ON $(N(k), \xi)$-SEMI-RIEMANNIAN 3-MANIFOLDS

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Abstract. The object of the present paper is to study 3-dimensional $(N(k), \xi)$-semi-Riemannian manifolds. We study $(N(k), \xi)$-semi-Riemannian 3-manifolds which are Ricci-semi-symmetric, locally ϕ-symmetric and have η-parallel Ricci tensor.

Key words and phrases: $(N(k), \xi)$-semi-Riemannian 3-manifold, Ricci-semi-symmetric, locally ϕ-symmetric, η-parallel Ricci tensor, η-Einstein manifold.

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1. Introduction

Let (M, g) be an n-dimensional semi-Riemannian manifold [12] equipped with a semi-Riemannian metric g. If index(g)=1 then g is a Lorentzian metric and (M, g) a Lorentzian manifold [4]. If g is positive definite then g is an usual Riemannian metric and (M, g) a Riemannian manifold. The notion of $(N(k), \xi)$-semi-Riemannian structure was introduced and studied by Tripathi and Gupta [21] to unify $N(k)$-contact metric [3], Sasakian [5], [14], (ϵ)-Sasakian [17], [22], Kenmotsu [10], para-Sasakian [15], (ϵ)-para-Sasakian structures [20].

In this paper we study 3-dimensional $(N(k),\xi)$-semi-Riemannian manifolds. The paper is organized as follows. Section 2 is devoted to some basic definitions and properties of almost contact metric, almost para contact metric and $(N(k),\xi)$-semi-Riemannian manifolds. Further, we prove that an $(N(k),\xi)$-semi-Riemannian 3-manifold is a space form if and only if the scalar curvature r of the manifold is equal to $6k$. In Section 3, we show that a Ricci-semi-symmetric $(N(k),\xi)$-semi-Riemannian 3-manifold is a space-form. In Section 4, a necessary and sufficient condition for an $(N(k),\xi)$-semi-Riemannian 3-manifold to be locally ϕ-symmetric is obtained. Section 5 contains some results on $(N(k),\xi)$-semi-Riemannian 3-manifold with η-parallel Ricci tensor.

2. Preliminaries

Let M be an n-dimensional differentiable manifold endowed with an almost contact structure (ϕ,ξ,η), where ϕ is a $(1,1)$-tensor field, ξ is a vector field and η is a 1-form on M satisfying

$$\eta(\xi)=1, \quad \phi^2=-I+\eta\otimes\xi, \tag{2.1}$$

where I denotes the identity transformation. It follows from (2.1) that

$$\eta\cdot\phi=0, \quad \phi(\xi)=0. \tag{2.2}$$

If there exists a semi-Riemannian metric g satisfying

$$g(\phi X,\phi Y)=g(X,Y)-\epsilon\eta(X)\eta(Y), \quad \forall X,Y\in\chi(M), \tag{2.3}$$

where $\epsilon=\pm1$, then the structure (ϕ,ξ,η,g) is called an (ϵ)-almost contact metric structure and M is called an (ϵ)-almost contact metric manifold. For an (ϵ)-almost contact metric manifold, we have

$$\eta(X)=\epsilon g(X,\xi) \ \textit{and} \ \epsilon=g(\xi,\xi) \ \forall X\in\chi(M). \tag{2.4}$$

When $\epsilon=1$ and index of g is 0 then M is the usual Sasakian manifold and M is a Lorentz-Sasakian manifold for $\epsilon=-1$ and index of g is 1.

If $d\eta(X,Y)=g(\phi X,Y)$, then M is said to have (ϵ)-contact metric structure (ϕ,ξ,η,g). For $\epsilon=1$ and g Riemannian, M is the usual contact metric manifold [5]. A contact metric manifold with $\xi\in N(k)$, is called a $N(k)$-contact metric manifold [1, 6]. If moreover, this structure is normal, that is, if

$$[\phi X,\phi Y]+\phi^2[X,Y]-\phi[X,\phi Y]-\phi[\phi X,Y]=-2d\eta(X,Y)\xi, \tag{2.5}$$

then the (ϵ)-contact metric structure is called an (ϵ)-Sasakian structure and the manifold endowed with this structure is called (ϵ)-Sasakian manifold. The physical importance of indefinite Sasakian manifolds have been pointed out by Duggal in [9].

An (ϵ)-almost contact metric structure (ϕ, ξ, η, g) is (ϵ)-Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \epsilon\eta(Y)X, \quad \forall X, Y \in \chi(M), \tag{2.6}$$

where ∇ is the Levi-Civita connection with respect to g. Also we have

$$\nabla_X \xi = -\epsilon\phi X \quad \forall X \in \chi(M). \tag{2.7}$$

An almost contact metric manifold is a Kenmotsu manifold [10] if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X. \tag{2.8}$$

By (2.8), we have

$$\nabla_X \xi = X - \eta(X)\xi.$$

If in (2.1), the condition $\phi^2 = -I + \eta \otimes \xi$ is replaced by

$$\phi^2 = I - \eta \otimes \xi \tag{2.9}$$

then (M, g) is called an (ϵ)-almost paracontact metric manifold equipped with an (ϵ)-almost paracontact metric structure (ϕ, ξ, η, g).

An (ϵ)-almost paracontact metric structure is called (ϵ)-para-Sasakian structure [20] if

$$(\nabla_X \phi)Y = -g(\phi X, \phi Y)\xi - \epsilon\eta(Y)\phi^2 X, \tag{2.10}$$

where ∇ is Levi-Civita connection with respect to the metric g. A manifold endowed with an (ϵ)-para-sasakian structure is called (ϵ)-para-Sasakian manifold [20]. For $\epsilon = 1$ and g Riemannian, M is the usual para-Sasakian manifold [15].

$(N(k), \xi)$-semi-Riemannian manifold

The *k-nullity distribution* [18] of (M, g) is the distribution

$$N(k) : p \to N_p(k) = \{Z \in T_pM : R(X,Y)Z = k(g(Y,Z)X - g(X,Z)Y)\}, \tag{2.11}$$

where k is a real number.

An $(N(k), \xi)$-semi-Riemannian manifold consists of a semi-Riemannian manifold (M, g), a k-nullity distribution $N(k)$ on (M, g) and a non-null unit vector field ξ in (M, g) belonging to $N(k)$. Throught the paper we assume that $X, Y, Z, U, V, W \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields in M, unless specifically stated otherwise. Let ξ be a non null unit vector field in (M, g) and η its associated 1-form. Thus

$$g(\xi, \xi) = \epsilon,$$

where $\epsilon = 1$ or -1 according as ξ is spacelike or timelike, and

$$a) g(X, \xi) = \epsilon\eta(X), \quad b) \eta(\xi) = 1. \tag{2.12}$$

In an n-dimensional $(N(k),\xi)$-semi-Riemannian manifold (M,g), the following relations hold [21]:

$$R(X,Y)\xi = \epsilon k\{\eta(Y)X-\eta(X)Y\}, \tag{2.13}$$

$$R(\xi,X)Y = \epsilon k\{\epsilon g(X,Y)\xi-\eta(Y)X\}, \tag{2.14}$$

$$\eta(R(X,Y)Z) = k\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\}, \tag{2.15}$$

$$S(X,\xi) = \epsilon k(n-1)\eta(X), \tag{2.16}$$

In a 3-dimensional Riemannian manifold we have

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX-g(X,Z)QY+S(Y,Z)X-S(X,Z)Y \\ &\quad -\frac{r}{2}\,[g(Y,Z)X-g(X,Z)Y], \end{aligned} \tag{2.17}$$

where Q is the Ricci operator, i.e., $g(QX,Y)=S(X,Y)$ and r is the scalar curvature of the manifold. Putting $Z=\xi$ in (2.17) and using (2.13) and (2.16), we have

$$\epsilon(\eta(Y)QX-\eta(X)QY)=\left(-\epsilon k+\frac{r}{2}\epsilon\right)(\eta(Y)X-\eta(X)Y). \tag{2.18}$$

Putting $Y=\xi$ in (2.18) and then using (2.12(b)) and (2.16) (for n=3), we get

$$QX=\frac{1}{2}\{(r-2k)X-(r-6k)\eta(X)\xi\}, \tag{2.19}$$

that is,

$$S(X,Y)=\frac{1}{2}\{(r-2k)g(X,Y)-\epsilon(r-6k)\eta(X)\eta(Y)\}. \tag{2.20}$$

An $(N(k),\xi)$-semi-Riemannian manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y) \tag{2.21}$$

for any vector fields X, Y where a,b are functions on M. Hence from (2.20) we can state the following:

Lemma 1 *A 3-dimensional $(N(k),\xi)$-semi-Riemannian manifold is an η-Einstein manifold.*

By using (2.19) and (2.20) in (2.17), we obtain

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2}-2k\right)\{g(Y,Z)X-g(X,Z)Y\} \\ &\quad -\left(\frac{r}{2}-3k\right)\{g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi \\ &\quad +\epsilon\eta(Y)\eta(Z)X-\epsilon\eta(X)\eta(Z)Y\}. \end{aligned} \tag{2.22}$$

An $(N(k),\xi)$-semi-Riemannian 3-manifold is a space of constant curvature then it is an indefinite space form.

Remark. Relations (2.19), (2.20) and (2.22) are true for

1. An $N(k)$-contact metric 3-manifold [8] if $\epsilon = 1$,
2. A Sasakian 3-manifold if $k = 1$ and $\epsilon = 1$,
3. A Kenmotsu 3-manifold [7] if $k = -1$ and $\epsilon = 1$,
4. An (ϵ)-Sasakian 3-manifold if $k = 1$ and $\epsilon k = 1$,
5. A para-Sasakian 3-manifold [2] if $k = -1$ and $\epsilon = 1$,
6. An (ϵ)-para-Sasakian 3-manifold [19] if $k = -\epsilon$ and $\epsilon k = -1$.

Lemma 2 *A 3-dimensional $(N(k),\xi)$-semi-Riemannian manifold is a space form if and only if the scalar curvature $r = 6k$.*

Consequently, for a 3-dimensional $(N(k),\xi)$-semi-Riemannian manifold, we have the following table:

M	**S =**	**r =**
N(k)-contact metric	$\frac{1}{2}\{(r-2k)g-(r-6k)\eta\otimes\eta\}$	$6k$
Sasakian	$\frac{1}{2}\{(r-2)g-(r-6)\eta\otimes\eta\}$	6
Kenmotsu	$\frac{1}{2}\{(r+2)g-(r+6)\eta\otimes\eta\}$	-6
(ϵ)-*Sasakian*	$\frac{1}{2}\{(r-2\epsilon)g-\epsilon(r-6\epsilon)\eta\otimes\eta\}$	6ϵ
para-Sasakian	$\frac{1}{2}\{(r+2)g-(r+6)\eta\otimes\eta\}$	-6
(ϵ)-*para Sasakian*	$\frac{1}{2}\{(r+2\epsilon)g-\epsilon(r+6\epsilon)\eta\otimes\eta\}$	-6ϵ

Proof. Let a 3-dimensional $(N(k),\xi)$-semi-Riemannian manifold be an indefinite space form. Then

$$R(X,Y)Z = c\{g(Y,Z)X - g(X,Z)Y\}, \qquad X,Y,Z \in \chi(M), \tag{2.23}$$

where c is the constant curvature of the manifold. By using the definition of Ricci curvature and (2.23) we have

$$S(X,Y) = 2cg(X,Y). \tag{2.24}$$

If we use (2.24) in the definition of the scalar curvature we get

$$r = 6c. \tag{2.25}$$

From (2.24) and (2.25) one can easily see that

$$S(X,Y) = \frac{r}{3}g(X,Y). \tag{2.26}$$

By plugging $X = Y = \xi$ in (2.20) and using (2.26) we obtain

$$r = 6k. \tag{2.27}$$

Conversely, if $r = 6k$, then from the equation (2.22) we can easily see that the manifold is a space form. This completes the proof. ■

3. Ricci-semi-symmetric $(N(k), \xi)$-semi-Riemannian 3-manifolds

A semi-Riemannian manifold M is said to be Ricci semi-symmetric [13] if its Ricci tensor S satisfies the condition

$$R(X,Y)\cdot S = 0, \qquad X, Y \in \chi(M), \tag{3.28}$$

where $R(X,Y)$ acts as a derivation on S. Ricci-semisymmetric manifold is a generalization of manifold of constant curvature, Einstein manifold, Ricci symmetric manifold, symmetric manifold and semisymmetric manifold. Ricci-semisymmetric condition for Kenmotsu 3-manifolds, (ϵ)-para-Sasakian 3-manifolds and LP-Sasakian 3-manifolds are studied in [7], [19] and [16] respectively.

Let M be a Ricci-semi-symmetric $(N(k), \xi)$-semi-Riemannian 3-manifold. From (3.28) we have

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0. \tag{3.29}$$

If we put $X = \xi$ in (3.29) and use (2.14), then we get

$$kg(Y,U)S(\xi,V) - \epsilon K\eta(U)S(Y,V) + kg(Y,V)S(U,\xi) - \epsilon K\eta(V)S(U,Y) = 0. \tag{3.30}$$

By using (2.16) in (3.30) we obtain

$$\epsilon K\{2kg(Y,U)\eta(V) - \eta(U)S(Y,V) - 2kg(Y,V)\eta(U) - \eta(V)S(U,Y)\} = 0. \tag{3.31}$$

Consider that $\{e_1, e_2, e_3\}$ be an orthonormal basis of the T_pM, $p \in M$. Then, by putting $X = U = e_i$ in (2.2) and taking the summation for $1 \leq i \leq 3$, we have

$$\epsilon k\{8k\eta(V) - \epsilon S(V,\xi) - r\eta(V)\} = 0. \tag{3.32}$$

Again, by using (2.16) in (3.32), we get

$$\epsilon k(6k - r)\eta(V) = 0, \tag{3.33}$$

which gives $r = 6k$. This implies, in view of Lemma 2, that the manifold is a space form.

Therefore, we have the following:

Theorem 1 *A Ricci-semi-symmetric $(N(k), \xi)$-semi-Riemannian 3-manifold is a space form.*

From Theorem 1 and the above table, we can state the following corollaries:

Corollary 1 *A Ricci-semi-symmetric $N(k)$-contact metric 3-manifold is a manifold of constant scalar curvature $6k$.*

Corollary 2 *A Ricci-semi-symmetric Sasakian* 3*-manifold is a manifold of constant positive scalar curvature* 6*.*

Corollary 3 [7] *A Ricci-semi-symmetric Kenmotsu* 3*-manifold is a manifold of constant negative scalar curvature* -6*.*

Corollary 4 *A Ricci-semi-symmetric* (ϵ)*-Sasakian* 3*-manifold is an indefinite space form.*

Corollary 5 [2] *A Ricci-semi-symmetric para-Sasakian* 3*-manifold is a manifold of constant negative scalar curvature* -6*.*

Corollary 6 [19] *A Ricci-semi-symmetric* (ϵ)*-para-Sasakian* 3*-manifold is an indefinite space form.*

4. Locally ϕ-symmetric $(N(k),\xi)$-semi-Riemannian 3-manifolds

Definition 1 An $(N(k),\xi)$-semi-Riemannian manifold is said to be locally ϕ-symmetric if

$$\phi^2(\nabla_W R)(X,Y,Z)=0,$$

for all vector fields W,X,Y,Z orthogonal to ξ. This notion was introduced for Sasakian manifolds by Takahashi [17].

Now, differentiating (2.22) covariantly with respect to W, we get

$$\begin{aligned}(\nabla_W R)(X,Y,Z)=&\frac{1}{2}(\nabla_W r)\{g(Y,Z)X-g(X,Z)Y-g(Y,Z)\eta(X)\xi\\&+g(X,Z_\eta(Y)\xi-\epsilon\eta(Y)\eta(Z)X+\epsilon\eta(X)\eta(Z)Y\}\\&-\frac{(r-6k)}{2}\{g(Y,Z)((\nabla_W\eta)(X)\xi+\eta(X)\nabla_W\xi)\\&-g(X,Z)((\nabla_W\eta)(Y)\xi+\eta(Y)\nabla_W\xi)\\&+\epsilon((\nabla_W\eta)(Y)\eta(Z)X+(\nabla_W\eta)(Z)\eta(Y)X)\\&-\epsilon((\nabla_W\eta)(X)\eta(Z)Y+(\nabla_W\eta)(Z)\eta(X)Y)\}.\end{aligned}$$

Taking W,X,Y,Z orthogonal to ξ, we have

$$\begin{aligned}(\nabla_W R)(X,Y,Z)=&\frac{1}{2}(\nabla_W r)\{g(Y,Z)X-g(X,Z)Y\}\\&-\frac{(r-6k)}{2}\{g(Y,Z)(\nabla_W\eta)(X)\xi-g(X,Z)(\nabla_W\eta)(Y)\xi\}.\end{aligned}\tag{4.34}$$

Applying ϕ^2 on both sides of the above equation and using $\phi\cdot\xi=0$, we have

$$\phi^2((\nabla_W R)(X,Y,Z))=\frac{1}{2}(\nabla_W r)\{g(Y,Z)\phi^2X-g(X,Z)\phi^2Y\}.\tag{4.35}$$

Now taking X, Y are orthogonal to ξ, we obtain

$$\phi^2((\nabla_W R)(X,Y,Z)) = -\frac{1}{2}(\nabla_W r)\{g(Y,Z)X - g(X,Z)Y\} \tag{4.36}$$

Hence from (4.36), we can state the following:

Theorem 2 *An $(N(k),\xi)$-semi-Riemannian 3-manifold is locally ϕ-symmetric if and only if the scalar curvature r is constant.*

If an $(N(k),\xi)$-semi-Riemannian 3-manifold is Ricci semi-symmetric, then we have showed that $r = 6k$, that is r is constant.

Therefore, from Theorem (2), we have

Theorem 3 *A Ricci-semi-symmetric $(N(k),\xi)$-semi-Riemannian 3-manifold is locally ϕ-symmetric.*

5. $(N(k),\xi)$-semi-Riemannian 3-manifold with η-parallel Ricci tensor

Definition 2 The Ricci tensor S of an $(N(k),\xi)$-semi-Riemannian manifold M is called η-parallel if it satisfies

$$(\nabla_Z S)(\phi X, \phi Y) = 0 \tag{5.37}$$

for all vector fields X, Y and Z. The notion of Ricci-η-parallelity for Sasakian manifolds was introduced by Kon in [11].

Now, let us consider a 3-dimensional $(N(k),\xi)$-semi-Riemannian manifold with η-parallel Ricci tensor. Then, from (2.20), we get

$$S(\phi X, \phi Y) = \frac{1}{2}(r-2k)[g(\phi X, \phi Y)]. \tag{5.38}$$

Differentiating (5.38) covariantly along Z, we have

$$(\nabla_Z S)(\phi X, \phi Y) = \frac{1}{2}dr(Z)g(\phi X, \phi Y). \tag{5.39}$$

If the Ricci tensor is η-parallel, then from (5.37) and (5.39) one can get

$$\frac{1}{2}dr(Z)g(\phi X, \phi Y) = 0.$$

From which, it follows that

$$dr(Z) = 0,$$

for all Z. This leads us to the following:

Theorem 4 *Let M be an $(N(k),\xi)$-semi-Riemannian 3-manifold with η-parallel Ricci tensor. The the scalar curvature r is constant.*

In view of Theorem (2) and Theorem (4), we have the following:

Theorem 5 *An $(N(k),\xi)$-semi-Riemannian 3-manifold with η-parallel Ricci tensor is locally ϕ-symmetric.*

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