STRONGLY PARACOMPACT MAPPINGS

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Abstract. In this paper, we define and study strongly paracompact mappings which are the fibrewise topological analogues of strongly paracompact spaces. Several characterizations and properties of strongly paractompact mappings are obtained.

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1. Preliminaries

Unless otherwise stated, Y is a fixed topological space with topology τ, collection of all neighborhoods $(nbd(s))$ of $y \in Y$ is denoted by $N(y)$. If $f : X \to Y$ is a mapping and $A \subseteq X$ and B ⊆ X, then $[A]_B$ means the closure of A in B. For continuous mappings $f : X \to Y$ and $g : Z \to Y$, a continuous mapping $\lambda : X \to Z$ such t etc. if λ is hat $f = g\ \lambda$ is called a morphism of f into g and is denoted by$\lambda : f \to g$. λ is called surjective, closed, perfect surjective, closed, perfect etc., respectively. Open covers will be denoted by $\hat{U}, \hat{V}, \hat{W}, \ldots.$

To proceed we need the following definitions and results. For more details one can consult [3] and [5].

Definition 1.1 A mapping $f : X \to Y$ is called a T_2-*mapping*, if there exist disjoint *nbds* in X for every $x, x^* \in f^{-1}y$ such that $x \neq x^*$.

Definition 1.2 If A and B are subsets of X, then we say that A and B are

1. *nbd* separated in $U \subseteq X$.

2. Functionally separated in $U \subseteq X$.

if, respectively, the sets $A \cap U$ and $B \cap U$.

1. Have disjoint *nbds* in U.
2. There exists a continuous function $f : U \to [0, 1]$ such that $A \cap U \subseteq f^{-1}(0)$ and $B \cap U \subseteq f^{-1}(1)$.

Definition 1.3 A mapping $f : X \to Y$is said to be completely regular (regular) if for every $x \in X$ and every closed set F in X such that $x \notin F$ there exists a neighborhood $O \in N(fx)$ such that $\{x\}$ and F are functionally separated (*nbd* separated) in $f^{-1}O$.

A completely regular (regular) T_0-mapping is called a Tychonoff or $T_{3\frac{1}{2}}$-mapping (regular or T_3-mapping).

Definition 1.4 A mapping $f : X \to Y$ is called functionally prenormal (prenormal) if for every $y \in Y$ and every disjoint closed (in X) sets F and H there exists a neighborhood O of y such that F and H are functionally separated (*nbd* separated) in $f^{-1}O$. If for every open subset O of Y the mapping $f\,|_{f^{-1}O} : f^{-1}O \to O$ is functionally prenoraml (prenormal), then f is called functionally normal (normal). A normal T_3-mapping is called T_4*-mapping.*

A mapping $g : A \to B$ is said to be a (closed, open, dense, etc.) submapping of the mapping $f : X \to Y$ if g is the restriction of f on the (closed, open, dense, etc.) subset A of the space X and $g(A) = f(A) \subseteq B \subseteq Y$. A mapping $f : X \to Y$ is said to be compact if and only if f is perfect. If $f : X \to Y$ is a compact T_2*-mapping* and $g : A \to B$ is a submapping of f where B is a closed subset of Y, then g is compact.

Definition 1.5 [3] A mapping $f : X \to Y$ is called paracompact if for every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha;\ \alpha \in \Delta\}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}y$ is covered by $\hat{U}$ and $(f^{-1}O_y \wedge \hat{U})$ has an open (in X) y-locally finite refinement in $f^{-1}O_y$.

Theorem 1.6 [3] *If $f : X \to Y$ is a regular mapping, then the following conditions are equivalent.*

1) *f is a paracompact T_2-mapping.*
2) *For every $y \in Y$ and every open (in X) cover $\hat{U}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}y$ is covered by $\hat{U}$ and $(f^{-1}O_y \wedge \hat{U})$ has an open (in X) σ-locally finite refinement $\hat{V}$ in $f^{-1}O_y$; that is $\hat{V} = \bigcup_{i<\omega} \hat{V}_i$, where $\hat{V}_i$ is locally finite in $f^{-1}O_y$ for every $i < \omega$.*

Definition 1.7 [1] Let $f : X \to Y$ be a mapping. Then f is called countably paracompact if for every $y \in Y$ and every countable open (in X) cover $\hat{U}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}y$ is covered by $\hat{U}$ and $f^{-1}O_y \wedge \hat{U}$ has an open (in X) y-locally finite refinement in $f^{-1}O_y$.

Theorem 1.8 [1] *Let $f : X \to Y$ be a mapping. Then the following conditions are equivalent.*

(i) *f is normal and countably paracompact.*

(ii) *For every $y \in Y$ and every countable open (in X) cover $\hat{U} = \{U_i;\ i \in N\}$ of $f^{-1}y$ there exists a neighborhood $O_y \in N(y)$ such that $f^{-1}O_y$ is covered by $\hat{U}$; furthermore, for every $i = 1, 2, \ldots$ there exists $O_{i(y)} \in N(y)$, where $O_{i(y)} \subseteq O_y$, and a closed (in $f^{-1}O_{i(y)}$) subset $F_i \subseteq f^{-1}O_{i(y)} \cap U_i$ such that $f^{-1}O_y = \bigcup_{i=1}^{\infty} F_i$.*

2. Strongly paracompact mappings

Definition 2.1 Let $f : X \to Y$ be a mapping. For every $y \in Y$ the open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ is said to be y-star-finite if for every $x \in f^{-1}y$ there exists $U_\alpha \in \hat{U}$ such that $x \in U_\alpha$ and the family $S(U_\alpha, \hat{U}) = \{U_\beta \in \hat{U}; U_\beta \cap U_\alpha \neq \Phi\}$ is finite.

Definition 2.2 Let $f : X \to Y$ be a mapping. Then f is said to be strongly paracompact if for every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a y-star-finite open (in X) refinement in $f^{-1}O_y$.

The following results can be derived easily from Definitions 2.1 and 2.2.

Theorem 2.3 *If $f : X \to Y$ is a strongly paracompact mapping, L a subset of Y and A is a closed subset of X, then*

1) *f is closed.*

2) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a star-finite open (in X) refinement $\hat{V} = \{V_\beta\}_{\beta\in\Lambda}$ in $f^{-1}O_y$; that is, V_β intersects finitely many elements of $\hat{V}$ for every $\beta \in \Lambda$.*

3) *f is paracompact.*

4) *If f is, also, Hausdorf, then it is normal.*

5) *$f|_A : A \to Y$ is strongly paracompact.*

6) *$f_L : f^{-1}L \to L$ is strongly paracompact.*

7) *Each fibre of f is a strongly paracompact space.*

If the fibres of a mapping f are strongly paracompact spaces, then f is not necessarily a strongly paracompact mapping, even if f is closed and Tychonoff.

Example 2.4 Let L be the Niemytzki plane and let L_1 be the line $y = 0$. Then L_1 is closed in L and so the quotient mapping $q : L \to L/L_1$ is closed. Since L is Tychonof space, q is Tychonof. Since each fibre is discrete, the fibres of f are strongly paracompact space. But f is still not strongly paracompact because it is not paracompact. See Buhagiar [1997].

We know that if $\{V_\alpha;\ \alpha \in \Delta\}$ is a locally finite open cover of a space X, then the cover $\{[V_\alpha]_X;\ \alpha \in \Delta\}$ is locally finite in X. The same holds for star-finite families; more precisely

Lemma 2.5 *If $\{V_\alpha;\ \alpha \in \Delta\}$ is a star-finite cover of the space X, then the corresponding cover $\{St(V_\alpha);\ \alpha \in \Delta\}$ such that $St(V_\alpha) = \bigcup\{V_\gamma;\ V_\gamma \cap V_\alpha \neq \Phi, \gamma \in \Delta\}$ is a star-finite cover of X.*

Corollary 2.6 *If $\{V_\alpha;\ \alpha \in \Delta\}$ is a star-finite open cover of the space X, then the corresponding cover $\{[V_\alpha]_X;\ \alpha \in \Delta\}$ is a star-finite cover of X.*

Now, we shall prove the following characterization of strongly paracompact mappings in the presence of regular mappings.

Theorem 2.7 *If $f : X \to Y$ is a regular mapping, then the following conditions are equivalent.*

1) *f is strongly paracompact.*
2) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup\limits_{\alpha\in\Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has y-closed star-finite locally finite refinement in $f^{-1}O_y$.*

Proof. (1) $\to$ (2). Let $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ be an open (in X) cover of $f^{-1}y$. For every $x \in f^{-1}y$ there exists W_x (open neighborhood of x) and $O_{y(x)} \in N(y)$ such that $[W_x]_{f^{-1}O_{y(x)}} \subseteq U_{\alpha(x)} \cap f^{-1}O_{y(x)}$, where $W_x \subseteq U_{\alpha(x)}$. Let $\hat{W} = \{W_x; x \in f^{-1}y\}$. Then $\hat{W}$ is an open cover of $f^{-1}y$, so that there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{W}$ and $\hat{W} \wedge f^{-1}O_y$ has a star-finite (and so locally finite) open (in X) refinement in $f^{-1}O_y$, say $\hat{V} = \{V_\beta; \beta \in \mathrm{B}\}$. Since $\hat{V}$ is star-finite and open (in $f^{-1}O_y$), we have (by the previous lemma) the family $\{[V_\beta]_{f^{-1}O_y}; \beta \in \mathrm{B}\}$is star-finite (and so locally finite) closed refinement of $\hat{W} \wedge f^{-1}O_y$ in $f^{-1}O_y$. For every $x \in X$ let $O^*_{y(x)} = O_y \cap O_{y(x)}$. But for every $\beta \in \mathrm{B}$ there exists $x \in X$ such that $V_\beta \subseteq W_x \cap f^{-1}O_y$; let $V^*_\beta = [V_\beta]_{f^{-1}O^*_{y(x)}}$. Since $V_\beta \subseteq W_x \cap f^{-1}O_y$, we've $V^*_\beta \subseteq [W_x \cap f^{-1}O_y]_{f^{-1}O^*_{y(x)}}$. Note that

$$
\begin{aligned}
[W_x \cap f^{-1}O_y]_{f^{-1}O^*_{y(x)}} &= [W_x \cap f^{-1}O_y]_X \cap f^{-1}O^*_{y(x)} \\
&\subseteq [W_x]_X \cap [f^{-1}O_y]_X \cap f^{-1}O^*_{y(x)} \\
&\subseteq [W_x]_X \cap f^{-1}O^*_{y(x)} \\
&\subseteq [W_x]_X \cap f^{-1}O_{y(x)} \\
&\subseteq U_{\alpha(x)} \cap f^{-1}O_{y(x)}
\end{aligned}
$$

for some $\alpha(x) \in \Delta$; hence $V^*_\beta \subseteq U_{\alpha(x)} \cap f^{-1}O_{y(x)}$ for some $\alpha(x) \in \Delta$. Let $\hat{V}^*_\beta = \{V^{**}_\beta; \beta \in \mathrm{B}\}$ such that $V^{**}_\beta = V^*_\beta \bigcup V_\beta$. It is clear that V^{**}_β is y-closed for every $\beta \in \mathrm{B}$. Since $V^*_\beta \subseteq f^{-1}O^*_{y(x)} \subseteq f^{-1}O_y$, $V^*_\beta \subseteq U_{\alpha(x)} \cap f^{-1}O_y$ and $V^{**}_\beta = V^*_\beta \bigcup V_\beta \subseteq U_{\alpha(x)} \cap f^{-1}O_y$; this implies that $\hat{V}^*$ is a refinement of $\hat{U} \wedge f^{-1}O_y$. One can readily prove that $V^{**}_\beta \subseteq [V_\beta]_{f^{-1}O_y}$ for every $\beta \in \mathrm{B}$. Since $\{[V_\beta]_{f^{-1}O_y}\}$ is star-finite and locally finite in $f^{-1}O_y$, we have $\hat{V}^*$ is star-finite locally finite in $f^{-1}O_y$. This completes the proof of (1) $\to$ (2).

Now, we shall show that (2) $\to$ (1). If $\hat{U}$ is an open cover of $f^{-1}y$, then there is $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a star-finite locally finite y-closed refinement in $f^{-1}O_y$, say $\{P_\alpha; \alpha \in \Delta\}$. For every $\alpha \in \Delta$ let $\hat{Z}_\alpha = \{P_{\alpha*}; P_{\alpha*} \cap P_\alpha = \Phi\}$ and $Z_\alpha = \bigcup\{P_{\alpha*}; P_{\alpha*} \cap P_\alpha = \Phi\}$. Now, for every $x \in f^{-1}y - Z_\alpha$ there exists an open (in $f^{-1}O_y$) neighborhood V_x of x such that V_x intersects finitely many elements of $\{P_\alpha; \alpha \in \Delta\}$, so that V_x intersects finitely many elements of $\hat{Z}_\alpha$, say $P_{\alpha 1}, P_{\alpha 2}, ..., P_{\alpha n}$. But $\bigcup_{i=1}^n P_{\alpha i}$ is closed in $f^{-1}O^*_y$ for some $O^*_y \in N(y)$. Set $G_x = V_x \cap (f^{-1}O^*_y - \bigcup_{i=1}^n P_{\alpha i})$. Then G_x is an open (in X) neighborhood of x and disjoint from Z_α. Let $\hat{W} = \{W_\alpha; \alpha \in \Delta\}$ where $W_\alpha = \bigcup\{G_x; x \in f^{-1}y - Z_\alpha\}$for every$\alpha \in \Delta$. Then $\hat{W}$ is an open (in X) cover of $f^{-1}y$ and W_α is contained in $St(P_\alpha) = \bigcup\{P_{\alpha*}; P_{\alpha*} \cap P_\alpha \neq \Phi\}$ for each $\alpha \in \Delta$. But the family $\{St(P_\alpha)\alpha \in \Delta\}$ is star-finite, by Lemma 2.1, so that $\hat{W}$ is a star-finite open (in X) cover of $f^{-1}y$. For every $\alpha \in \Delta$ there exists $U_\alpha \in \hat{U}$ such that P_α is contained in U_α. Let $W^*_\alpha = W_\alpha \cap U_\alpha$ and let $\hat{W}^* = \{W^*_\alpha; \alpha \in \Delta\}$. Then $\hat{W}^*$ is a star-finite open (in X) refinement of $\hat{U} \wedge f^{-1}O_y$. To complete the proof note that *ii)* implies that f is closed, so that there exists $O^*_y \in N(y)$ such that $f^{-1}O^*_y \subseteq \bigcup \hat{W}^*$; hence the cover $\hat{W}^* \wedge f^{-1}O^*_y$ is a star-finite open (in X) refinement of $\hat{U} \wedge f^{-1}O^*_y$ in $f^{-1}O^*_y$. This completes the proof. ■

For the sake of a later application we shall prove the following theorem.

Theorem 2.8 *If $f : X \to Y$ is a regular mapping, then the following conditions are equivalent.*

1) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup\limits_{\alpha\in\Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a y-closed star-countable locally finite refinement in $f^{-1}O_y$.*

2) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup\limits_{\alpha\in\Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a star-countable open $(in\ X)$ refinement in $f^{-1}O_y$.*

Proof. (1) $\rightarrow$ (2). If $\hat{U}$is an open (in X) cover of $f^{-1}y$, then there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a locally finite star-countable y-closed refinement in $f^{-1}O_y$, say $\hat{V}$. We can write $\hat{V} = \bigcup_{t\in T} \hat{V}_t$ such that $\{\hat{V}_t; t \in T\}$ is the set of all components of $\hat{V}$ and $\hat{V}_t = \{V_{t,i}; i \in N\}$ for every $t \in T$. Set $C_t = \bigcup_{i\in N} V_{t,i}$. The family $\{C_t; t \in T\}$ covers $f^{-1}O_y$. We shall show that, C_t is closed and open in $f^{-1}O_y$ for every $t \in T$. Since $\{C_t; t \in T\}$ is a pairwise disjoint family, it is sufficient to show that C_t is open in $f^{-1}O_y$ for each $t \in T$. Let $x \in C_t$. Then, by locally finiteness, there exists V_x(open neighborhood of x) such that V_x intersects finitely many elements of $\hat{V} - \hat{V}_t$, say $P_{\alpha 1}, P_{\alpha 2}, ..., P_{\alpha n}$. There exists $O^*_y \in N(y)$ such that $\bigcup_{i=1}^n P_{\alpha i}$ is closed in $f^{-1}O^*_y$. If $G_x = (f^{-1}O^*_y - \bigcup_{i=1}^n P_{\alpha i})$, then G_x is an open neighborhood of x contained in C_t; this implies that C_t is open in $f^{-1}O_y$ and so closed and open in $f^{-1}O_y$. For every $t \in T$ and every natural number i let $U(t,i)$ in $\hat{U}$ such that $V_{t,i} \subseteq U(t,i)$ and $\hat{V}^* = \{C_t \cap U(t,i); t \in T, i \in N\}$. It is clear that $\hat{V}^*$is an open (in X) star-countable refinement of $f^{-1}O_y \wedge \hat{U}$ in $f^{-1}O_y$. This completes the proof of (1) $\rightarrow$ (2).

Now, we prove (2) $\rightarrow$ (1). First, we shall show that f is paracompact. Let $\hat{U}$ be an open (in X) cover of $f^{-1}y$. Then there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a starcountable open refinement $\hat{V}$ in $f^{-1}O_y$. Since $\hat{V}$ is star-countable, we have $\hat{V} = \bigcup_{t\in T} \hat{V}_t$ such that $\{\hat{V}_t; t \in T\}$ is the set of all components of $\hat{V}$ and $\hat{V}_t = \{V_{t,i}; i \in N\}$ for every $t \in T$. Let $\hat{V}_i = \{V_{t,i}; t \in T\}$. $\hat{V}_i$ is locally finite in $f^{-1}O_y$, so that $\hat{V} = \bigcup_{t\in T} \hat{V}_t = \bigcup_{i\in N} \hat{V}_i$ is a σ-locally finite open (in X) refinement of $\hat{U}$in $f^{-1}O_y$. But f is regular; hence(by Theorem 1.6) f is paracompact.

(2) $\rightarrow$ (1). Let $\hat{U}$ be an open cover of $f^{-1}y$. For every $x \in f^{-1}y$ there exists W_x open (in X) neighborhood of x and $O_{y(x)} \in N(y)$ such that $[W_x]_{f^{-1}O_{y(x)}} \subseteq U_{(x)} \cap f^{-1}O_{y(x)}$, where $W_x \subseteq U_{(x)}$ for some $U_{(x)} \in \hat{U}$. Let $\hat{W} = \{W_x; x \in f^{-1}y\}$. $\hat{W}$ is an open cover of $f^{-1}y$, so that there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{W}$ and $\hat{W} \wedge f^{-1}O_y$ has a star-countable open (in X) refinement in $f^{-1}O_y$, say $\hat{V} = \{V_\alpha; \alpha \in \Delta\}$. Since $\hat{V}$ is an open cover of $f^{-1}y$ and f is a paracompact mapping, there exists $O^*_y \in N(y)$ such that $O^*_y \subseteq O_y$, $f^{-1}O_y \subseteq \bigcup \hat{V}$ and $\hat{V} \wedge f^{-1}O_y$ has a locally finite open (in X) refinement in $f^{-1}O^*_y$, say $\hat{V}^*$. If $V \in \hat{V}^*$, then there exists $\alpha(V) \in \Delta$ such that $V \subseteq V_{\alpha(V)}$. For every $V \in \hat{V}^*$ fix such an $\alpha(V)$ and let $M_\alpha = \bigcup\{V;\ \alpha(V) = \alpha$ and $V \in \hat{V}^*\}$. It is clear that $M_\alpha \subseteq V_\alpha$for every $\alpha \in \Delta$; hence $\{M_\alpha; \alpha \in \Delta\}$ is an open (in X) cover of $f^{-1}O^*_y$. Since $\hat{V} = \{V_\alpha; \alpha \in \Delta\}$is star-countable and $M_\alpha \subseteq V_\alpha$,the family $\{M_\alpha; \alpha \in \Delta\}$is star-countable, and since $\hat{V}^*$ is locally finite in $f^{-1}O^*_y$, there exist U_x (open subset of $f^{-1}O^*_y$) such that U_x intersects finitely many elements of $\hat{V}^*$ for every $x \in f^{-1}O^*_y$(and so finitely many elements of $\{M_\alpha; \alpha \in \Delta\}$); hence the family $\{M_\alpha; \alpha \in \Delta\}$ is a star-countable

locally finite open (in X) refinement of $\hat{V} \wedge f^{-1}O_y^*$ in $f^{-1}O_y^*$, so that the family $\{[M_\alpha]_{f^{-1}O_y^*}\}_{\alpha\in\Delta}$ is a star-countable locally finite (in X) refinement of $\hat{V} \wedge f^{-1}O_y^*$ in $f^{-1}O_y^*$. For every $x \in X$ let $O_{y(x)}^* = O_y^* \cap O_{y(x)}$, and for every $\alpha \in \Delta$ let $M_\alpha^* = [M_\alpha]_X \cap f^{-1}O_{y(x)}^*$ and $M_\alpha^{**} = M_\alpha^* \bigcup M_\alpha$. Then, one readily proves that $\hat{M} = \{M_\alpha^{**}\}_{\alpha\in\Delta}$ is a star-countable locally finite refinement of $\hat{U} \wedge f^{-1}O_y^*$ in$f^{-1}O_y^*$ consisting of y-closed subsets. This completes the proof. ∎

The next two lemmas will be used to show that the conditions in Theorems 2.7 and 2.8 are equivalent in the realm of functionally normal mappings. First, we shall define functionally open sets for mappings.

Definition 2.9 Let $f : X \to Y$ be any mapping. For a subset A of X we say that A is y-functionally open if there exists $O_y \in N(y)$ and a continuous real valued function $g : f^{-1}O_y \to I$ such that $g^{-1}((0,1]) = A \cap f^{-1}O_y$.

Lemma 2.10 *If $f : X \to Y$ is a continuous mapping and $\{U_i^*; i \in N\}$ is an open cover of $f^{-1}y$ such that each U_i^* is y-functionally open, then there exists a star-finite open cover of $f^{-1}y$ and refines $\{U_i^*;\ i \in N\}$.*

Proof. For every $i{\in}N$ there exists $O_i \in N(y)$ and a continuous real valued function $\alpha_i : f^{-1}O_i \to I$ such that $\alpha_i^{-1}((0,1]) = U_i^* \cap f^{-1}O_i$. Let $U_i = U_i^* \cap f^{-1}O_i$. It is clear that $\{U_i; i \in N\}$ is an open cover of $f^{-1}y$ refinement of $\{U_i^*; i \in N\}$. For every $i \in N$ let $f_i : X \to I$ be defined such that

$$f_i(x) = \begin{cases} \alpha_i(x); & x \in f^{-1}O_i \\ 0 & x \in X - f^{-1}O_i \end{cases}$$

Note that f_i is not necessarily a continuous function. Let $g : X \to I$ be defined such that $g(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{2^i}$. g is not necessarily continuous. For every $k \in N$ set $V_k = g^{-1}((\frac{1}{k}, 1])$ and $F_k = g^{-1}([\frac{1}{k}, 1])$. Note that V_k is not necessarily an open subset of X, and F_k is not necessarily a closed subset of X. For every $i \leq k$ let $U_{i,k} = U_i \cap (V_{k+1} - F_{k-1})$. We shall show that the family $\hat{U} = \{Int(U_{i,k}); i,k \in N, i \leq k\}$ is a star-finite open cover of $f^{-1}y$ and refines $\{U_i^*; i \in N\}$. First, one can readily prove that $\{U_{i,k}; i,k \in N\}$ is a star-finite refinement of $\{U_i^*; i \in N\}$, so that the family $\hat{U} = \{Int(U_{i,k}); i,k \in N \text{ and } i \leq k\}$ is a star-finite refinement of $\{U_i^*; i \in N\}$. It remains to show that the family $\hat{U} = \{Int(U_{i,k}); i,k \in N \text{ and } i \leq k\}$ covers $f^{-1}y$. If $x \in f^{-1}y$, then $x \in U_i$ for some natural number i; which implies that $f_i(x) \neq 0$ and so $g(x) \neq 0$, so that there exists a natural number m such that $g(x) > \frac{1}{m}$, i.e., $x \in V_m \subseteq F_m$ .Let k be the smallest natural number such that $x \in F_k$. Since $F_k \subseteq V_{k+1}$, we have $x \in V_{k+1}$ and $x \notin F_{k-1}$. It is clear that $F_k \subseteq \bigcup_{i\leq k} U_i$, so that there exists $j \leq k$ such that $x \in U_j$; which implies that $x \in U_{j,k}$. Now, we shall show that $x \in int(U_{j,k})$. Since $x \in V_{k+1}$, we have $g(x) = \sum_{i=1}^{\infty} \frac{f_i(x)}{2^i} > \frac{1}{k+1}$, so that there exists $n_o \in N$ such that $\sum_{i=1}^{n_o} \frac{f_i(x)}{2^i} > \frac{1}{k+1}$. For every $i = 1, 2, ..., n_o$let $s_i = 0$ whenever $f_i(x) = 0$, and if $f_i(x) \neq 0$ let s_i be any real number such that $s_i < f_i(x)$ and $\sum_{i=1}^{\infty} \frac{s_i}{2^i} > \frac{1}{k+1}$; furthermore let

$$M_i = \begin{cases} f_i^{-1}O_i & \text{if} \quad f_i(x) = 0 \\ f_i^{-1}((s_i, 1] = \alpha_i^{-1}((s_i, 1] & \text{if} \quad f_i(x) \neq 0 \end{cases}$$

Set $H_x = \cap_{i=1}^{n_o} M_i$. It is clear that H_x is an open subset of X and contains x. We shall show that $H_x \subseteq V_{k+1}$. But if $t \in H_x$, then $t \in f_i^{-1}([s_i,1])$ for every $i = 1,2,...,n_o$, so that $f_i(t) \geq s_i$ and $g(t) = \sum_{i=1}^{\infty} \frac{f_i(t)}{2^i} \geq \sum_{i=1}^{n_o} \frac{f_i(t)}{2^i} \geq \sum_{i+1}^{n_o} \frac{s_i}{2^i} > \frac{1}{k+1}$; hence $t \in g^{-1}((\frac{1}{k+1},1]) = V_{k+1}$, which implies that $H_x \subseteq V_{k+1}$. On the other hand, since $x \notin F_{k-1}$, we've $g(x) < \frac{1}{k-1}$. Let $s \in (g(x), \frac{1}{k-1})$ and $n_o \in N$ such that $\sum_{i=n_o+1}^{\infty} \frac{1}{2^i} < \frac{1}{k-1} - s$. For every $i \leq n_o$ let $s_i = 1$ whenever $f_i(x) = 1$ and let $s_i \in (f_i(x), 1)$ such that $\sum_{i=1}^{n_o} \frac{s_i}{2^i} < s$ whenever $f_i(x) \neq 1$; furthermore let

$$A_i = \begin{cases} f_i^{-1}O_i = \alpha_i^{-1}([0,1]) & \text{if} f_i(x) = 1, \\ \alpha_i^{-1}([0,s_i]) & \text{if} f_i(x) \neq 1. \end{cases}$$

Set $H_x^* = \cap_{i=1}^{n_o} A_i$. It is clear that H_x^* is open in X and contains x. We shall show hat $H_x^* \cap F_{k-1} = \Phi$. If $t \in H_x^*$, then $t \in A_i$ for every $i = 1,2,..,n_0$, so that $f_i(x) \leq s_i$ for every s_i , which implies that

$$\begin{aligned} g(t) &= \sum_{i=1}^{\infty} \frac{f_i(t)}{2^i} = \sum_{i=1}^{n_o} \frac{f_i(t)}{2^i} + \sum_{i=n_o+1}^{\infty} \frac{f_i(t)}{2^i} \\ &\leq \sum_{i=1}^{n_o} \frac{f_i(t)}{2^i} + \sum_{i=n_o+1}^{\infty} \frac{1}{2^i} \\ &< \sum_{i=1}^{n_o} \frac{s_i}{2^i} + \frac{1}{k-1} - s \\ &< s + \frac{1}{k-1} - s = \frac{1}{k-1} \end{aligned}$$

So that $t \notin g^{-1}([\frac{1}{k-1},1]) = F_{k-1}$ and $H_x^* \cap F_{k-1} = \Phi$. Let $H_x^{**} = H_x^* \cap H_x \cap U_j$. H_x^{**} is an open (in X) neighborhood of x contained in $U_{j,k}$; more precisely $x \in Int(U_{j,k})$, so that the family $\hat{U} = \{Int(U_{i,k}); i,k \in N, i \leq k\}$ covers $f^{-1}y$. This completes the proof. ∎

Lemma 2.11 *If $f : X \to Y$ is a countably paracompact functionally normal mapping, then for every open (in X) cover $\hat{U} = \{U_i; i \in N\}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a star-finite open (in X) refinement in $f^{-1}O_y$.*

Proof. By Theorem 1.8, there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a refinement $\{F_i; i \in N\}$ in $f^{-1}O_y$ such that $F_i \subseteq U_i$ for every i, and there exists $O_i \in N(y)$ for every i such that $O_i \subseteq O_y$, F_i is a closed subset of $f^{-1}O_i$ and $\bigcup_{i=1}^{\infty} F_i = f^{-1}O_y$. Since f is functionally normal, the mapping $f_{O_i} : f^{-1}O_i \to O_i$ is functionally normal for every i, so that there exists $O_i^* \in N(y)$ such that $O_i^* \subseteq O_i$ and there exists a continuous function $\alpha_i : f^{-1}O_i^* \to I$ such that $\alpha_i(F_i \cap f^{-1}O_i^*) = 1$ and $\alpha_i((f^{-1}O_i^*) - U_i) = 0$. Set $V_i = \alpha_i^{-1}((0,1])$. Since $F_i \cap f^{-1}O_i^* \subseteq V_i^*$, we have the family $\hat{V} = \{V_i; i \in N\}$ is an open cover of $f^{-1}y$ consisting of y-functionally open subsets. By Lemma 2.10, there exists a star-finite open (in X) cover $\hat{W}$ of $f^{-1}y$ refines $\hat{V} = \{V_i; i \in N\}$. Since f is closed, there exists $O_y^* \in N(y)$ such that$f^{-1}O_y^* \subseteq \bigcup \hat{W}$. It is clear that $f^{-1}O_y^* \wedge \hat{W}$ is a star-finite open (in X) refinement of $f^{-1}O_y^* \wedge \hat{U}$ in $f^{-1}O_y^*$. This completes the proof. ∎

Question 2.12 *Is Lemma 2.11 true if the functional normality is replaced by normality?*

The following theorem is a characterization of strongly paracompact mappings.

Theorem 2.13 *If $f : X \to Y$ is a functionally normal mapping, then the following conditions are equivalent.*

1) *f is strongly paracompact.*
2) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup\limits_{\alpha\in\Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a y-closed star-finite locally finite refinement in $f^{-1}O_y$.*
3) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup\limits_{\alpha\in\Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a star-countable y-closed locally finite refinement in $f^{-1}O_y$.*
4) *For every $y \in Y$ and every open (in X) cover $\hat{U} = \{U_\alpha\}_{\alpha\in\Delta}$ of $f^{-1}y$ there exists $O_y \in N(y)$ such that $f^{-1}O_y \subseteq \bigcup\limits_{\alpha\in\Delta} U_\alpha$ and $\hat{U} \wedge f^{-1}O_y$ has a star-countable open (in X) refinement in $f^{-1}O_y$.*

Proof. By Theorems 2.7 and 2.8 we have 1) is equivalent to 2) and 3) is equivalent to 4). It is clear that 2) implies 3). It suffices to show that 4) implies 1).

Assume 4). First, note that f is paracompact (see the proof of Theorem 2.7). Now, we shall show that f is a strongly paracompact mapping.

Let $\hat{U}$ be an open (in X) cover of $f^{-1}y$. Then there exists $O_y \in N(y)$such that $f^{-1}O_y \subseteq \bigcup \hat{U}$ and $\hat{U} \wedge f^{-1}O_y$ has a starcountable open refinement$\hat{V}$in $f^{-1}O_y$. We can write $\hat{V} = \bigcup_{t\in T} \hat{V}_t$ such that $\{\hat{V}_t \in T\}$ is the set of all components of $\hat{V}$. But for every $t \in T$ we have $\hat{V}_t = \{V_{t,i};\ i \in N\}$. Set $C_t = \bigcup_{i\in N} V_{t,i}$. Then C_t is a closed and open subset of $f^{-1}O_y$ for every $t \in T$. Since f is a paracompact functionally normal mapping, we have $f_{O_y} : f^{-1}O_y \to O_y$ is a paracompact functionally normal mapping, and since C_t is a closed subset of $f^{-1}O_y$, we have $f_t = f\,|_{C_t} : C_t \to O_y$ is a paracompact (and so a countably paracompact) functionally normal mapping for every $t \in T$. But $\hat{V}_t$ is an open cover of C_t so it is an open cover of $f_t^{-1}y$ and (by Lemma 2.11) there exists $O_t \in N(y)$ such that $f_t^{-1}O_t \subseteq \bigcup\limits_{i=1}^{\infty} V_{t,i}$ and $\hat{V}_t \wedge f_t^{-1}O_t$ has a star-finite open (in X) refinement $\hat{V}_t^*$ in $f_t^{-1}O_t$. It is clear that $\hat{V}^* = \bigcup\limits_{t\in T} \hat{V}_t^*$ is an open (in X) cover of $f^{-1}y$. Since the family $\hat{C} = \{C_t; t \in T\}$ is pairwise disjoint and $\hat{V}_t^*$ is star-finite for every $t \in T$, we have $\hat{V}^* = \bigcup_{t\in T} \hat{V}_t^*$ is star-finite, and since f is closed , there exists $O_y^* \in N(y)$ such that $f^{-1}O_y^* \subseteq \bigcup \hat{V}^*$, so that $f^{-1}O_y^* \wedge \hat{V}^*$ is a star-finite open (in X) refinement of $f^{-1}O_y^* \wedge \hat{U}$ in $f^{-1}O_y^*$. This completes the proof. ∎

Corollary 2.14 *Every Lindelöf functionally normal mapping is strongly paracompact.*

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