon.
\end{aligned}
$$

Thus, $\{Bu : u \in S_1\}$ is a bounded and uniformly Cauchy subset implies BS_1 is relatively compact. Hence the proof. ∎

Lemma 3.2. *Assume that condition* (I) *holds. Then a solution of* (2.5) *is in* S_1 *for* $t \in \mathbb{N}_{a+n_1+1}$.

Proof. Clearly, A is a contraction mapping with the constant 0, implies condition (1) of Theorem 2.5 holds. Using Lemma 3.1, the operator B is continuous and BS_1 is a compact subset of $\mathbb{R}$, implies condition (2) of Theorem 2.5 holds. Also

$Pu = Au + Bu$ and u is the solution of (2.5)–(2.6) if it is a fixed point of the operator P. Now we prove condition (3) of Theorem 2.5. Let us suppose, for a fixed $v \in S_1$, $u = Au + Bv$. Applying condition (I) and (3.6), we have

$$
\begin{aligned}
|u(t)| &\le |Au(t)| + |Bv(t)| \\
&\le \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}|u_0| + \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}|f(s,v(s))| \\
&\le \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}|u_0| + \frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(t_1-a)^{\overline{\alpha-\beta_1}} \\
&= \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{-\gamma_1}}(t-a-\gamma_1)^{\overline{1/2(\alpha+\beta_1)-1}} \\
&+\frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(t-a)^{\overline{-\gamma_1}}(t-a-\gamma_1)^{\overline{-\gamma_1}} \\
&= \Big[\frac{|u_0|}{\Gamma(\alpha)}(n_1-\gamma_1)^{\overline{1/2(\alpha+\beta_1)-1}} \\
&+\frac{L_1\Gamma(1-\beta_1)}{\Gamma(1+\alpha-\beta_1)}(n_1-\gamma_1)^{\overline{-\gamma_1}}\Big](t-a)^{\overline{-\gamma_1}} \\
&\le (t-a)^{\overline{-\gamma_1}},
\end{aligned}
\tag{3.8}
$$

for $t \in \mathbb{N}_{a+n_1+1}$. Thus, $u \in S_1$. According to Theorem 2.5, u is a fixed point of P implies u is the solution of (2.5)–(2.6). Hence the proof. ∎

Theorem 3.3. *Assume that condition* (I) *holds, then the zero solution of* (2.5) *is attractive.*

Proof. Using Lemma 3.1, the solutions of (2.5) exist in S_1 and tend to 0 as $t \to \infty$. Hence the proof. ∎

Theorem 3.4. *Assume that the following condition is satisfied:*

(II) *There exist constants* $\beta_2 \in (\alpha, 1)$ *and* $L_2 \ge 0$ *such that*

$$|f(t,u(t)) - f(t,v(t))| \le L_2(t-a)^{\overline{-\beta_2}}|u-v|, \quad \text{for } t \in \mathbb{N}_{a+1}. \tag{3.9}$$

Then, the solutions of (2.5) *are stable provided that*

$$c = L_2\Gamma(1-\beta_2) < 1. \tag{3.10}$$

Proof. Let $u(t)$ be the solution of (2.5)–(2.6), and let $\widetilde{u(t)}$ be the solution of (2.5) with the initial condition $\widetilde{u(0)} = \widetilde{u_0}$. Applying condition (II), we have

$$
\begin{aligned}
|u(t)-\widetilde{u}(t)| &\leq \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}|u_0-\widetilde{u_0}| \\
&+ \tfrac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}|f(s,u(s))-f(s,\widetilde{u(s)})| \\
&\leq \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}|u_0-\widetilde{u_0}| + \frac{L_2}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(s-a)^{\overline{-\beta_2}}|u-\widetilde{u}| \\
&= \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}|u_0-\widetilde{u_0}| + L_2\nabla_a^{-\alpha}(t-a)^{\overline{-\beta_2}}|u-\widetilde{u}| \\
&= \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}|u_0-\widetilde{u_0}| + \frac{L_2\Gamma(1-\beta_2)}{\Gamma(1+\alpha-\beta_2)}(t-a)^{\overline{\alpha-\beta_2}}|u-\widetilde{u}| \\
&\leq \frac{(2)^{\overline{\alpha-1}}}{\Gamma(\alpha)}|u_0-\widetilde{u_0}| + \frac{L_2\Gamma(1-\beta_2)}{\Gamma(1+\alpha-\beta_2)}(1)^{\overline{\alpha-\beta_2}}|u-\widetilde{u}| \\
&= \alpha|u_0-\widetilde{u_0}| + L_2\Gamma(1-\beta_2)|u-\widetilde{u}| \\
&< \alpha|u_0-\widetilde{u_0}| + c|u-\widetilde{u}|,
\end{aligned}
$$

which implies that

$$
|u-\widetilde{u}| < \frac{\alpha}{1-c}|u-\widetilde{u_0}|. \tag{3.11}
$$

For any given $\varepsilon > 0$, let $\delta = \frac{(1-c)}{\alpha}\epsilon$. Then, $|u-\widetilde{u_0}| < \delta$ implies $|u-\widetilde{u}| < \varepsilon$, which shows that the solutions of (2.5) are stable. This completes the proof. ∎

Theorem 3.5. *Assume that conditions* (I) *and* (II) *hold, then the zero solution of* (2.5) *is asymptotically stable provided that* (3.10) *holds.*

Theorem 3.6. *Assume that the following condition is satisfied:*

(III) *There exist constants* $\beta_3 \in \left(\alpha, \frac{1+\alpha}{2}\right)$, $\gamma_2 = \frac{1}{2}(1-\alpha)$ *and* $L_3 \geq 0$ *such that*

$$
|f(t,u(t))| \leq L_3(t-a+\gamma_2)^{\overline{-\beta_3}}|u(t)| \quad \text{for } t \in \mathbb{N}_{a+1}. \tag{3.12}
$$

Then, the zero solution of (2.5) *is attractive.*

Proof. Set $S_2 = \{u(t) : |u(t)| \leq (t-a)^{\overline{-\gamma_2}}$ for $t \in \mathbb{N}_{a+n_2+1}\}$, where $n_2 \in \mathbb{N}$ such that

$$
\frac{|u_0|}{\Gamma(\alpha)}(n_2-\gamma_2)^{\overline{-\gamma_2}} + \frac{L_3\Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)}(n_2-\gamma_2)^{\overline{\alpha-\beta_3}} \leq 1. \tag{3.13}
$$

Clearly $\beta_3 - \alpha > 0$. Using Lemma 2.1, for $t \in \mathbb{N}_{a+1}$,

$$
(t-a)^{\overline{\alpha-\beta_3}} = \frac{\Gamma(t-a+\alpha-\beta_3)}{\Gamma(t-a)} = (t-a)^{-(\beta_3-\alpha)}\Big[1+O\Big(\frac{1}{t-a}\Big)\Big].
$$

Further, $(t-a)^{-\overline{\gamma}_2} \to 0$ as $t \to \infty$. Then, there exists $n_2 \in \mathbb{N}$ such that $(t-a)^{\overline{\alpha-\beta_3}} \to 0$ for $(t-a) > n_2$. Clearly, $\beta_3 - \alpha < \gamma_2$ and $n_2 - \gamma_2 < t - a$.

Using Lemma 2.2, we get $(n_2-\gamma_2)^{\overline{-\gamma_2}} \le (n_2-\gamma_2)^{\overline{\alpha-\beta_3}} \le (t-a)^{\overline{\alpha-\beta_3}} \to 0$ and $(n_2-\gamma_2)^{\overline{\alpha-\beta_3}} \le (t-a)^{\overline{\alpha-\beta_3}} \to 0$. Thus, we have the inequality (3.13) which implies that the set S_2 exists. Now, we prove condition (3) of Theorem 2.5. Suppose, for a fixed $v \in S_2$ and for all $u \in \mathbb{R}$, $u = Au + Bv$. Applying condition (III), we have

$$\begin{aligned}
|u(t)| &\le |Au(t)| + |Bv(t)| \\
&\le \frac{|u_0|}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}|f(s,v(s))| \\
&\le \frac{|u_0|}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}L_3(s-a+\gamma_2)^{\overline{-\beta_3}}|v(s)| \\
&\le \frac{|u_0|}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{L_3}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(s-a+\gamma_2)^{\overline{-\beta_3}}(s-a)^{\overline{-\gamma_2}} \\
&= \frac{|u_0|}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{L_3}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(s-a)^{\overline{-\beta_3-\gamma_2}} \\
&\le \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{\alpha-1}} + \frac{L_3\Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)}(t-a)^{\overline{\alpha-\beta_3-\gamma_2}} \\
&= \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{-\gamma_2-\gamma_2}} + \frac{L_3\Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)}(t-a)^{\overline{\alpha-\beta_3-\gamma_2}} \\
&= \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{-\gamma_2}}(t-a-\gamma_2)^{\overline{-\gamma_2}} \\
&+ \frac{L_3\Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)}(t-a)^{\overline{-\gamma_2}}(t-a-\gamma_2)^{\overline{\alpha-\beta_3}} \\
&= \left[\frac{|u_0|}{\Gamma(\alpha)}(t-a-\gamma_2)^{\overline{-\gamma_2}} + \frac{L_3\Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)}(t-a-\gamma_2)^{\overline{\alpha-\beta_3}}\right](t-a)^{\overline{-\gamma_2}} \\
&\le \left[\frac{|u_0|}{\Gamma(\alpha)}(n_2-\gamma_2)^{\overline{-\gamma_2}} + \frac{L_3\Gamma(1-\beta_3-\gamma_2)}{\Gamma(1+\alpha-\beta_3-\gamma_2)}(n_2-\gamma_2)^{\overline{\alpha-\beta_3}}\right](t-a)^{\overline{-\gamma_2}} \\
&\le (t-a)^{\overline{-\gamma_2}}
\end{aligned}$$

for $t \in \mathbb{N}_{a+n_2+1}$. Thus, condition (3) of Theorem 2.5 holds. The proof of condition (2) of Theorem 2.5 is similar to that of Lemma 3.1, and we omit it. Therefore, by Theorem 2.5, P has a fixed point $u \in S_2$ implies u is the solution of (2.5)–(2.6). Moreover, all functions in S_2 tend to 0 as $t \to \infty$, then the zero solution of (2.5) tends to zero as $t \to \infty$, which shows that the zero solution of (2.5) is attractive. This completes the proof. ∎

Theorem 3.7. *Assume that conditions* (II) *and* (III) *hold, then the zero solution of* (2.5) *is asymptotically stable provided that* (3.10) *holds.*

Theorem 3.8. *Assume that the following condition is satisfied:*

(IV) *There exist constants* $\eta \in (0,1)$, $\beta_4 \in \left(\alpha, \frac{2+\alpha\eta}{2+\eta}\right)$ *and* $L_4 \ge 0$ *such that*

$$|f(t,u(t))| \le L_4(t-a+1)^{\overline{-\beta_4}}\left|u\left(t+\frac{\beta_4-\alpha}{2}\right)\right|^{\eta} \text{ for } t \in \mathbb{N}_{a+1}. \tag{3.14}$$

Then, the zero solution of (2.5) *is attractive.*

Proof. Set

$$S_3 = \{u(t) : |u(t)| \leq (t-a)^{\overline{-\gamma_3}} \text{ for } t \in \mathbb{N}_{a+n_3+1}\} \tag{3.15}$$

where $\gamma_3 = (1/2)(\beta_4 - \alpha)$ and $n_3 \in \mathbb{N}$ satisfies that

$$\frac{|u_0|}{\Gamma(\alpha)}(n_3-\gamma_3)^{\overline{\beta_4-\gamma_3-1}} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(n_3-\gamma_3)^{\overline{-\gamma_3}} \leq 1 \tag{3.16}$$

Clearly $\gamma_3 > 0$. Using Lemma 2.1, for $t \in \mathbb{N}_{a+1}$,

$$(t-a)^{\overline{-\gamma_3}} = \frac{\Gamma(t-a-\gamma_3)}{\Gamma(t-a)} = (t-a)^{-\gamma_3}\Big[1 + O\Big(\frac{1}{t-a}\Big)\Big].$$

Further, $(t-a)^{-\overline{\gamma}_3} \to 0$ as $t \to \infty$. Then, there exists $n_3 \in \mathbb{N}$ such that $(t-a)^{-\overline{\gamma}_3} \to 0$ for $(t-a) > n_3$. Clearly $\gamma_3 < 1 + \gamma_3 - \beta_4$ and $n_3 - \gamma_3 < t - a$. Using Lemma 2.2, we get $(n_3-\gamma_3)^{\overline{\beta_4-\gamma_3-1}} \leq (n_3-\gamma_3)^{\overline{-\gamma_3}} \leq (t-a)^{-\overline{\gamma}_3} \to 0$ and $(n_3-\gamma_3)^{\overline{-\gamma_3}} \leq (t-a)^{-\overline{\gamma}_3} \to 0$. Thus, we have the inequality (3.16) which implies that the set S_3 exists. Now, we prove condition (3) of Theorem 2.5 only, and the remaining part of the proof is similar to that of Theorem 3.6. Since $\eta \in (0,1)$, $\beta_4 \in \Big(\alpha, \frac{2+\alpha\eta}{2+\eta}\Big)$ and $\gamma_3 = (1/2)(\beta_4 - \alpha)$, then γ_3, $\gamma_3\eta$, $\alpha + \gamma_3 \in (0,1)$ and $\beta_4 + \gamma_3\eta \in (\alpha, 1)$. Suppose, for a fixed $v \in S_3$ and for all $u \in \mathbb{R}$, $u = Au + Bv$. Applying condition (IV), Lemma 2.2 and (3.16), we have

$$\begin{aligned}
&|u(t)| \leq |Au(t)| + |Bv(t)| \\
&\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}L_4(s-a+1)^{\overline{-\beta_4}}\Big|v\Big(s+\frac{\beta_4-\alpha}{2}\Big)\Big|^{\eta} \\
&\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{L_4}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(s-a+\gamma_3\eta)^{\overline{-\beta_4}}\big((s-a+\gamma_3)^{\overline{-\gamma_3}}\big)^{\eta} \\
&\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-1}} + \frac{L_4}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(s-a+\gamma_3\eta)^{\overline{-\beta_4}}(s-a+\gamma_3\eta)^{\overline{-\gamma_3\eta}} \\
&\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{\alpha-1}} + \frac{L_4}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(s-a+\gamma_3\eta)^{\overline{-\beta_4-\gamma_3\eta}} \\
&\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{\alpha-1}} + \frac{L_4}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(s-a)^{\overline{-\beta_4-\gamma_3\eta}} \\
&= \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{\beta_4-\gamma_3-\gamma_3-1}} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(t-a)^{\overline{\alpha-\beta_4-\gamma_3\eta}} \\
&\leq \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{\beta_4-\gamma_3-\gamma_3-1}} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(t-a)^{\overline{\alpha-\beta_4}}
\end{aligned}$$

$$= \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{\beta_4-\gamma_3-\gamma_3-1}} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(t-a)^{\overline{-\gamma_3-\gamma_3}}$$
$$= \frac{|u_0|}{\Gamma(\alpha)}(t-a)^{\overline{-\gamma_3}}(t-a-\gamma_3)^{\overline{\beta_4-\gamma_3-1}} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(t-a)^{\overline{-\gamma_3}}(t-a-\gamma_3)^{\overline{-\gamma_3}}$$
$$= \Big[\frac{|u_0|}{\Gamma(\alpha)}(t-a-\gamma_3)^{\overline{\beta_4-\gamma_3-1}} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(t-a-\gamma_3)^{\overline{-\gamma_3}}\Big](t-a)^{\overline{-\gamma_3}}$$
$$\leq \Big[\frac{|u_0|}{\Gamma(\alpha)}(n_3-\gamma_3)^{\overline{\beta_4-\gamma_3-1}} + \frac{L_4\Gamma(1-\beta_4-\gamma_3\eta)}{\Gamma(1+\alpha-\beta_4-\gamma_3\eta)}(n_3-\gamma_3)^{\overline{-\gamma_3}}\Big](t-a)^{\overline{-\gamma_3}}$$
$$\leq (t-a)^{\overline{-\gamma_3}}$$

for $t \in \mathbb{N}_{a+n_3+1}$. Thus, condition (3) of Theorem 2.5 holds. Hence the proof. ∎

4. Caputo type fractional difference equation

Let l_a^∞ be the set of all real sequences $u = \{u(t)\}_{t=a}^\infty$ with norm $\|u\| = \sup_{t\in\mathbb{N}_a} |u(t)|$, then l_a^∞ is a Banach space. Define the operator

$$Tu(t) = u_0 + \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,u(s)), \tag{4.1}$$

Obviously, $u(t)$ is a solution of (2.8)–(2.9) if it is a fixed point of the operator T.

Lemma 4.1. *Assume that the following condition is satisfied:*

(V) *there exist constants γ_4, $L_5 > 0$ such that*

$$\Big|u_0 + \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}f(s,u(s))\Big| \leq L_5(t-a)^{\overline{-\gamma_4}} \text{ for } t \in \mathbb{N}_{a+1}. \tag{4.2}$$

Then there exists a solution for (2.8).

Proof. Define the set

$$S_4 = \{u(t) : |u(t)| \leq L_5(t-a)^{\overline{-\gamma_4}}\} \text{ for } t \in \mathbb{N}_{a+1}. \tag{4.3}$$

Clearly, S_4 is a closed, bounded and convex subset of $\mathbb{R}$. Using Lemma 2.1, we have

$$(t-a)^{\overline{-\gamma_4}} = \frac{\Gamma(t-a-\gamma_4)}{\Gamma(t-a)} = (t-a)^{-\gamma_4}\Big[1 + O\Big(\frac{1}{t-a}\Big)\Big]$$

for $t \in \mathbb{N}_{a+1}$. Then, $(t-a)^{-\overline{\gamma}_4} \to 0$ as $t \to \infty$. To prove that T has a fixed point, first we show that T maps S_4 in S_4. For $t \in \mathbb{N}_{a+1}$, from condition (V), we have $|Tu(t)| \leq L_5(t-a)^{\overline{-\gamma_4}}$ implies $TS_4 \subset S_4$. Next, we show that T is continuous on S_4. Let $\epsilon > 0$ be given. Then there exists a $N_1 \in \mathbb{N}_{a+1}$ such that $t > N_1$ and $L_5(t-a)^{\overline{-\gamma_4}} < \frac{\epsilon}{2}$. Let $\{u_n\}$ be a sequence such that $u_n \to u$. For $t \in \{a+1, a+2, a+3, ..., N_1\}$, applying the continuity of f, we have

$$\begin{aligned}|Tu_n(t)-Tu(t)| &\leq \frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}|f(s,u_n(s))-f(s,u(s))|\\ &\leq \left[\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}\right]\max_{s\in\{a+1,a+2,a+3,\ldots,N_1\}}|f(s,u_n(s))-f(s,u(s))|\\ &= \frac{\Gamma(t-a+\alpha)}{\Gamma(\alpha+1)\Gamma(t-a)}\max_{s\in\{a+1,a+2,a+3,\ldots,N_1\}}|f(s,u_n(s))-f(s,u(s))|\to 0\\ &\qquad\qquad \text{as } n\to\infty.\end{aligned}$$

For $t\in\{N_1+1,N_1+2,...\}$, we have

$$\begin{aligned}|Tu_n(t)-Tu(t)| &\leq \left|\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-\rho(s))^{\overline{\alpha-1}}(f(s,u_n(s))-f(s,u(s))\right|\\ &\leq 2L_5(t-a)^{\overline{-\gamma_4}}<\epsilon.\end{aligned}$$

Thus, for all $t\in\mathbb{N}_{a+1}$, $|Tu_n(t)-Tu(t)|\to 0$ as $n\to\infty$, which means that T is continuous. Finally, we show that TS_4 is relatively compact. Let t_1, $t_2\in\mathbb{N}_{a+1}$ and $t_2>t_1\geq N_1$, we have

$$\begin{aligned}|Tu(t_2)-Tu(t_1)| &\leq \left|\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t_2}(t_2-\rho(s))^{\overline{\alpha-1}}f(s,u(s))\right|\\ &+ \left|\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t_1}(t_1-\rho(s))^{\overline{\alpha-1}}f(s,u(s))\right| \qquad (4.4)\\ &\leq L_5(t_2-a)^{\overline{-\gamma_1}}+L_5(t_1-a)^{\overline{-\gamma_1}}<\epsilon.\end{aligned}$$

Therefore, $\{Tu : u\in S_4\}$ is a bounded and uniformly Cauchy subset. Hence, by Theorem 2.4, TS_4 is relatively compact. According to Theorem 2.6, T has a fixed point in S_4 which is the solution of (2.8) - (2.9). Hence the proof. ∎

Theorem 4.2. *Assume that condition* (V) *holds. Then the zero solution of* (2.8) *is attractive.*

Proof. By Lemma 4.1, the solutions of (2.8) exist and are in S_4. But all functions in S_4 tend to 0 as $t\to\infty$. Then, the zero solution of (2.8) tend to zero as $t\to\infty$. This completes the proof. ∎

Theorem 4.3. *Assume that the following condition is satisfied:*

(VI) *there exist constants* $\gamma_5\in(\alpha,1)$ *and* $L_6>0$ *such that*

$$|f(t,u(t))-f(t,v(t))|\leq L_6(t-a)^{\overline{-\gamma_5}}|u-v| \text{ for } t\in\ \mathbb{N}_{a+1}. \qquad (4.5)$$

Then, the solutions of (2.8) *are stable provided that*

$$c=L_6\Gamma(1-\gamma_5)<1. \qquad (4.6)$$

Proof. Let $u(t)$ be the solution of (2.8)–(2.9), and let $\widetilde{u(t)}$ be the solution of (2.8) satisfying the initial condition $\widetilde{u(0)} = \widetilde{u_0}$. Applying condition (VI), we have

$$\begin{aligned}
|u(t) - \widetilde{u}(t)| &\leq |u_0 - \widetilde{u_0}| + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} |f(s,u(s)) - f(s,\widetilde{u(s)})| \\
&\leq |u_0 - \widetilde{u_0}| + \frac{L_6}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} (s-a)^{\overline{-\gamma_5}} |u - \widetilde{u}| \\
&= |u_0 - \widetilde{u_0}| + L_6 \nabla_a^{-\alpha} (t-a)^{\overline{-\gamma_5}} |u - \widetilde{u}| \\
&= |u_0 - \widetilde{u_0}| + \frac{L_6 \Gamma(1-\gamma_5)}{\Gamma(1+\alpha-\gamma_5)} (t-a)^{\overline{\alpha-\gamma_5}} |u - \widetilde{u}| \\
&\leq |u_0 - \widetilde{u_0}| + L_6 \Gamma(1-\gamma_5) |u - \widetilde{u}| \\
&< |u_0 - \widetilde{u_0}| + c|u - \widetilde{u}|
\end{aligned}$$

which implies that

$$|u - \widetilde{u}| < \frac{\alpha}{1-c} |u - \widetilde{u_0}|. \tag{4.7}$$

For any given $\varepsilon > 0$, let $\delta = \frac{(1-c)}{\alpha}\epsilon$. Then, $|u - \widetilde{u_0}| < \delta$ implies $|u - \widetilde{u}| < \varepsilon$, which shows that the solutions of (2.8) are stable. This completes the proof. ∎

Theorem 4.4. *Assume that conditions* (V) *and* (VI) *hold. Then, the zero solution of* (2.8) *is asymptotically stable provided that* (4.6) *holds.*

Lemma 4.5. *Assume that the following condition is satisfied:*

(VII) *there exist constants* $\gamma_6 \in (\alpha, 1)$ *and* $L_7 > 0$ *such that*

$$\left| \frac{(t-a)^{\overline{\alpha}}}{\Gamma(1-\alpha)} u_0 + f(t, u(t)) \right| \leq L_7 (t-a)^{\overline{-\gamma_6}} \text{ for } t \in \mathbb{N}_{a+1}. \tag{4.8}$$

Then, there exists a solution for (2.8).

Proof. Define the set

$$S_5 = \left\{ u(t) : |u(t)| \leq \frac{L_7 \Gamma(1-\gamma_6)}{\Gamma(1+\alpha-\gamma_6)} (t-a)^{\overline{\alpha-\gamma_6}} \right\} \text{ for } t \in \mathbb{N}_{a+1}. \tag{4.9}$$

From the definition, clearly, S_5 is a closed, bounded and convex subset of $\mathbb{R}$. First, we show that T maps S_5 in S_5. From condition (VII), we have

$$\begin{aligned}
|Tu(t)| &= \left|u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s,u(s))\right| \\
&= \left|\frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \frac{(s-a)^{\overline{\alpha}}}{\Gamma(1-\alpha)} u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s,u(s))\right| \\
&= \left|\frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \Big[\frac{(s-a)^{\overline{\alpha}}}{\Gamma(1-\alpha)} u_0 + f(s,u(s))\Big]\right| \\
&\le \frac{L_7}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} (s-a)^{\overline{-\gamma_6}} \\
&= L_7 \nabla_a^{-\alpha} (t-a)^{\overline{-\gamma_6}} \\
&= \frac{L_7 \Gamma(1-\gamma_6)}{\Gamma(1+\alpha-\gamma_6)} (t-a)^{\overline{\alpha-\gamma_6}}
\end{aligned}$$

for $t \in \mathbb{N}_{a+1}$ implies $TS_5 \subset S_5$. The remaining proof of TS_5 to be relatively compact is similar to that of Lemma 4.1, and we omit it. By Theorem 2.6, T has a fixed point in S_5 which is the solution of (2.8)–(2.9). Hence the proof. ∎

Theorem 4.6. *Assume that condition* (VII) *holds. Then the zero solution of* (2.8) *is attractive.*

Theorem 4.7. *Assume that conditions* (VI) *and* (VII) *hold. Then the zero solution of* (2.8) *is asymptotically stable provided that* (4.6) *holds.*

Lemma 4.8. *Assume that the following condition is satisfied:*

(VIII) *there exist constants* $\beta_5 > \dfrac{1}{1-\alpha}$, $\dfrac{\alpha}{\beta_5 - 1} < \gamma_7 < \dfrac{1}{\beta_5}$ *and* $L_8 > 0$, *such that*

$$\left|\frac{(t-a)^{\overline{\alpha}}}{\Gamma(1-\alpha)} u_0 + f(t,u(t))\right| \le L_8 |u(t+\gamma_7)|^{\beta_5} \text{ for } t \in \mathbb{N}_{a+1}. \tag{4.10}$$

Then, there exists a solution for (2.8) *provided that*

$$\frac{L_8 \Gamma(1-\beta_5\gamma_7)}{\Gamma(1+\alpha-\beta_5\gamma_7)} \frac{\Gamma(1+\beta_5\gamma_7)}{[\Gamma(1+\gamma_7)]^{\beta_5}} \le 1. \tag{4.11}$$

Proof. From $\beta_5 > \dfrac{1}{1-\alpha}$, we have that $\dfrac{\alpha}{\beta_5 - 1} < \dfrac{1}{\beta_5}$ which implies that γ_7 exists. In addition, $\gamma_7 < \dfrac{1}{\beta_5}$ means that $\Gamma(1-\beta_5\gamma_7) > 0$ and $\Gamma(1+\alpha-\beta_5\gamma_7) > 0$, $\dfrac{\alpha}{\beta_5 - 1} < \gamma_7$ implies that $\alpha - \beta_5\gamma_7 < -\gamma_7$. Define the set

$$S_6 = \left\{u(t) : |u(t)| \le (t-a)^{\overline{-\gamma_7}} \text{ for } t \in \mathbb{N}_{a+1}\right\}. \tag{4.12}$$

We show that T maps S_6 in S_6. Applying condition (VIII), Lemma 2.2 and (4.11), we have

$$\begin{aligned}
|Tu(t)| &= \left| u_0 + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s,u(s)) \right| \\
&= \left| \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} \Big[\frac{(s-a)^{\overline{\alpha}}}{\Gamma(1-\alpha)} u_0 + f(s,u(s)) \Big] \right| \\
&\leq \frac{L_8}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} |u(s+\gamma_7)|^{\beta_5} \\
&\leq \frac{L_8}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} [(t-a+\gamma_7)^{\overline{-\gamma_7}}]^{\beta_5} \\
&\leq \frac{L_8}{\Gamma(\alpha)} \frac{\Gamma(1+\beta_5\gamma_7)}{[\Gamma(1+\gamma_7)]^{\beta_5}} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} (t-a+\beta_5\gamma_7)^{\overline{-\beta_5\gamma_7}} \\
&\leq \frac{L_8}{\Gamma(\alpha)} \frac{\Gamma(1+\beta_5\gamma_7)}{[\Gamma(1+\gamma_7)]^{\beta_5}} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} (t-a)^{\overline{-\beta_5\gamma_7}} \\
&= L_8 \frac{\Gamma(1+\beta_5\gamma_7)}{[\Gamma(1+\gamma_7)]^{\beta_5}} \nabla_a^{-\alpha} (t-a)^{\overline{-\beta_5\gamma_7}} \\
&= \frac{L_8\Gamma(1-\beta_5\gamma_7)}{\Gamma(1+\alpha-\beta_5\gamma_7)} \frac{\Gamma(1+\beta_5\gamma_7)}{[\Gamma(1+\gamma_7)]^{\beta_5}} (t-a)^{\overline{\alpha-\beta_5\gamma_7}} \\
&\leq \frac{L_8\Gamma(1-\beta_5\gamma_7)}{\Gamma(1+\alpha-\beta_5\gamma_7)} \frac{\Gamma(1+\beta_5\gamma_7)}{[\Gamma(1+\gamma_7)]^{\beta_5}} (t-a)^{\overline{-\gamma_7}} \leq (t-a)^{\overline{-\gamma_7}}
\end{aligned}$$

for $t \in \mathbb{N}_{a+1}$ implies $TS_6 \subset S_6$. The remaining part of the proof is similar to that of Lemma 4.1, so we omit it. ■

Theorem 4.9. *Assume that the condition* (VIII) *and* (4.11) *hold, then the zero solution of* (2.8) *is attractive.*

Theorem 4.10. *Assume that the condition* (VI) *and* (VIII) *hold, then the zero solution of* (2.8) *is asymptotically stable provided that* (4.6) *and* (4.11) *hold.*

5. Examples

Example 5.1.

$$\nabla_{-1}^{0.5} u(t) = (0.3) t^{\overline{-0.25}} \sin(u(t)), \quad \nabla_{-1}^{-0.5} u(t)\Big|_{t=0} = u(0), \quad t \in \mathbb{N}_1. \tag{5.1}$$

Here, $f(t, u(t)) = (0.3)t^{\overline{-0.25}} \sin(u(t)), t \in \mathbb{N}_1$. Clearly, $|f(t, u(t))| \leq (0.3)t^{\overline{-0.25}}$ implies condition (I) holds. Further, $|f(t, u(t)) - f(t, v(t))| \leq (0.3)t^{\overline{-0.25}}|u - v|$ implies condition (II) is satisfied. Here $L_2 = 0.3$ and $\beta_2 = 0.25$ and we have $c = L_2\Gamma(1 - \beta_2) = (0.3)\Gamma(0.75) \approx (0.3)(1.2254) < 1$, implies that the inequality (3.10) holds. Hence, the solution of (5.1) is asymptotically stable by Theorem 3.5.

Example 5.2.

$$\nabla^{0.5}_{-1}u(t) = (0.1)(t+1)t^{\overline{-0.6}}u(t), \quad \nabla^{-0.5}_{-1}u(t)\Big|_{t=0} = u(0), \quad t \in \mathbb{N}_1. \tag{5.2}$$

Here $f(t, u(t)) = (0.1)(t+1)^{\overline{-0.6}}u(t), t \in \mathbb{N}_1$, where $\beta_3 = 0.6$ and $\alpha = 0.5$. Clearly, $\beta_3 \in \left(\alpha, \dfrac{1+\alpha}{2}\right)$, $\gamma_2 = \dfrac{1}{2}(1 - \alpha) = 0.25$ and $L_3 = 0.1$ such that

$$|f(t, u(t))| = |(0.1)(t+1)^{\overline{-0.6}}u(t)| \leq (0.1)(t+0.25)^{\overline{-0.6}}|u(t)|$$

which implies that condition (III) is satisfied. Further,

$$|f(t, u(t)) - f(t, v(t))| \leq (0.1)(t+1)^{\overline{-0.6}}|u - v| \leq (0.1)(t)^{\overline{-0.6}}|u - v|,$$

implies that condition (II) is satisfied. Here, $L_2 = 0.1$, $\beta_2 = 0.6$ and $\alpha = 0.5$. Then, we have $c = L_2\Gamma(1 - \beta_2) = (0.1)\Gamma(0.4) \approx (0.1)(2.2181) < 1$, implies that the inequality (3.10) holds. Hence, the solution of (5.2) is asymptotically stable by Theorem 3.7.

Example 5.3.

$$\nabla^{0.5}_{-1}u(t) = (0.2)(t+1.5)^{\overline{-0.6}}[u(t+0.05)]^{0.3}, \quad \nabla^{-0.5}_{-1}u(t)\Big|_{t=0} = u(0), \quad t \in \mathbb{N}_1. \tag{5.3}$$

Here, $f(t, u(t)) = (0.2)(t+1.5)^{\overline{-0.6}}[u(t+0.05)]^{0.3}$, where $\beta_4 = 0.6$, $\eta = 0.3$, $L_4 = 0.2$ and $\alpha = 0.5$. Clearly, $\eta \in (0, 1)$, $\beta_4 \in \left(\alpha, \dfrac{2+\alpha\eta}{2+\eta}\right)$ such that

$$|f(t, u(t))| = |(0.2)(t+1.5)^{\overline{-0.6}}[u(t+0.05)]^{0.3}| \leq (0.2)(t+1)^{\overline{-0.6}}|u(t+0.05)|^{0.3}$$

which implies condition (IV) is satisfied. Hence, the zero solution of (5.3) is attractive by Theorem 3.8.

Example 5.4.

$$\nabla^{0.5}_{0*}u(t) = (0.2)t^{\overline{-0.75}}sin(u(t)), \quad u(0) = 0, \quad t \in \mathbb{N}_1. \tag{5.4}$$

Using (2.10), we get

$$u(t) = \frac{(0.2)}{\Gamma(0.5)}\sum_{s=1}^{t}(t - \rho(s))^{\overline{-0.5}}s^{\overline{-0.75}}\sin(u(s)).$$

Now, consider

$$\Big|\frac{(0.2)}{\Gamma(0.5)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{-0.5}}s^{\overline{-0.75}}\sin(u(s))\Big| \leq \frac{(0.2)}{\Gamma(0.5)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{-0.5}}s^{\overline{-0.75}} \leq t^{\overline{-0.25}}$$

implies that (V) holds. Further, $|f(t,u(t))-f(t,v(t))| \leq (0.2)t^{\overline{-0.75}}|u-v|$ implies condition (VI) is satisfied. Here, $L_6 = 0.2$, $\gamma_5 = 0.75$ and we have $c = L_6\Gamma(1-\gamma_5) = 0.2\Gamma(0.25) \approx (0.2)(3.6256) < 1$, implies (4.6) holds. Hence, the zero solution of (5.4) is asymptotically stable by Theorem 4.4.

Example 5.5.

$$\nabla_{0*}^{0.5}u(t) = (0.2)t^{\overline{-0.75}}\sin(u(t)) - u_0\frac{t^{\overline{0.5}}}{\Gamma(0.5)}, \quad u(0) = u_0, \quad t \in \mathbb{N}_1. \tag{5.5}$$

Using (2.10), we get

$$u(t) = u_0 + \frac{1}{\Gamma(0.5)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{-0.5}}\Big[(0.2)s^{\overline{-0.75}}\sin(u(s)) - u_0\frac{s^{\overline{0.5}}}{\Gamma(0.5)}\Big].$$

Now, consider

$$\Big|u_0\frac{t^{\overline{0.5}}}{\Gamma(0.5)} + f(t,u(t))\Big| \leq (0.2)t^{\overline{-0.75}}|\sin(u(s))| \leq (0.2)t^{\overline{-0.75}} \leq t^{\overline{-0.75}}$$

implies that (VII) holds. Further, $|f(t,u(t))-f(t,v(t))| \leq 0.2t^{\overline{-0.75}}|u-v|$ implies condition (VI) is satisfied. Here, $L_6 = 0.2$, $\gamma_5 = 0.75$ and we have $c = L_6\Gamma(1-\gamma_5) = 0.2\Gamma(0.25) \approx (0.2)(3.6256) < 1$, implies (4.6) holds. Hence, the zero solution of (5.5) is asymptotically stable by Theorem 4.7.

Example 5.6.

$$\nabla_{0*}^{0.1}u(t) = (0.5)[u(t+0.2)]^2 - u_0\frac{t^{\overline{0.5}}}{\Gamma(0.5)}, \quad u(0) = u_0, \quad t \in \mathbb{N}_1. \tag{5.6}$$

Using (2.10), we get

$$u(t) = \frac{1}{\Gamma(0.5)}\sum_{s=1}^{t}(t-\rho(s))^{\overline{-0.5}}\Big[(0.5)[u(s+0.2)]^2 - u_0\frac{s^{\overline{0.5}}}{\Gamma(0.5)}\Big].$$

Now, consider

$$\Big|u_0\frac{t^{\overline{0.5}}}{\Gamma(0.5)} + f(t,u(t))\Big| = \Big|(0.5)[u(t+0.2)]^2\Big| \leq (0.5)|u(t+0.2)|^2$$

implies that (VIII) holds. Here, $L_8 = 0.5$, $\beta_5 = 2$ and $\gamma_7 = 0.2$ and we have

$$\frac{L_8\Gamma(1-\beta_5\gamma_7)}{\Gamma(1+\alpha-\beta_5\gamma_7)}\frac{\Gamma(1+\beta_5\gamma_7)}{[\Gamma(1+\gamma_7)]^{\beta_5}} = \frac{(0.5)\Gamma(0.6)}{\Gamma(0.7)}\frac{\Gamma(1.4)}{[\Gamma(1.2)]^2} \approx 0.6039 < 1,$$

implies (4.11) is satisfied. Hence, the zero solution of (5.6) is attractive by Theorem 4.9.

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