STRONGLY DUO AND DUO RIGHT S-ACTS

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Abstract. New kinds of acts, namely strongly duo and duo acts over a monoid are introduced and investigated. This leads to the study of the relation between these kinds of acts and other classes of acts, such as injective, projective and multiplication. It is shown that a projective act is duo if and only if it is a multiplication act. In addition if S is commutative, then a cyclic S-act is strongly duo if and only if it is cyclic quasi-injective.

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1. Introduction and preliminaries

In this paper, S is a monoid and an S-act A_S (or A) is a unitary right S-act. Let A be a right S-act and let B be a subact of A. We say that B is *fully invariant* if $f(B) \subseteq B$ for every endomorphism f of A and A is called *duo* if every subact of A is fully invariant. This notion generalizes the concept of right duo semigroups (semigroups for which every right ideal is two sided) see [1]. Also a right S-act A is called *strongly duo* if for every subact B of A the trace of B in A is equal to B, i.e., $tr(B,A) = \bigcup_{f\in Hom(B,A)} f(B) = B$ (see the properties of trace in [4] page 146). It is clear that every strongly duo act is duo. In this paper several equivalent conditions to being strongly duo are given. For instance, it is shown that a right S-act A is strongly duo if and only if it is duo and is quasi-injective relative to all inclusions from its cyclic subacts. Also we prove that a projective right S-act A is duo if and only if it is multiplication (i.e., every subact of A is of the from AI for some right ideal I of S). It is proved that if S_S is strongly duo, then all right S-acts are torsion free (divisible) and S is left reversible. To obtain a characterization for strongly duo acts, we need the following definition (see Definition 1.4.20 of [4]).

Definition 1.1 Suppose S is a monoid and A is a right S-act. For an element $a \in A$ we define the *annihilator* of a as ann(a):=$\{(s,t) \in S \times S : as = at\}$.

Theorem 1.2 *Let S be a monoid and let A be a right S-act. Then the following are equivalent:*

(i) *A is strongly duo.*

(ii) *Every subact of A is strongly duo.*

(iii) *Every finitely generated subact of A is strongly duo.*

(iv) *If B, C are subacts of A and B is a homomorphic image of C, then $B \subseteq C$.*

(v) *If $ann(a) \subseteq ann(b)$ for some $a, b \in A$, then $b \in aS$.*

Proof. (i)$\longrightarrow$(ii) and (ii)$\longrightarrow$(iii) are clear.

(iii)$\longrightarrow$(i) Suppose B is a subact of A and $f : B \longrightarrow A$ is a homomorphism. Let b be an element of B and let $C = bS \cup f(b)S$. If $g = f|_{bS}$, then, clearly, $f(b) = g(b) \in tr(bS, C)$. By assumption, C is strongly duo and so $f(b) \in tr(bS, C) = bS$. It follows that $tr(B, A) = B$.

(i)$\longrightarrow$(iv) If B, C are two subacts of A and $f : C \longrightarrow B$ is an epimorphism, then $B = Im(f) \subseteq tr(C, A) = C$.

(iv)$\longrightarrow$(v) Suppose $ann(a) \subseteq ann(b)$ for some $a, b \in A$. Define $f : aS \longrightarrow bS$ by $f(as) = bs$ for every $s \in S$. Clearly, f is a well-defined epimorphism and so $bS \subseteq aS$ by assumption.

(v)$\longrightarrow$(i) Suppose B is a subact of A and $f \in Hom(B, A)$. If $b \in B$, then $ann(b) \subseteq ann(f(b))$ and hence $f(b) \in bS \subseteq B$ by assumption. Consequently, $tr(B, A) = B$. ∎

The following corollary shows that, there are acts which are not strongly duo.

Corollary 1.3 *Suppose S is a monoid and A is a strongly duo right S-act. If S_S is a subact of A, then $A_S = S_S$.*

Proof. By part (ii) of the previous theorem, S_S is strongly duo. Also every cyclic right S-act is a homomorphic image of S_S. Thus the result holds by part (iv) of Theorem 1.2. ∎

Definition 1.4 A right S-act A is called *completely ordered* if for any two subacts B, C of A, either $B \subseteq C$ or $C \subseteq B$.

Proposition 1.5 *Suppose S is a monoid and A is a completely ordered right S-act. If A satisfies the descending chain condition on cyclic subacts, then A is strongly duo. In particular if S is a finite principal right ideal monoid, then S_S is strongly duo.*

Proof. For elements $a, b \in A$ with condition $ann(a) \subseteq ann(b)$, we show that $b \in aS$. If $b \notin aS$, then by assumption $aS \subseteq bS$ and so $a = bs$ for some $s \in S$. Since $bsS \supseteq bs^2S \supseteq ...$, by the hypothesis there exists $n \in \mathbb{N}$ such that $(bs^n)S = (bs^{n+1})S$ and so $bs^n = bs^{n+1}t$ for some $t \in S$. Thus $as^{n-1} = as^n t$. Since $ann(a) \subseteq ann(b), bs^{n-1} = bs^n t$. Repeat this argument to obtain $a = ast$. Thus $b = bst = (bs)t = at \in aS$, which is a contradiction. Therefore, $bS \subseteq aS$ and A_S is strongly duo by Theorem 1.2. In case that S is a principal right ideal monoid, S_S is completely ordered. Hence the finite condition on S implies that S_S is strongly duo. ∎

Corollary 1.6 *Suppose S is a monoid which satisfies the descending chain condition on principal right ideals. If A is a completely ordered right S-act, then A is strongly duo.*

Proof. We show that A satisfies the descending chain condition on cyclic subacts. If $a, b \in A$ and $bS \subseteq aS$, then there exists $s \in S$ such that $bS = asS$. Thus every descending chain of cyclic subacts of A has the form, $aS \supseteq as_1S \supseteq as_1s_2S \supseteq ....$ Now consider the descending chain, $S \supseteq s_1S \supseteq s_1s_2S \supseteq ....$ By assumption there exists $n \in \mathbb{N}$ such that $s_1...s_nS = s_1...s_{n+1}S$. It follows that A satisfies the descending chain condition on cyclic subacts and by the previous proposition is strongly duo. ∎

The following proposition reveals some similarities between strongly duo acts and cohopfian modules.

Proposition 1.7 *If S is a monoid and A is a strongly duo right S-act, then every monomorphism $f : A \longrightarrow A$ is an epimorphism.*

Proof. Suppose $f : A \longrightarrow A$ is a monomorphism and define $g : f(A) \longrightarrow A$ by $g(f(a)) = a$ for every $a \in A$. Since f is a monomorphism, g is a well-defined homomorphism and clearly $g(f(A)) = A$. Since A is strongly duo, $g(f(A)) \subseteq tr(f(A), A) = f(A)$ and so $f(A) = A$, that is f is an epimorphism. ∎

Now, we study the properties of duo acts. As the following lemma shows a good source of duo acts is provided by multiplication acts. By [3] an S-act A is called *multiplication* if every subact of A is of the form AI, for some right ideal I of S. Recall that a monoid S is said to be *left reversible* if every two right ideals of S have a nonempty intersection. Also for a right S-act A and subact B of A, the factor act A/B is defined as A/ρ_B, where ρ_B is the Rees congruence (see [4]).

Lemma 1.8 *Over a monoid S the following statements hold:*

(i) *A right S-act A is duo if and only if for each endomorphism f of A and each element a of A, $f(a) = at$ for some $t \in S$. In particular, if S is commutative and A is a duo right S-act, then $End(A)$ is a commutative monoid.*

(ii) *Let $B \subseteq C$ be subacts of a right S-act A. If B and C/B are fully invariant subacts of A and A/B respectively, then C is fully invariant in A.*

(iii) *If S_S is duo, then for any two elements $s, t \in S, st = tx$ for some $x \in S$.*

(iv) *S_S is duo if and only if every right ideal of S is a two sided ideal.*

(v) *If S_S is duo, then S is left reversible.*

(vi) *Every multiplication right S-act is duo.*

Proof. (i) Note that if A is duo, then $f(aS) \subseteq aS$ for all $f \in End(A)$ and $a \in A$. In particular, if S is commutative and A is a duo right S-act, then the first part of (i) implies that $End(A)$ is commutative.

(ii) Let $f \in End(A)$ and define $\bar{f} : A/B \longrightarrow A/B$ by $\bar{f}(\bar{a}) = \overline{f(a)}$. As B is fully invariant in A, $\bar{f}$ is a well-defined homomorphism and $\overline{f(C)} = \bar{f}(C/B)$ and so $f(C) \subseteq C$.

(iii) For every $s \in S$ define $\lambda_s : S \longrightarrow S$ by $\lambda_s(t) = st$ for every $t \in S$. Thus by part (i), $\lambda_s(t) \in tS$ for every $t \in S$ and the result follows. (iv) and (v) are obvious by part (i). (vi) If B is a subact of a multiplication right S-act A, then $B = AI$ for some right ideal I of S and so for every endomorphism f of A, $f(B) = f(AI) = f(A)I \subseteq AI = B$. ∎

Corollary 1.9 *Suppose T is a proper submonoid of a monoid S. Then S_T the right T-act S is not duo.*

Proof. Suppose $s \in S \backslash T$ and define $f : S_T \longrightarrow S_T$ by $f(x) = sx$ for every $x \in S$. If S_T is duo, then by the previous lemma, $s = f(1) = 1t$ for some $t \in T$ and so $s \in T$, which is a contradiction. ∎

Proposition 1.10 *Suppose S is a monoid and A is a completely ordered right S-act. If S_S is duo and A satisfies the ascending chain condition on cyclic subacts, then A is duo.*

Proof. Let $f : A \longrightarrow A$ be an endomporphism of A and let $a_1 \in A$. Suppose for some elements $a_2, a_3, ...$ of A, $f(a_1) = a_2$, $f(a_2) = a_3$,.... Since A is completely ordered, for every $n \in \mathbb{N}$, either $a_n \in a_{n+1}S$ or $a_{n+1} \in a_nS$. If for every $n \in \mathbb{N}$, $a_nS \subset a_{n+1}S$, then we have the ascending chain $a_1S \subset a_2S \subset a_3S...$, which is a contradiction. Thus there exists the smallest $n \in \mathbb{N}$ such that $a_{n+1}S \subseteq a_nS$. Hence $a_{n-1} \in a_nS, a_{n-2} \in a_{n-1}S, ..., a_1 \in a_2S$ and so $a_1 \in a_nS$. If $a_1 = a_ns$ for some $s \in S$, then $f(a_1) = f(a_n)s = a_{n+1}s$. Since $a_{n+1} \in a_nS$, $a_{n+1} = a_nt$ for some $t \in S$ and so $f(a_1) = a_nts$. By Lemma 1.8.(iii), $ts = sx$ for some $x \in S$ and hence $f(a_1) = a_nsx = a_1x$. Now the result follows by Lemma 1.8.(i). ∎

Now, the behavior of duo acts under taking subacts and homomorphic images is considered.

Definition 1.11 Let S be a monoid. Then an S-act A is called (*cyclic*) *quasi-injective* if for every (cyclic) subact B of A and for every homomorphism $f \in Hom(B, A)$, there exists a homomorphism $g \in Hom(A, A)$ which extends f, i.e., $g|_B = f$. Also A is called *quasi-projective* if for a given epimorphism $g : A \longrightarrow C$ and homomorphism $f : A \longrightarrow C$, there exists a homomorphism $h : A \longrightarrow A$ such that $gh = f$.

Proposition 1.12 *Suppose S is a monoid and A is a duo right S-act. Then the following statements hold:*

(i) *If A is quasi-injective, then every subact of A is duo and quasi-injective.*

(ii) *If A is quasi-projective, then every homomorphic image of A is duo and every Rees factor of A is quasi-projective.*

Proof. (i) Suppose B is a subact of A and C is a subact of B. Let f be an endomorphism of B. Since A is quasi-injective, f can be lifted to an endomorphism $\bar{f}$ of A. Thus $\bar{f}(C) = f(C)$, which is contained in C because A is duo. Also it is easy to see that B is quasi-injective.

(ii) Let B be a homomorphic image of A and let g be an endomorphism of B. Suppose $B = \frac{A}{\rho}$ for some right congruence ρ on A and $\frac{C}{\rho}$ is a subact of B. Since A is quasi-projective, there exists an endomorphism g^* of A such that $\pi \circ g^* = g \circ \pi$, where π is the natural epimorphism. Since A is duo, $g^*(C) \subseteq C$ and so $g(\frac{C}{\rho}) \subseteq \frac{C}{\rho}$. Thus B is duo. By a similar proof as part (ii) of Lemma 1.8, we can conclude that every Rees factor of A is quasi-projective. ∎

It is easy to see that every homomorphic image of any multiplication right S-act is multiplication. Also by Proposition 1.12.(ii), if S_S is duo, then every cyclic right S-act is duo. Thus, by Lemma 1.8, we have the following result.

Corollary 1.13 *Over a monoid S the following statements are equivalent:*

(i) *S_S is duo.*

(ii) *S_S is multiplication.*

(iii) *Every cyclic right S-act is multiplication.*

(iv) *Every cyclic right S-act is duo.*

Proposition 1.14 *Suppose S is a monoid and A is a right S-act. Then the following hold:*

(i) *If every countably generated subact of A is duo, then A is duo.*

(ii) *If A is duo and cyclic quasi-injective, then A is strongly duo.*

Proof. (i) Suppose $a \in A$ and $f \in End(A)$. Let $B = aS \cup f(a)S \cup f(f(a))S \cup \ldots$. Clearly, B is a countably generated subact of A and $g := f|_B$ is an element of $End(B)$. Thus, by Lemma 1.8.(i) and by assumption, $g(a) = as$ for some $s \in S$. Hence $f(a) = g(a) = as$ for some $s \in S$ and by Lemma 1.8.(i) A is duo.

(ii) Let B be a subact of A. Since A is cyclic quasi-injective, for every $b \in B$ and every homomorphism $f : B \longrightarrow A$, there exists $\bar{f} : A \longrightarrow A$ such that $\bar{f}(bS) = f(bS)$. Hence the duo condition on A implies that $f(b) = \bar{f}(b) \in bS \subseteq B$, proving that A is strongly duo. ∎

Note that, by Propositions 1.12.(i), 1.14.(ii) and Corollary 1.3, over every monoid S there exist right S-acts which are not strongly duo (duo). To investigate the relation between projective acts and duo acts we need the concept of the "dual basis" for acts. In [3], Khaksari et al generalized the concept of dual basis from modules to acts over commutative monoids. But it is easy to see that the condition "commutativity" is not necessary for this result. Regarding this observation we have the following theorem.

Theorem 1.15 *Let S be a monoid and let A be a projective right S-act. Then A is duo if and only if A is multiplication.*

Proof. If A is multiplication, then by Lemma 1.8.(vi), A is duo. Conversely, suppose A is duo and B is a subact of A. Let $A^* = Hom(A, S)$. Since A is projective, by Theorem 1 of [3], there exists a subset $T = \{(x_\alpha : f_\alpha) : \alpha \in \Lambda\}$ of $A \times A^*$ such that for every $x \in A, x = x_\alpha f_\alpha(x)$, where $(x_\alpha, f_\alpha) \in T$. Let I be the right ideal generated by the elements of the form $f_\alpha(x)$ for $x \in B$ and $\alpha \in \Lambda$. We claim that $B = AI$. If $x \in B \subseteq A$, then $x = x_\alpha f_\alpha(x)$ for some $\alpha \in \Lambda$ and $(x_\alpha, f_\alpha) \in T$ and hence $x \in AI$. Now suppose $x \in B, a \in A$ and $\alpha \in \Lambda$. Thus $\lambda_a \circ f_\alpha \in End(A)$, where $\lambda_a : S \longrightarrow A$ is defined by $\lambda_a(s) = as$ for every $s \in S$. Consequently, $af_\alpha(x) = \lambda_a(f_\alpha(x)) \in \lambda_a(f_\alpha(xS)) \subseteq xS$ because A is duo. Hence $AI \subseteq B$ and so $B = AI$. ∎

Corollary 1.16 *Suppose S is an idempotent monoid (i.e., $I^2 = I$ for every right ideal I of S) and A is a projective right S-act. Then the following statements are equivalent:*

(i) *A is multiplication.*

(ii) *A is strongly duo.*

(iii) *A is duo.*

Proof. (i)$\longleftrightarrow$(iii) holds by the previous theorem.

(i)$\longrightarrow$(ii) Suppose for some right ideal I of S, $B = AI$ is a subact of A. Then for every $f \in Hom(B, A), f(B) = f(AI) = f(AI^2) = f(AI)I \subseteq AI = B$ and so A is strongly duo.

(ii)$\longrightarrow$(iii) is clear. ∎

Lemma 1.17 *Suppose S is a monoid and S_S is strongly duo. If I is a projective right ideal of S, then $I^2 = I$.*

Proof. By Theorem 1 of [3], $Itr(I, S) = I$ for every projective right ideal I of S. Since S_S is strongly duo, $tr(I, S) = I$ and so $I^2 = I$. ∎

A monoid S is called *right hereditary* if all right ideals of S are projective.

Theorem 1.18 *Suppose S is a commutative monoid. Then the following hold:*

(i) *A right S-act A is strongly duo if and only if it is duo and cyclic quasi-injective. In particular, a cyclic right S-act A is strongly duo if and only if it is cyclic quasi-injective.*

(ii) *If $E(S)$ is the injective envelope of the right S-act S_S, then S_S is injective if and only if $E(S)$ is a duo right S-act.*

(iii) *If S is right hereditary, then S_S is strongly duo if and only if $I^2 = I$ for every right ideal I of S.*

Proof. (i) By Proposition 1.14.(ii), every duo cyclic quasi-injective right S-act is strongly duo. Now let A be a strongly duo right S-act. Clearly, A is duo. We show that A is cyclic quasi-injective. For this, let $f : aS \longrightarrow A$ be a homomorphism where $a \in A$. Then $f(a) \in aS$ because A is strongly duo. Thus $f(a) = as$ for some $s \in S$. Define $\bar{f} : A \longrightarrow A$ by $f(x) = xs$ for every $x \in A$. Since S is commutative, $\bar{f}$ is a well-defined homomorphism which is an extension of f. Now the result follows by the definition of cyclic quasi-injectivity. Also over a commutative monoid S, every cyclic right S-act is duo. Thus a cyclic right S-act A is strongly duo if and only if it is cyclic quasi-injective.

(ii) If $E(S)$ is a duo right S-act, then by Proposition 1.14.(ii), $E(S)$ is strongly duo and by Corollary 1.3, $S_S = E(S)$ and so S_S is injective. The converse is obvious because S is commutative and so it is duo.

(iii) The necessity follows by Lemma 1.17. Conversely, by Theorem 2 of [3], every right ideal I of S is multiplication and so it is strongly duo by Corollary 1.16. ∎

Remark. Note that, in the previous theorem, the condition "cyclic quasi-injectivity" is necessary. Indeed, if $S = (\mathbb{Z}, .)$, then S_S is clearly duo, but by Theorem 1.2, S_S is not strongly duo. In fact S_S is not cyclic quasi-injective.

A right S-act A is called *torsion free* if for any $a, b \in A$ and for any right cancelable element $c \in S$ the equality $ac = bc$ implies $a = b$.

Proposition 1.19 *Suppose S is a monoid and S_S is strongly duo. Then the following hold:*

(i) *Every left cancelable element of S is invertible.*

(ii) *All right S-acts are divisible (torsion free).*

(iii) *Every right cancelable element of S is invertible.*

(iv) *If S is left (right) cancelative, then S is a group.*

Proof. (i) If c is a left cancelable element of S, then $ann(c) \subseteq ann(s)$ for every $s \in S$. Thus by Theorem 1.2, $s \in cS$ for every $s \in S$ and therefore $S = cS$. Consequently $cx = 1$ for some $x \in S$. Also since S_S is duo, by Lemma 1.8.(iii), $1 = cx = xy$ for some $y \in S$, which implies c is invertible.

(ii) By (i) all right S-acts are torsion free. Also by (i) S_S is divisible and hence all right S-acts are divisible by Proposition 3.2.2 of [4].

(iii) is clear by part (ii) and Theorem 4.6.1 of [4] and Lemma 1.8.(iii).

(iv) holds by parts (i) and (iii). ■

Proposition 1.20 *Suppose S is a monoid and $\{A_i : i \in I\}$ is a family of right S-acts. Then the following hold:*

(i) *If $\prod_{i\in I} A_i$ $\left(\coprod_{i\in I} A_i\right)$ is duo, then for every $i \in I$, A_i is duo.*

(ii) *If $\prod_{i\in I} A_i$ $\left(\coprod_{i\in I} A_i\right)$ is strongly duo, then for every $i \in I$, A_i is strongly duo.*

Proof. (i) Suppose $A = \prod_{i\in I} A_i$ is duo. For a fixed element $j \in I$, let $f_j \in End(A_j)$. Define $f : A \longrightarrow A$, by $f(\{a_i\}_{i\in I}) = \{b_i\}_{i\in I}$, where

$$b_i = \begin{cases} a_i, & i \neq j; \\ f_j(a_j), & i = j, \end{cases} \tag{1.1}$$

for every element $\{a_i\}_{i\in I}$ of $A = \prod_{i\in I} A_i$. Clearly $f \in End(A)$ and by Lemma 1.8.(i), $f(\{a_i\}_{i\in I}) = \{a_i\}_{i\in I}s = \{a_is\}_{i\in I}$ for some $s \in S$. Thus, for every, $a_j \in A_j$, $f_j(a_j) = a_js$, for some $s \in S$, and so A_j is duo by Lemma 1.8.(i). The proof for $A = \coprod_{i\in I} A_i$ is straightforward and will be omitted.

(ii) is obvious by Theorem 1.2. ■

In general, if $\{A_i : i \in I\}$ is a family of duo (strongly duo) right S-acts, then $\prod_{i\in I} A_i$ and $\coprod_{i\in I} A_i$ are not duo (strongly duo). See the following example.

Example 1.21

(i) If S is a monoid and θ_S is the one element right S-act, then clearly θ_S is duo (strongly duo). But we can easily see that $\theta_S \sqcup \theta_S$ is not duo (strongly duo).

(ii) Suppose S is a commutative monoid which is not a group and S_S is principally weakly injective. By Theorem 1.18.(i), S_S is strongly duo and so it is duo, but $(S \times S)_S$ is not duo (strongly duo). To see this, define $f : (S \times S)_S \longrightarrow (S \times S)_S$ by $f(x, y) = (y, x)$, for every $(x, y) \in S \times S$. It is clear that f is a homomorphism. Thus if $(S \times S)_S$ is duo, then by Lemma 1.8.(i), for every $(x, y) \in S \times S, f(x, y) = (x, y)s$ for some $s \in S$. It implies $S = xS = Sx$ for every $x \in S$, and hence S is a group, a contradiction.

2. Strongly duo and duo acts over monoids with a zero element

In this section, we study the behavior of duo and strongly duo acts under the formation of 0-decomposition of an act. We assume that S is a monoid with a zero. If A is a right S-act, then A is called *centered* if A has a unique zero element. Let A be a centered right S-act with a zero element θ_A. By a 0-decomposition of A we mean an expression $A = \bigcup_{i \in I} A_i$ of A such that for every i, A_i is a centered subact of A and for all distinct $i, j \in I, A_i \cap A_j = \theta_A$. In what follows, we will use the notation $A = \coprod_{i \in I}^{0} A_i$ for this concept. By [2], over monoids with a zero element, every centered right S-act has a 0-decomposition.

Proposition 2.1 *Suppose S is a monoid and A is a right S-act. If $A = \coprod_{i \in I}^{0} A_i$ is a 0-decomposition of A, then the following hold:*

(i) *For every $i \in I$, A_i is a fully invariant subact of A if and only if $Hom(A_i, A_k) = 0$ for all distinct $i, k \in I$.*

(ii) *If A is duo, then for every $i \in I, A_i$ is duo.*

Proof. (i) Suppose $i \in I$ and A_i is a fully invariant subact of A. For $k \neq i$ in I, let $f \in Hom(A_i, A_k)$ and suppose $j_k : A_k \longrightarrow A$ is the inclusion map and $\pi_i : A \longrightarrow A_i$ is the canonical projection. Clearly $\bar{f} = j_k \circ f \circ \pi_i$ is an endomorphism of A and so by assumption $\bar{f}(A_i) \subseteq A_i$. Since $\bar{f}(A_i) = f(A_i) \subseteq A_k$, $f(A_i) \subseteq A_i \cap A_k = 0$ and hence $f = 0$. Conversely, suppose $i \in I$ and for every $k \neq i$ in $I, Hom(A_i, A_k) = 0$. We show that A_i is a fully invariant subact of A. For this let $f \in End(A)$ and suppose $j_i : A_i \longrightarrow A$ is the inclusion map and $\pi_i : A \longrightarrow A_i$ is the canonical projection. It is clear that for every $k \neq i$ in $I, \pi_k \circ f \circ j_i \in Hom(A_i, A_k) = 0$, where $\pi_k : A \longrightarrow A_k$ is the canonical projection. Thus $f(A_i) \subseteq \bigcup_{k \in I} (\pi_k f j_i)(A_i) \subseteq \pi_i f j_i(A_i) \subseteq A_i$.

(ii) Suppose $i \in I$ and $f_i \in End(A_i)$. If $\pi_i : A \longrightarrow A_i$ is the canonical projection and $j_i : A_i \longrightarrow A$ is the inclusion map, then $f = j_i \circ f_i \circ \pi_i \in End(A)$. Thus for every subact B of A_i, $f_i(B) = f(B) \subseteq B$ because A is duo. ∎

Theorem 2.2 *Suppose S is a monoid and A is a right S-act. If $A = \coprod^{0}_{i \in I} A_i$ is a 0-decomposition of A, then A is duo if and only if for every $i \in I, A_i$ is duo and $Hom(A_i, A_k) = 0$ for all distinct $i, k \in I$.*

Proof. The necessity follows by the previous proposition. Conversely, suppose B is a subact of A and $f \in End(A)$. Let $\pi_k : A \longrightarrow A_k$ denote the canonical projection and $j_k : A_k \longrightarrow A$ denote the inclusion map for every $k \in I$. Thus $\pi_k \circ f \circ j_k(A_k \cap B) \subseteq A_k \cap B$ because $\pi_k \circ f \circ j_k \in End(A_k)$ and A_k is duo for every $k \in I$. Also by assumption, $\pi_k \circ f \circ j_i(A_k \cap B) = 0$ for all distinct $k, i \in I$. Since $B = \bigcup_{k \in I}(A_k \cap B)$, $f(B) \subseteq \bigcup_{k \in I} f(A_k \cap B) \subseteq \bigcup_{k \in I} \pi_k f j_k(A_k \cap B) \subseteq \bigcup_{k \in I}(A_k \cap B) \subseteq B$, i.e., B is fully invariant, which implies A is duo. ∎

Theorem 2.3 *Suppose S is a commutative monoid and A is a right S-act. If $A = \coprod^{0}_{i \in I} A_i$ is a 0-decomposition of A, then A is strongly duo if and only if $A_i \overset{0}{\sqcup} A_j$ is strongly duo for all distinct $i, j \in I$.*

Proof. The necessity follows by Theorem 1.2. Conversely, suppose for every $i \neq j \in I$, $A_i \overset{0}{\sqcup} A_j$ is strongly duo. Since $A_i \overset{0}{\sqcup} A_j$ is duo, by Proposition 2.1.(i), $Hom(A_i, A_j) = 0$ for every $i \neq j \in I$. Also by Theorem 1.2, for every $i \in I$, A_i is strongly duo and so is duo. Hence by Theorem 2.2, A is duo. Now we show that A is cyclic quasi-injective. Suppose aS is a cyclic subact of A and $f \in Hom(aS, A)$. If $a \in A_i$ and $f(a) \in A_j$ for some $i \neq j \in I$, then $a, f(a) \in A_i \overset{0}{\sqcup} A_j$. Since $ann(a) \subseteq ann(f(a))$, by Theorem 1.2, $f(a) \in aS \subseteq A_i$ and so $f(a) \in A_i \cap A_j = 0$. Define $\bar{f} : A \longrightarrow A$ by $\bar{f}(x) = \theta_A$ for every $x \in A$. Clearly $\bar{f}$ is an extension of f and so in this case, A is cyclic quasi-injective. Now suppose $a, f(a) \in A_i$ for some $i \in I$. Since A_i is strongly duo, A_i is cyclic quasi-injective by Theorem 1.18.(i). Hence there exists $\bar{f} : A_i \longrightarrow A_i$ such that $\bar{f}(a) = f(a)$. Define $g : A \longrightarrow A$ by

$$g(x) = \begin{cases} \bar{f}(x), & x \in A_i; \\ \theta_A, & x \notin A_i. \end{cases} \tag{2.1}$$

It is easy to see that g is a well-defined homomorphism and is an extension of f. It follows that A is cyclic quasi-injective and the proof is complete by Theorem 1.18.(i). ∎

Definition 2.4 Suppose S is a monoid and A is a right S-act. We say that A is *weakly duo* if every non-zero subact of A contains a non-zero subact which is fully invariant in A.

Theorem 2.5 *Let S be a monoid and A be a quasi-projective right S-act. The following are equivalent:*

(i) *A is a duo right S-act.*

(ii) *Every factor of A is weakly duo.*

Proof. (i)$\longrightarrow$ (ii) By Proposition 1.12.(ii), every factor of A is duo and so is weakly duo.

(ii)$\longrightarrow$(i) Suppose B is a non-zero subact of A such that B is not fully invariant. By the hypothesis, there exists a non-zero subact $C \subset B$ which is fully invariant in A. Let $D = \bigcup\{L : L$ is a fully invariant subact of A and $L \subset B\}$. Clearly D is a fully invariant subact of A and $D \neq B$. Thus B/D is a non-zero subact of A/D and so by assumption there exists a subact $E \subseteq B$ such that E/D is a non-zero fully invariant subact of A/D. Thus $D \subset E \subseteq B$. By Lemma 1.8.(ii), E is fully invariant in A and by the choice of D we must have $B = E$, a contradiction. ∎

By the previous theorem we give the following result.

Corollary 2.6 *If S is a monoid, then S_S is duo if and only if every cyclic right S-act is weakly duo.*

Theorem 2.7 *If S is a monoid and A is a projective right S-act, then the following are equivalent:*

(i) *A is duo.*

(ii) *A is multiplication.*

(iii) *Every factor of A is weakly duo.*

Proof. The result follows by Theorems 1.15 and 2.5. ∎

Recall that a subact B of an S-act A is called *essential* in A, if $B \cap C \neq 0$ for each $0 \neq C \leq A$ (see [5]).

Proposition 2.8 *If S is a monoid and A is a duo right S-act, then for every monomorphism $f : A \longrightarrow A$, $f(A)$ is an essential subact of A.*

Proof. Suppose $f : A \longrightarrow A$ is a monomorphism and $0 \neq B$ is a subact of A such that $B \cap f(A) = 0$. Since A is duo, $f(B) \subseteq B$, and so $f(B) \subseteq B \cap f(A)$. Thus $f(B) = 0$ and so $B = 0$, a contradiction. ∎

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