MORE PROPERTIES ON FLEXIBLE GRADED MODULES

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Abstract. In this paper, we study the structure of flexible graded modules over various types of graded rings such as first strong and augmented graded rings. Also, we introduce the notions of flexibly simple and flexibly Noetherian modules, and investigate various properties of such modules.

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1. Introduction

Strongly graded modules were a reasonable generalization of strongly graded rings, which play an important role in producing interesting theorems and results in Graded Ring Theory and Graded Module Theory. Of course, this is due to the nice structure of a ring or a module when it is equipped with a strong graduation. A natural question arises about whether we can extend the category of strongly graded rings and modules in order to increase the application area of these nice results. As a step on this path, first and second strongly graded rings were presented by Refai [5] to generalize strongly graded rings. Another step was presented by Refai and Moh'D [10], where first strongly graded rings were produced to generalize strongly graded modules. A second generalization of strongly graded modules was the flexible graded modules, exhibited in [12].

This paper continues the work done in [12], where we further investigate about flexible modules. In particular, we study the structure of these modules over various types of graded rings, and observe the behavior of the submodules of such modules. In addition, several characterizations related to flexible modules are listed in different places in the article. The main result in this paper demonstrates that over augmented graded rings, flexible and augmented modules are the same.

In Section 2, we list some important facts about flexible modules. These facts will be useful in the sequel sections.

In Section 3, we study flexible graded modules when the ring is first strong. We study various properties in this stage. In particular, we give a partial answer to the validity of the statement "A module is flexible iff every submodule is flexible". Also, a partial answer of the question "whether every module can be graduated to be flexible?" is presented. On the other hand, we generalize some results of [3]. The main result of this section asserts that for flexible modules over commutative graded rings, every submodule can be graduated to become flexible. This assertion can be interpreted as "In flexible modules over commutative rings, every discussion about a property concerning submodules can be carried out regarding them as flexible submodules". This fact allows us to use the nice structure and the smoothness of flexible modules.

In Section 4, we further study the structure of flexible modules over augmented graded rings. The main result in this section shows that flexible modules and augmented graded modules are the same over augmented graded rings. This result allows us to build augmented graded modules through flexible modules, and avoid checking out the rather complicated conditions of augmented graded modules.

In Section 5, we introduce the notions of flexibly simple and flexibly Noetherian modules. These concepts generalize the concepts of flexible simple and flexible Noetherian modules [14], which were defined only for flexible modules. We show that in flexible modules over commutative graded rings, "flexibly simple" (resp. "flexibly Noetherian") property implies "simple" (resp. "Noetherian") property.

2. Preliminaries

This section presents some necessary background of graded rings and graded modules considered in this paper. More details can be found in [5, 6, 7, 12]. For a general background of graded rings and graded modules, we advise the reader to look in [1, 2, 3]. Throughout this paper, unless otherwise stated, G is a group with identity e, $R = \bigoplus_{g\in G} R_g$ is a $G-$graded ring with unity 1, and $M = \bigoplus_{g\in G} M_g$ is a $G-$graded left $R-$module. The set $supp(R,G) = \{g \in G : \; R_g \neq 0\}$ is called the support of R. The support of M, $supp(M,G)$, is defined similarly. To avoid repetition, we assume $R \neq 0$, and $M \neq 0$. One more thing, all modules considered in this article are left modules.

Definition 2.1 [12] Let R be a $G-$graded ring, and M a $G-$graded $R-$module. Then M is said to be flexible if $M_g = R_g\, M_e$, for every $g \in G$.

Proposition 2.2 [12] *Let R be a $G-$graded ring, and M a $G-$graded $R-$module. Then M is flexible iff $M = R\, M_e$.*

Remark 2.3 [12] Let M be a flexible $G-$graded $R-$module. Then

1. $M \neq 0$ iff $M_e \neq 0$ iff $e \in spp(M, G)$.
2. $supp(M, G) \subseteq supp(R, G)$.

Definition 2.4 [12] Let R be a $G-$graded ring, and M be a flexible $R-$module. A $G-$graded $R-$submodule N of M is said to be flexible, if N is a flexible $R-$ module.

Proposition 2.5 [12] *Let R be a $G-$graded ring, and M be a $G-$graded $R-$module. If X is an R_e-submodule of M_e, then RX is a flexible R-submodule of M.*

3. Flexible modules over first strongly graded rings

This section is devoted to study the structure of flexible modules over first strongly graded rings. In addition, some different properties are considered. For instance, flexibility (i.e., to become flexible) results such as Theorem 3.8 and Proposition 3.10.

Definition 3.1 [5] A $G-$graded ring R is said to be first strong iff $R_gR_h = R_{gh}$, for all $g, h \in supp(R, G)$ iff $R_gR_{g^{-1}} = R_e$, for all $g \in supp(R, G)$ iff $1 \in R_gR_{g^{-1}}$, for all $g \in supp(R, G)$.

Proposition 3.2 [5] *If R is a first strongly $G-$graded ring, then $supp(R, G)$ is a subgroup of G.*

Lemma 3.3 [10] *Let R be a first strongly $G-$graded ring, and M be an $R-$module. If N and L are R_e-submodules of M, then $R_g\,(N \cap L) = R_g\, N \cap R_g\, L$, for every $g \in G$.*

The following proposition tells that on first strongly graded rings, flexibility of a graded module is completely determined by the behavior of the supports of the ring and the module.

Proposition 3.4 [10] *Let R be a first strongly $G-$graded ring, and M be a $G-$graded $R-$module. Then M is flexible iff $supp(M, G) = supp(R, G)$.*

Lemma 3.5 [10] *Suppose R is a first strongly $G-$graded ring, and M is an $R-$module. If X is an $R-$submodule of M, then $R_g\ X = X$, for every $g \in supp(R, G)$.*

Proposition 3.6 [14] *Let R be a first strongly $G-$graded ring, and M be a flexible $R-$module. Then every $G-$graded $R-$submodule of M is also flexible.*

Proof. We have $R_{g^{-1}}M_g = M_e$, for every $g \in supp(R,G)$. Let $X = \bigoplus_{g\in G} X_g$ be a $G-$graded R-submodule of M. Fix $g \in supp(R,G)$. By Lemma 3.5, $R_gX = X$. So, Lemma 3.3 implies $X_g = R_gR_{g^{-1}}(X \cap M_g) = R_g(X \cap M_e) = R_gX_e$. If $g \notin supp(R,G)$, Proposition 3.4 yields $X_g = R_gX_e$. ∎

Proposition 3.6 asserts that for flexible modules over first strongly graded rings, there is no graded submodule of M , which is not flexible. In the next two results, if R is commutative, we prove two facts. The first fact states that we can drop the "first strong" condition from Proposition 3.6. The second fact states that not only are graded submodules in this proposition flexible, but also that every submodule is graded and flexible.

Lemma 3.7 *Let R be a commutative $G-$graded ring, and M be a $G-$graded $R-$module. If N is an $R-$submodule of $R\,M_e$, then there exists an R_e-submodule X of M_e such that $N = R\,X$. Further, N is a flexible $G-$graded $R-$submodule of M.*

Proof. Define the set X by $X = \{x \in M_e :\ r\,x \in N,\ \text{for some } r \in R\}$. Then X is an R_e-submodule of M_e. Moreover, $N = R\,X$, and it is a $G-$graded flexible $R-$submodule of M, with $N_g = R_g\,X$, for each $g \in G$. ∎

Theorem 3.8 *Let R be a commutative $G-$graded ring, and M be a flexible $G-$graded $R-$module. Then every $R-$submodule of M is $G-$graded and flexible $R-$submodule of M.*

Proof. Apply Lemma 3.7 to M. ∎

Remark 3.9 Theorem 3.8 shows how strong the structure of flexible modules is over commutative rings. In such modules, any property of submodules can be considered as a property of graded or even flexible submodules. This fact gives us the opportunity to use the nice properties of flexible modules. Hence, a question arises "Given an $R-$module M and a group G, Does a graduation of M by G exist such that M turns into a flexible $R-$module?" We give a partial answer to this question. Hopefully, we will continue searching for more answers in the near future.

Although the following proposition is not related to modules over first strongly graded rings, we prefer to put it in this section to keep the flow of the article homogeneous, especially since it was, in the first place, an indirect outcome of Proposition 3.6.

Proposition 3.10 *Let R be a $G-$graded ring, and M be a free $R-$module. Then there exists a graduation of M by G such that M is a flexible $G-$graded $R-$module.*

Proof. Let T be a basis of M. We have $X = R_e T$ is an R_e-submodule of M. For each $g \in G$, set $M_g = R_g X$. It is not difficult to see that $M_g = R_g M_e$, for each $g \in G$, and $M = \sum_{g\in G} M_g$. Let $z \in M_h \cap (\sum_{g\in G-\{h\}} M_g)$. Then there exist $r_g \in R_g$, and $x^g \in X$, where $g \in G$, such that $z = r_h x^h = \sum_{g\in G-\{h\}} r_g x^g$. So, $r_h x^h - \sum_{g\in G-\{h\}} r_g x^g = 0$. Since $x^g \in X$, there exist $t^g \in T$, and $r^g \in R_e$ such that $x^g = r^g t^g$, where $g \in G$. Thus, $r_h r^h t^h - \sum_{g\in G-\{h\}} r_g r^g t^g = 0$. Since T is linearly independent, we have $r_h r^h = 0$, which implies $z = 0$. Therefore, $M = \bigoplus_{g\in G} M_g$. Consequently, M is a flexible $G-$graded $R-$module. ∎

Recall that if M is an $R-$module, $S \subseteq R$, and $0 \neq x \in M$. Then x is called $S-$torsion free, if $rx \neq 0$, whenever $0 \neq r \in S$. Otherwise, x is said to be $S-$torsion. In addition, A subset N of M is $S-$torsion free, if every element of N is $S-$torsion free. In the upcoming work, we give a partial converse of Proposition 3.6, as shown in Corollary 3.14.

Lemma 3.11 *Let R be a $G-$graded ring, and M be a flexible $R-$module. If M_e contains an $R-$torsion free element, then supp$(M,G) =$ supp(R,G).*

Proof. Let $g \in G$. Since M_e has a torsion free element, say x, we have $M_g = 0$ iff $R_g = 0$. Actually, if $M_g = R_g M_e = 0$, then $R_g x = 0$, and hence $R_g = 0$. Therefore, *supp*$(M,G) =$ *supp*(R,G). ∎

Proposition 3.12 *Let R be a $G-$graded ring, and M be a flexible $R-$module such that M_e contains an $R-$torsion free element. If $g \in$ supp(R,G), then R_g contains a regular element (an element which is not a zero divisor) iff M_g contains an $R-$torsion free element.*

Proof. Suppose R_g contains an element r_g, which is not a zero divisor. By assumption, let $x_e \in M_e$ be an $R-$torsion free element. We have $0 \neq r_g x_e \in M_g$. Moreover, if $\alpha \in R$, and $\alpha\,(r_g x_e) = 0$, then $(\alpha\, r_g)\, x_e = 0$, which implies in turn $\alpha\, r_g = 0$. Thus, $\alpha = 0$. Hence, the element $r_g x_e$ is an $R-$torsion free in M_g.

For the converse, assume M_g contains an $R-$torsion free element. By Lemma 3.11, $g \in$ *supp*(M,G). Let $x_g = r_g\, x_e \in M_g$ be an $R-$torsion free element, where $r_g \in R_g$ and $x_e \in M_e$. Let $h \in$ *supp*(R,G), and $\alpha \in R_h$ such that $\alpha\, r_g = 0$. Then $(\alpha\, r_g) x_e = 0$ or $\alpha(r_g x_e) = 0$. Hence, $\alpha\, x_g = 0$. Since x_g is $R-$torsion free, we obtain $\alpha = 0$. Consequently, R_g has no zero divisors in R. ∎

In fact, Proposition 3.12 allows us to switch the discussion between torsion free elements in M, and regular elements in R.

Proposition 3.13 *Let R be a $G-$graded ring, and M be a flexible $R-$module such that M_g has a torsion free element, for every $g \in$ supp(M,G). Then R is first strong iff every $G-$graded cyclic submodule of M, with a homogeneous generator, is flexible.*

Proof. Assume every $G-$graded cyclic submodule of M, with a homogeneous generator, is flexible. By Lemma 3.11, $supp(M,G) = supp(R,G)$. Let $g \in supp(R,G)$, and x_g be an $R-$torsion free element in M_g. By assumption, $R\,x_g$ is flexible. Fix $h \in supp(R,G)$. We have $R_{h\,g^{-1}}\,x_g = (R\,x_g)_h = R_h(R\,x_g)_e = R_h R_{g^{-1}}\,x_g$. Since x_g is $R-$torsion free, we get $R_{h\,g^{-1}} = R_h R_{g^{-1}}$. Now, put $h = g$ in the last equation to obtain $R_e = R_g R_{g^{-1}}$. Therefore, R is first strong. The converse follows by Proposition 3.6 . ∎

Corollary 3.14 *Let R be a $G-$graded ring, and M be a flexible $R-$module such that M_g has a torsion free element, for every $g \in supp(M,G)$. Then R is first strong iff every $G-$graded $R-$submodule of M is flexible.*

As final applications of flexible modules over first strongly graded rings, we curve away to homological algebra to generalize some results of [3]. The proof of Proposition 3.15 resembles the proof of the similar result of [3]. So, we will not put it in the body of the article. On the other hand, the proof of Proposition 3.16 is inferred by combining Proposition 2.1 of [11], and the same result of [3], which is valid for strongly graded rings.

Proposition 3.15 [14] *Let R be a first strongly $G-$graded ring such that $supp(R,G)$ is finite, M be an $R-$module, and $N = \bigoplus_{g\in G} N_g$ be a flexible $R-$module. Then*

1. *For each $g \in supp(R,G)$, the function $\varphi : Hom_R(M,N) \longrightarrow Hom_{R_e}(M,N_g)$ defined by $\varphi(f) = \pi_g \circ f$ is a group isomorphism, where $\pi_g : N \longrightarrow N_g$ is the canonical projection on the $g-th$ component of N.*

2. *For each $g \in supp(R,G)$, the function $\psi : Hom_R(N,M) \longrightarrow Hom_{R_e}(N_g,M)$ defined by $\varphi(f) = f \circ i_g$ is a group isomorphism, where $i_g : N_g \longrightarrow N$ is the canonical inclusion of the $g-th$ component of N.*

Proposition 3.16 [14] *Let R be a first strongly $G-$graded ring such that $supp(R,G)$ is finite, M be an $R-$module, and $N = \bigoplus_{g\in G} N_g$ be a flexible $R-$module. Then*

1. *For each $g \in supp(R,G)$, $Ext^n_R(M,N) \cong Ext^n_{R_e}(M,N_g)$.*

2. *For each $g \in supp(R,G)$, $Ext^n_R(N,M) \cong Ext^n_{R_e}(N_g,M)$.*

4. Flexible modules over augmented graded rings

In this section, we further study the structure of flexible modules over augmented $G-$graded rings. We show in particular that over augmented graded rings, flexible modules and augmented modules are the same.

Definition 4.1 [7] A $G-$graded ring R is said to be augmented if the following conditions hold:

1. $R_e = \bigoplus_{g \in G} R_{e-g}$ is a $G-$graded ring.
2. For each $g \in G$, there exists $r_g \in R_g$ such that $R_g = R_e r_g$. We assume $r_e = 1$.
3. If $r_g \neq 0$ and $r_h \neq 0$ are as in (2), then $r_g r_h = r_{gh}$.
4. If $r_g \neq 0$ and $r_h \neq 0$ are as in (2), and $x, y \in R_e$, then $(xr_g)(yr_h) = xyr_{gh}$.

An r_g that appears in condition (2) of Definition 4.1 is called a $g-$representative. The set of all selected nonzero representatives is denoted by $\Lambda(R, G)$. Thus, the set $\Lambda(R, G)$ may vary as the representatives vary. However, once an augmented graded ring is under consideration, we fix $\Lambda(R, G)$.

Proposition 4.4 below modifies Proposition 1.12 of [9], where it shows that augmented graded rings are a subcategory of first strongly graded rings. We begin with the following lemma, which we omit its proof, because it is trivial.

Lemma 4.2 *Let R be an augmented $G-$graded ring. Then $g \in supp(R, G)$ iff $r_g \in \Lambda(R, G)$.*

Remark 4.3 [7] If R is an augmented $G-$graded ring, then

1. For every $g \in G$, R_g is a $G-$graded R_e-module, with the graduation $R_{g-h} = R_{e-h}\, r_g$.
2. For every $g, h, g', h' \in G$, $R_{g-h}\, R_{g'-h'} \subseteq R_{gg'-hh'}$.
3. Condition (4) of Definition 4.1 is equivalent to the condition

$$(x_{e-h}\, r_g)(x_{e-h'}\, r_{g'}) = x_{e-h} x_{e-h'}\, r_g r_{g'}\,,$$

for all $h, h' \in G$, and $r_g, r_{g'} \in \Lambda(R, G)$.

Proposition 4.4 *Every augmented $G-$graded ring is first strong.*

Proof. Assume R is an augmented $G-$graded ring. Let $g, h \in supp(R, G)$. By Lemma 4.2, $R_g R_h = R_e r_g\, R_e r_h = R_e R_e\, r_g r_h = R_e\, r_{gh} = R_{gh}$. Therefore, R is first strong. ∎

Corollary 4.5 *Let R be an augmented $G-$graded ring. Then*

1. *$supp(R, G)$ is a subgroup of G.*
2. *$\Lambda(R, G)$ is a multiplicative group.*
3. *$supp(R, G)$ is group-isomorphic to $\Lambda(R, G)$.*
4. *$R_g = r_g R_e = R_e\, r_g$, for every $g \in supp(R, G)$.*

Proof. 1. Follows directly from Propositions 3.2 and 4.4.

2. Apply Lemma 2.3 of [9] along with condition (1).
3. Apply Lemma 2.3 and conditions (1) and (2).
4. Apply (1) and condition (4) of Definition 4.1. ■

Remark 4.6 Corollary 4.5 gives us the right to cancel the condition "$supp(R, G)$ is a subgroup of G" from all results in [9].

Definition 4.7 [6] Let R be an augmented $G-$graded ring. A $G-$graded $R-$module M is said to be augmented if the following conditions hold:

1. $M_g = \bigoplus_{h\in G} M_{g-h}$ is a $G-$graded R_e-module.
2. $R_{g-h}\, M_{g'-h'} \subseteq M_{gg'-hh'}$, for every $g, h, g', h' \in G$.

Given an augmented $G-$graded ring, we let λ be the sum of elements of $\Lambda(R, G)$. If we set $r_g = 0$, for every $g \notin supp(R, G)$, and $r_g \in \Lambda(R, G)$, otherwise, we can write $\lambda = \sum_{g\in G} r_g$. The next proposition shows that a flexible module over an augmented ring is completely determined by M_e and λ.

Proposition 4.8 *Let R be an augmented $G-$graded ring, and M be a $G-$graded $R-$module. Then M is flexible iff $M = \lambda\, M_e$.*

Proof. Suppose M is flexible $R-$module. It follows from Propositions 3.4 and 4.4 that $supp(M, G) = supp(R, G)$. Let $g \in G$. We have $M_g = R_g\, M_e = r_g\, M_e$. Hence, $M = \bigoplus_{g\in G}(r_g\, M_e) = \lambda\, M_e$.

Conversely, Suppose $M = \lambda\, M_e$. Then $M = (\sum_{g\in G} r_g)\, M_e = \bigoplus_{g\in G} R_g\, M_e$. Since $R_g\, M_e \subseteq M_g$, we obtain $M_g = R_g\, M_e$, for every $g \in G$. Therefore, M is flexible. ■

Corollary 4.9 *Let R be an augmented $G-$graded ring, and M be an augmented $G-$graded $R-$module. If $supp(M, G) = supp(R, G)$, then M is flexible, and $M_{g-h} = r_g\, M_{e-h}$, for every $g, h \in G$.*

Proof. Assume $supp(M, G) = supp(R, G)$. By Propositions 3.4 and 4.4, M is flexible. Proposition 4.8 implies $M_g = r_g\, M_e$, for all $g \in G$, where $r_g \in \Lambda(R, G)$, if $g \in supp(R, G)$, and $r_g = 0$, otherwise. Thus, $M_{g-h} = r_g\, M_{e-h}$, for every $g, h \in G$. ■

The following proposition gives the converse of Proposition 4.8.

Proposition 4.10 *Let R be an augmented $G-$graded ring, and M be a flexible $G-$graded $R-$module such that M_e is a $G-$graded R_e-module. Then M is an augmented $G-$graded $R-$module.*

Proof. First, we prove that M_g is a $G-$graded R_e-module, with $M_{g-h} = r_g\, M_{e-h}$, for all $g, h \in G$. Propositions 3.4 and 4.4 yield $supp(M, G) = supp(R, G)$. Let $g \in supp(R, G)$ and $h \in G$. Then $\sum_{h\in G} r_g\, M_{e-h} \subseteq M_g$. On the other hand, let $x \in M_g$. Then $x = \sum_{h\in G} r_g\, m_{e-h} \in \sum_{h\in G} r_g\, M_{e-h}$ and hence $M_g = \sum_{h\in G} r_g\, M_{e-h}$. Let $\sigma \in G$, and $x \in r_g\, M_{e-\sigma} \cap \sum_{h\in G-\{\sigma\}} r_g\, M_{e-h}$. Then $x = r_g\, m_{e-\sigma} = \sum_{h\in G-\{\sigma\}} r_g\, m_{e-h}$, where $m_{e-h} \in M_{e-h}$, for every $h \in G$. By Corollary 4.5, r_g is a unit. So we get $m_{e-\sigma} - \sum_{h\in G-\{\sigma\}} m_{e-h} = 0$. Thus, we obtain $m_{e-\sigma} = 0$, and then $x = 0$. Therefore, $r_g\, M_{e-\sigma} \cap \sum_{h\in G-\{\sigma\}} r_g\, M_{e-h} = 0$. Consequently, $M_g = \bigoplus_{h\in G} r_g\, M_{e-h}$. The same result obviously holds if $g \notin supp(R, G)$. For each $g, h \in G$, set $M_{g-h} = r_g\, M_{e-h}$. Let $\sigma \in G$. Then $R_{e-\sigma}\, M_{g-h} = r_g(R_{e-\sigma}\, M_{e-h}) \subseteq M_{g-\sigma\, h}$. As a result, we obtain M_g is a $G-$graded R_e-module.

Now, we show that $R_{g-h}\, M_{\sigma-\tau} \subseteq M_{g\sigma-h\tau}$, for every $g, h, \sigma, \tau \in G$. In fact, if $g, \sigma \in supp(R, G)$, we have $R_{g-h}\, M_{\sigma-\tau} = r_g R_{e-h}\; r_\sigma M_{e-\tau} = r_g r_\sigma R_{e-h}\; M_{e-\tau} \subseteq r_g r_\sigma\, M_{e-h\tau} = M_{g\sigma-h\tau}$. The case where either $g \notin supp(R, G)$ or $\sigma \notin supp(R, G)$ is easy.

As a conclusion, M is an augmented $G-$graded $R-$module. ∎

The following result is the main result in this section. It sums up both Corollary 4.9 and Proposition 4.10 in the statement "Over augmented graded rings, flexible modules and augmented modules are the same". The proof is a quick application of Corollary 4.9 and Proposition 4.10.

Theorem 4.11 *Let R be an augmented $G-$graded ring, and M be a $G-$graded $R-$module such that M_e is a $G-$graded R_e-module. Then M is flexible $R-$module iff M is an augmented $G-$graded $R-$module, and $supp(R, G) = supp(M, G)$.*

5. Flexibly simple and flexibly Noetherian modules

In this section, We define flexibly simple and flexibly Noetherian modules, and give an analogous study to that exhibited in [8]. We start by noticing that the combination of Propositions 2.2 and 2.5 yields the fact which says "given a $G-$graded ring R, and a $G-$graded $R-$module M, then a $G-$graded submodule N of M is flexible iff there exists an R_e-submodule N_e of M_e such that $N = RN_e$". So, regardless M is flexible or not, we deduce that RM_e is the largest flexible $R-$submodule of M, and 0 is the smallest flexible $R-$submodule of M.

Now, we move to our definition of flexibly simple modules. To avoid any confusion, it is important to draw the reader's attention to the fact that Definition 5.1 is a generalization of the definition of flexible simple modules [14], which is only defined for flexible modules.

Definition 5.1 Let R be a $G-$ graded ring, and M be a $G-$graded $R-$module. Then M is called flexibly simple if 0 and RM_e are the only flexible $R-$submodules of M.

The structure of flexible modules indicates a strong relationship between flexibly simple $R-$modules and simple R_e-modules, as the following proposition illustrates.

Proposition 5.2 *Let R be a $G-$ graded ring, and M be a $G-$graded $R-$module. Then M is flexibly simple iff M_e is a simple R_e-module.*

Proof. Suppose M is a flexibly simple $R-$module. Let $X \neq 0$ be an R_e-submodule of M_e. By Proposition 2.5, RX is a nonzero flexible $R-$submodule of M. Therefore, $RX = RM_e$, and hence $X = M_e$. Consequently, M_e is a simple R_e-module.

Conversely, suppose M_e is a simple R_e-module. Let $N \neq 0$ be a flexible $R-$submodule of M. We have $N_e \neq 0$ is an R_e-submodule of M_e. Thus, $N_e = M_e$, which implies that $N = RM_e$. ∎

In the following corollary, the definition of $gr-$simple modules (i.e., graded simple modules) can be found in [2].

Corollary 5.3 *Let R be a first strongly $G-$ graded ring, and M be a $G-$graded $R-$module such that $e \in supp(M,G)$. Then the following statements are equivalent:*

1. *RM_e is $gr-$simple $R-$module.*
2. *M_e is simple R_e-module.*
3. *M is flexibly simple $R-$module.*

Proof. The equivalence of (1) and (3) follows from Proposition 5.2. The implication "(1) $\Rightarrow$ (2)" is a well known result of Graded Module Theory [2]. For "(2) $\Rightarrow$ (1)", apply Propositions 3.6 and 5.2. ∎

In the following result, we show that nontrivial flexibly simple $R-$modules are only valid over non-strongly graded rings. In fact, this is one reason that urges mathematicians to seek well-structured graded rings, rather than strongly graded rings, over which modules obtain nice properties. Examples of such structures are first and second strongly graded rings, as well as augmented graded rings.

Proposition 5.4 *Suppose R is a strongly $G-$graded ring, and M is non-cyclic $G-$graded $R-$module. Then M is not flexibly simple. In particular, if M is flexibly simple $R-$module, then R is not strong.*

Proof. Since R is strong, M is flexible and $M_e \neq 0$. Pick $0 \neq x \in M_e$. Set $N = Rx$. We have $0 \neq N \neq M$ is a $G-$graded $R-$submodule of M. Moreover, N is flexible because if $g \in G$, $N_g = R_g x = (R_g R_e)x = R_g(R_e x) = R_g N_e$. ∎

Next we study flexibly Noetherian $R-$modules. Again, Definition 5.5 generalizes the definition of flexible Noetherian modules [14], which is defined only for flexible modules.

Definition 5.5 Let R be a $G-$ graded ring, and M be a $G-$graded $R-$module. Then M is said to be flexibly Noetherian if every ascending chain of flexible $R-$submodules of M terminates.

An immediate consequence of Definition 5.5 is that every flexibly simple $R-$module is flexibly Noetherian.

Proposition 5.6 *Let R be a $G-$ graded ring, and M be a $G-$graded $R-$module. Then M is a flexibly Noetherian $R-$module iff M_e is a Noetherian R_e-module.*

Proof. Suppose M is a flexibly Noetherian $R-$module. Let $X^{(1)} \subseteq X^{(2)} \subseteq \ldots$ be an ascending chain of R_e-submodules of M_e. Then by Proposition 2.5, $R\,X^{(1)} \subseteq R\,X^{(2)} \subseteq \ldots$ is the corresponding ascending chain of flexible $R-$submodules of M. By assumption, there exist $n \in \mathbb{N}$ such that $R\,X^{(n)} = R\,X^{(n+1)} = \ldots$. Hence, we obtain $X^{(n)} = X^{(n+1)} = \ldots$. Therefore, M_e is a Noetherian R_e-module.

For the converse, assume M_e is a Noetherian R_e-module.

Let $X^{(1)} \subseteq X^{(2)} \subseteq \ldots$ be an ascending chain of flexible $R-$submodules of M. We have $X_e^{(1)} \subseteq X_e^{(2)} \subseteq \ldots$ is an ascending chain of R_e-submodules of M_e. By assumption, there exists $n \in \mathbb{N}$ such that $X_e^{(n)} = X_e^{(n+1)} = \ldots$. Hence,

$$R\,X_e^{(n)} = R\,X_e^{(n+1)} = \ldots, \qquad \text{or} \qquad X^{(n)} = X^{(n+1)} = \ldots.$$

That is, M is flexibly Noetherian. ∎

The following corollaries generalize Corollary 4.2.6 and Remark 4.2.10 of [14], respectively. The proof of the first one follows obviously from Corollary 5.3.

Corollary 5.7 *Let R be a first strongly $G-$ graded ring, and M be a flexible $G-$graded $R-$module. Then the following statements are equivalent:*

1. *M is gr$-$simple (resp. gr-Noetherian) $R-$module.*
2. *M_e is simple (resp. Noetherian) R_e- module.*
3. *M is flexibly simple (resp. flexibly Noetherian) $R-$module.* □

Corollary 5.8 *Let R be a commutative $G-$ graded ring, and M be a flexible $G-$graded $R-$module. Then the following statements are equivalent:*

1. *M is simple (resp. Noetherian) $R-$module.*
2. *M is gr$-$simple (resp. gr-Noetherian) $R-$module.*
3. *M_e is simple (resp. Noetherian) R_e- module.*
4. *M is flexibly simple (resp. flexibly Noetherian) $R-$module.*

Proof. Apply Theorem 3.8 (resp. Proposition 5.6) and Proposition 5.2. ∎

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