DESCRIBING GREEN'S RELATIONS IN ORDERED Γ-GROUPOIDS USING A NEW CONCEPT: FUZZY SUBSETS[1]

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Abstract. This work extends the idea of the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and the usual relation $\mathcal{N}$ of ordered groupoids to ordered Γ-groupoids in terms of fuzzy subsets, the concept of the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and the usual relation $\mathcal{N}$ of ordered Γ-groupoids is introduced and investigated, which is an interesting for ordered Γ-groupoids and some interesting characterizations of the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and the usual relation $\mathcal{N}$ of ordered Γ-groupoids are obtained. Also, we define relations $\mathcal{R}^F, \mathcal{L}^F, \mathcal{I}^F$ and $\mathcal{N}^F$ of ordered Γ-groupoids in terms of fuzzy subsets and we prove that $\mathcal{R}^F = \mathcal{R}$, $\mathcal{L}^F = \mathcal{L}$, $\mathcal{I}^F = \mathcal{I}$ and $\mathcal{N}^F = \mathcal{N}$ in the same way as of the results of Kehayopulu and Tsingelis [8].

Keywords: Green's relations; ordered Γ-groupoid; fuzzy subset.
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1. Introduction and prerequisites

The notion of the Green's relations of semigroups introduced by Green [5] in 1951, several researches have characterized the many type of the Green's relations on the algebraic structures such as: In 2002, Petro and Pasku [14] studied the

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Green-Kehayopulu relation $\mathcal{H}$ in le-semigroups mimics the definition of the usual Green relation $\mathcal{H}$ in semigroups. They proved that certain properties of $\mathcal{H}$-classes essentially differ from those of $\mathcal{H}$-classes need not constitute a subsemigroup. In 2007, Kehayopulu and Tsingelis [8] characterized the Green's relations $\mathcal{R}, \mathcal{L}$ and $\mathcal{I}$ of ordered groupoids in terms of fuzzy subsets. For an ordered groupoid S, they defined relations $\mathcal{R}^F, \mathcal{L}^F$ and $\mathcal{I}^F$ in terms of fuzzy subsets and showed that each of them coincides with the corresponding Green's relations on S. In 2008, Chinram and Siammai [2] studied the Green's relations for Γ-semigroups and reductive Γ-semigroups. In this year, Siripitukdet and Iampan [21] introduced the concept of the Green-Kehayopulu relations in le-Γ-semigroups mimics the definition of the Green-Kehayopulu relations in le-semigroups. They proved that an $\mathcal{H}_\gamma$-class of an le-Γ-semigroup M satisfies Green's condition if and only if it contains a γ-idempotent and an $\mathcal{H}_\gamma$-class of an le-Γ-semigroup M is a subgroup of $< M_\gamma, \circ >$ if and only if it consists of a single idempotent.

A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [22] in 1965. The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the concept of fuzzy sets by Zadeh [22], several researches were conducted on the generalizations of the notion of fuzzy set and application to many algebraic structures such as: In 1971, Rosenfeld [15] was the first who studied fuzzy sets in the structure of groups. Fuzzy semigroups have been first considered by Kuroki [9], [10], [11], [12], and fuzzy ordered groupoids and ordered semigroups, by Kehayopulu and Tsingelis [6], [7]. In 2007, Kehayopulu and Tsingelis [8] characterized the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ of ordered groupoids in terms of fuzzy subsets. In 2008, Shabir and Khan [19] defined fuzzy bi-ideal subsets and fuzzy bi-filters in ordered semigroups and characterized ordered semigroups in terms of fuzzy bi-ideal subsets and fuzzy bi-filters. In 2009, Majumder and Sardar [13] studied fuzzy ideals and fuzzy ideal extensions in ordered semigroups. In 2010, Chinram and Saelee [1] studied fuzzy ternary subsemigroups (left ideals, right ideals, lateral ideals, ideals) and fuzzy left filters (right filters, lateral filters, filters) of ordered ternary semigroups. In 2010, Shah and Rehman [20] introduced Γ-ideals and Γ-bi-ideals of Γ-AG-groupoids which are in fact a generalization of ideals and bi-ideals of AG-groupoids and studied some characteristics of Γ-ideals and Γ-bi-ideals of Γ-AG-groupoids. In 2010, Iampan [3] characterized the relationship between the fuzzy ordered ideals (fuzzy ordered filters) and the characteristic mappings of fuzzy ordered ideals (fuzzy ordered filters) in ordered Γ-semigroups. In 2011, Saelee and Chinram [16] studied rough, fuzzy and rough fuzzy bi-ideals of ternary semigroups.

As we know, Γ-groupoids are a generalization of groupoids. This paper is a sequel to our study [7] of fuzzification in ordered groupoids. The Green's relations defined in terms of ordered right ideals, ordered left ideals, ordered ideals and the concept of fuzzy sets play an important role in studying the structure of ordered groupoids. Now, we characterize the Green's relations $\mathcal{R}, \mathcal{L}, \mathcal{I}$ and the usual relation $\mathcal{N}$ defined in terms of ordered filters on ordered Γ-groupoids in terms of fuzzy subsets. To present the main theorems we discuss some elementary definitions that we use later.

Definition 1.1. Let M be a set. A *fuzzy subset* of M is an arbitrary mapping $f\colon M \to [0,1]$ where $[0,1]$ is the unit segment of the real line.

Definition 1.2. Let M be a set and $A \subseteq M$. The *characteristic mapping* $f_A\colon M \to [0,1]$ defined via

$$x \mapsto f_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

By the definition of characteristic mapping, f_A is a mapping of M into $\{0,1\} \subset [0,1]$. Hence f_A is a fuzzy subset of M.

Definition 1.3. Let M and Γ be any two nonempty sets. Then (M,Γ) is called a Γ-*groupoid* [18] if there exists a mappings $M \times \Gamma \times M \to M$, written as $(a,\gamma,b) \mapsto a\gamma b$. A nonempty subset K of M is called a Γ-*subgroupoid* of M if $K\Gamma K \subseteq K$.

Definition 1.4. A partially ordered Γ-groupoid $(M,\Gamma,\leq)$ is called an *ordered* Γ-*groupoid* [17] if for any $a,b,c \in M$ and $\gamma \in \Gamma, a \leq b$ implies $a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$.

Definition 1.5. Let $(M,\Gamma,\leq)$ be an ordered Γ-groupoid. For each nonempty subset H of M, we define

$$(H] = \{t \in M \mid t \leq h \text{ for some } h \in H\}.$$

Definition 1.6. Let $(M,\Gamma,\leq)$ be an ordered Γ-groupoid. A nonempty subset A of M is called an *ordered left ideal* of M if

L1. $M\Gamma A \subseteq A$.

L2. For any $b \in M$ and $a \in A, b \leq a$ implies $b \in A$.

A nonempty subset A of M is called an *ordered right ideal* of M if

R1. $A\Gamma M \subseteq A$.

R2. For any $b \in M$ and $a \in A, b \leq a$ implies $b \in A$.

A nonempty subset A of M is called an *ordered ideal* of M if it is both an ordered left ideal and an ordered right ideal of M. That is,

I1. $M\Gamma A \subseteq A$ and $A\Gamma M \subseteq A$.

I2. For any $b \in M$ and $a \in A, b \leq a$ implies $b \in A$.

We denote by $R(x), L(x), I(x)$ the ordered right ideal, ordered left ideal, ordered ideal of M, respectively, generated by x.

Definition 1.7. The Greens relations $\mathcal{R}, \mathcal{L}$ and $\mathcal{I}$ are the equivalence relations on ordered Γ-groupoid M, defined as follows:

$$\begin{aligned} \mathcal{R} &:= \{(x,y) \mid R(x) = R(y)\}, \\ \mathcal{L} &:= \{(x,y) \mid L(x) = L(y)\}, \\ \mathcal{I} &:= \{(x,y) \mid I(x) = I(y)\}. \end{aligned}$$

Definition 1.8. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. A Γ-subgroupoid F of M is called an *ordered filter* of M if

F1. For any $a, b \in M$ and $\gamma \in \Gamma, a\gamma b \in F$ implies $a, b \in F$.

F2. For any $b \in M$ and $a \in F, a \leq b$ implies $b \in F$.

For an element $x \in M$, we denote by $N(x)$ the ordered filter of M generated by x, and by $\mathcal{N}$ the relation on M defined by

$$\mathcal{N} := \{(x,y) \mid N(x) = N(y)\}.$$

We can easily prove that for two subsets A and B of ordered Γ-groupoid M, we have $A \subseteq B$ if and only if $f_A \subseteq f_B$ and $A = B$ if and only if $f_A = f_B$. We denote by $F(M)$ the set of all fuzzy subsets of M. We define an order relation "$\subseteq$" on $F(M)$ as follows: For $f, g \in F(M)$, we define $f \subseteq g$ if and only if $f(x) \leq g(x)$ for all $x \in M$. Clearly, $(F(M), \subseteq)$ is an ordered set. Let $0_F, 1_F$ be the fuzzy subsets of M defined by: $0_F\colon M \to [0,1] \mid x \mapsto 0_F(x) := 0$ and $1_F\colon M \to [0,1] \mid x \mapsto 1_F(x) := 1$. It is clear that the mapping 0_F is the least element of $F(M)$ and the mapping 1_F is the greatest element $F(M)$.

Definition 1.9. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. A fuzzy subset f of M is called a *fuzzy ordered left ideal* of M if

FL1. For any $a, b \in M, a \leq b$ implies $f(a) \geq f(b)$.

FL2. $f(a\gamma b) \geq f(b)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

A fuzzy subset f of M is called a *fuzzy ordered right ideal* of M if

FR1. For any $a, b \in M, a \leq b$ implies $f(a) \geq f(b)$.

FR2. $f(a\gamma b) \geq f(a)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

A fuzzy subset f of M is called a *fuzzy ordered ideal* of M if it is both a fuzzy ordered left and a fuzzy ordered right ideal of M. That is,

FI1. For any $a, b \in M, a \leq b$ implies $f(a) \geq f(b)$.

FI2. $f(a\gamma b) \geq f(b)$ and $f(a\gamma b) \geq f(a)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition 1.10. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. A fuzzy subset f of M is called a *fuzzy ordered filter* of M if

FF1. For any $a, b \in M, a \leq b$ implies $f(a) \leq f(b)$.

FF2. $f(a\gamma b) = \min\{f(a), f(b)\}$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition 1.11. Let (M, Γ) be a Γ-groupoid. A fuzzy subset f of M is called a *fuzzy* Γ*-subgroupoid* of M if $f(a\gamma b) \geq \min\{f(a), f(b)\}$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Definition 1.12. Let (M, Γ) be a Γ-groupoid and f a fuzzy subset of M. The mapping

$$f' \colon M \to [0,1] \text{ defined via } f'(x) = 1 - f(x)$$

is a fuzzy subset of M called the *complement* of f in S.

Definition 1.13. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. A fuzzy subset f of M is called *prime* if $f(a\gamma b) \leq \max\{f(a), f(b)\}$ for all $a, b \in M$ and $\gamma \in \Gamma$.

2. The Green's relations in terms of Fuzzy Subsets

In this section, we give some interesting properties of fuzzy subsets and study the Green's Relations of ordered Γ-groupoids in terms of fuzzy subsets.

Proposition 2.1. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and $\{f_i \mid i \in I\}$ a nonempty family of fuzzy subsets of M. Then*

$$\bigwedge_{i\in I} f_i \in F(M) \text{ and } \bigvee_{i\in I} f_i \in F(M).$$

Proof. Let $x \in M$. The set $\{f_i(x) \mid i \in I\}$ is a nonempty lower bounded subset of $\mathbb{R}$, so there exists the $\inf\{f_i(x) \mid i \in I\}$ in $\mathbb{R}$. Since $0 \leq f_i(x) \leq 1$ for each $i \in I$, we have $0 \leq \inf\{f_i(x) \mid i \in I\} \leq 1$. If $x, y \in M$ is such that $x = y$, then clearly $(\bigwedge_{i\in I} f_i)(x) = (\bigwedge_{i\in I} f_i)(y)$. Hence $\bigwedge_{i\in I} f_i \in F(M)$. Similarly, $\bigvee_{i\in I} f_i \in F(M)$. ∎

Remark 2.2. By Proposition 2.1, we have $(F(M), \subseteq)$ is a complete lattice.

Remark 2.3. The following two statements hold true:

(i) $\bigwedge_{i\in I} f_i \subseteq f_j$ for all $j \in I$,

(ii) $f_j \subseteq \bigvee_{i\in I} f_i$ for all $j \in I$.

Proposition 2.4. *Let $(M,\Gamma,\leq)$ be an ordered Γ-groupoid and $\{f_i \mid i\in I\}$ a nonempty family of fuzzy subsets of M. Then we have the following:*

(i) $\bigwedge_{i\in I} f_i = \inf\{f_i \mid i\in I\}$,

(ii) $\bigvee_{i\in I} f_i = \sup\{f_i \mid i\in I\}$.

Proof. (i) By Proposition 2.1, $\bigwedge_{i\in I} f_i \in F(M)$ and by Remark 2.3, $\bigwedge_{i\in I} f_i \subseteq f_i$ for all $i\in I$. Thus $\bigwedge_{i\in I} f_i$ is a lower bound of $\{f_i \mid i\in I\}$. Let $g\in F(M)$ be such that $g\subseteq f_i$ for all $i\in I$. Then $g(x)\leq f_i(x)$ for all $x\in M$ and $i\in I$, so $g(x)$ is a lower bound of $\{f_i(x)\mid i\in I\}$ for all $x\in M$. By the proof of Proposition 2.1, we have

$$g(x)\leq \inf\{f_i(x)\mid i\in I\}:=(\bigwedge_{i\in I} f_i)(x) \text{ for all } x\in M.$$

Thus $g\subseteq \bigwedge_{i\in I} f_i$, so $\bigwedge_{i\in I} f_i = \inf\{f_i\mid i\in I\}$. The proof of (ii) is similar. ∎

Proposition 2.5. *Let $(M,\Gamma,\leq)$ be an ordered Γ-groupoid and $\{f_i \mid i\in I\}$ a nonempty family of fuzzy subsets of M. Then we have the following:*

(i) *If f_i is a fuzzy ordered right ideal of M for all $i\in I$, then $\bigwedge_{i\in I} f_i$ and $\bigvee_{i\in I} f_i$ are fuzzy ordered right ideals of M.*

(ii) *If f_i is a fuzzy ordered left ideal of M for all $i\in I$, then $\bigwedge_{i\in I} f_i$ and $\bigvee_{i\in I} f_i$ are fuzzy ordered left ideals of M.*

(iii) *If f_i is a fuzzy ordered ideal of M for all $i\in I$, then $\bigwedge_{i\in I} f_i$ and $\bigvee_{i\in I} f_i$ are fuzzy ordered ideals of M.*

(iv) *If f_i is a fuzzy Γ-subgroupoid of M for all $i\in I$, then $\bigwedge_{i\in I} f_i$ is a fuzzy Γ-subgroupoid of M.*

(v) *If f_i is a fuzzy ordered filter of M for all $i\in I$, then $\bigwedge_{i\in I} f_i$ is a fuzzy ordered filter of M.*

Proof. (i) By Proposition 2.1, $\bigwedge_{i\in I} f_i$ is a fuzzy subset of M. Let $x,y\in M$ and $\gamma\in\Gamma$. Since f_i is a fuzzy ordered right ideal of M, we have $f_i(x\gamma y)\geq f_i(x)$ for all $i\in I$. Then

$$(\bigwedge_{i\in I} f_i)(x\gamma y):=\inf\{f_i(x\gamma y)\mid i\in I\}\geq \inf\{f_i(x)\mid i\in I\}:=(\bigwedge_{i\in I} f_i)(x).$$

Let $x, y \in M$ be such that $x \leq y$. Since f_i is a fuzzy ordered right ideal of M, we have $f_i(x) \geq f_i(y)$ for all $i \in I$. Then

$$(\bigwedge_{i\in I} f_i)(x):=\inf\{f_i(x)\mid i\in I\}\geq \inf\{f_i(y)\mid i\in I\}:=(\bigwedge_{i\in I} f_i)(y).$$

Hence $\bigwedge_{i\in I} f_i$ is a fuzzy ordered right ideals of M. Similarly, $\bigvee_{i\in I} f_i$ is a fuzzy ordered right ideals of M.
(ii) It is proved similarly.
(iii) It follows from (i) and (ii).
(iv) Let $x, y \in M$ and $\gamma \in \Gamma$. By Proposition 2.1, $\bigwedge_{i\in I} f_i$ is a fuzzy subset of M. Since f_i is a fuzzy Γ-subgroupoid of M, we have $f_i(x\gamma y) \geq \min\{f_i(x), f_i(y)\}$ for all $i \in I$. Then $f_i(x\gamma y) \geq f_i(x)$ or $f_i(x\gamma y) \geq f_i(y)$ for all $i \in I$. Since $f_i(x) \geq \inf\{f_i(x) \mid i \in I\}$ and $f_i(y) \geq \inf\{f_i(y) \mid i \in I\}$, we have $f_i(x\gamma y) \geq \inf\{f_i(x) \mid i \in I\}:= (\bigwedge_{i\in I} f_i)(x)$ or $f_i(x\gamma y) \geq \inf\{f_i(y) \mid i \in I\}:= (\bigwedge_{i\in I} f_i)(y)$ for all $i \in I$. Thus $f_i(x\gamma y) \geq \min\{(\bigwedge_{i\in I} f_i)(x), (\bigwedge_{i\in I} f_i)(y)\}$ for all $i \in I$. Hence

$$(\bigwedge_{i\in I} f_i)(x\gamma y):=\inf\{f_i(x\gamma y)\mid i\in I\}\geq \min\{(\bigwedge_{i\in I} f_i)(x),(\bigwedge_{i\in I} f_i)(y)\}.$$

Therefore $\bigwedge_{i\in I} f_i$ is a fuzzy Γ-subgroupoid of M.
(v) Let $x, y \in M$ and $\gamma \in \Gamma$. By Proposition 2.1, $\bigwedge_{i\in I} f_i$ is a fuzzy subset of M. Since f_i is a fuzzy ordered filter of M, we have $f_i(x\gamma y) = \min\{f_i(x), f_i(y)\}$ for all $i \in I$. Then $f_i(x\gamma y) = \min\{f_i(x), f_i(y)\} \leq f_i(x), f_i()y$ for all $i \in I$. Thus $(\bigwedge_{i\in I} f_i)(x\gamma y):= \inf\{f_i(x\gamma y) \mid i \in I\} \leq \inf\{f_i(x) \mid i \in I\}:= (\bigwedge_{i\in I} f_i)(x)$ and $(\bigwedge_{i\in I} f_i)(x\gamma y):= \inf\{f_i(x\gamma y) \mid i \in I\} \leq \inf\{f_i(y) \mid i \in I\}:= (\bigwedge_{i\in I} f_i)(y)$. Hence

$$(\bigwedge_{i\in I} f_i)(x\gamma y)\leq \min\{(\bigwedge_{i\in I} f_i)(x),(\bigwedge_{i\in I} f_i)(y)\}.$$

On the other hand, by (iv), the mapping $\bigwedge_{i\in I} f_i$ is a fuzzy Γ-subgroupoid of M, which means that $(\bigwedge_{i\in I} f_i)(x\gamma y) \geq \min\{(\bigwedge_{i\in I} f_i)(x), (\bigwedge_{i\in I} f_i)(y)\}$. Thus

$$(\bigwedge_{i\in I} f_i)(x\gamma y)= \min\{(\bigwedge_{i\in I} f_i)(x),(\bigwedge_{i\in I} f_i)(y)\}.$$

Let $x, y \in M$ be such that $x \leq y$. Since f_i is a fuzzy ordered filter of M, we have $f_i(x) \leq f_i(y)$ for all $i \in I$. Then

$$(\bigwedge_{i\in I} f_i)(x) := \inf\{f_i(x) \mid i \in I\} \leq \inf\{f_i(y) \mid i \in I\} := (\bigwedge_{i\in I} f_i)(y).$$

Therefore $\bigwedge_{i\in I} f_i$ is a fuzzy ordered filter of M. ∎

Notation 2.6. For an ordered Γ-groupoid $(M, \Gamma, \leq)$ and a fuzzy subset f of M, we denote

$$\begin{aligned} R_f &:= \{g \mid g \text{ fuzzy ordered right ideal of } M \text{ and } g \supseteq f\}, \\ L_f &:= \{g \mid g \text{ fuzzy ordered left ideal of } M \text{ and } g \supseteq f\}, \\ I_f &:= \{g \mid g \text{ fuzzy ordered ideal of } M \text{ and } g \supseteq f\}, \\ N_f &:= \{g \mid g \text{ fuzzy ordered filter of } M \text{ and } g \supseteq f\}. \end{aligned}$$

Remark 2.7. Clearly, $1_F \in R_f, 1_F \in L_f, 1_F \in I_f$ and $1_F \in N_f$. Since R_f, L_f, I_f and N_f are nonempty subsets of $F(M)$, by Proposition 2.1, the elements $\bigwedge_{g\in R_f} g$, $\bigwedge_{g\in L_f} g$, $\bigwedge_{g\in I_f} g$ and $\bigwedge_{g\in N_f} g$ exist in $F(M)$. Moreover, by Proposition 2.5, we have the following:

(i) $\bigwedge_{g\in R_f} g$ is a fuzzy ordered right ideal of M,

(ii) $\bigwedge_{g\in L_f} g$ is a fuzzy ordered left ideal of M,

(iii) $\bigwedge_{g\in I_f} g$ is a fuzzy ordered ideal of M,

(iv) $\bigwedge_{g\in N_f} g$ is a fuzzy ordered filter of M.

Proposition 2.8. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and f a fuzzy subsets of M. Then we have the following:*

(i) $\bigwedge_{g\in R_f} g \in R_f$ *(resp.* $\bigwedge_{g\in L_f} g \in L_f$, $\bigwedge_{g\in I_f} g \in I_f$ *and* $\bigwedge_{g\in N_f} g \in N_f$*),*

(ii) $\bigwedge_{g\in R_f} g \subseteq h$ *(resp.* $\bigwedge_{g\in L_f} g \subseteq h$, $\bigwedge_{g\in I_f} g \subseteq h$ *and* $\bigwedge_{g\in N_f} g \subseteq h$*) for all* $h \in R_f$ *(resp.* $h \in L_f, I_f, N_f$*).*

Proof. (i) By Remark 2.7, the element $\bigwedge_{g\in R_f} g$ is a fuzzy ordered right ideal of M. Let $x \in M$ and $g \in R_f$. Since $g \supseteq f$, we have $f(x) \leq g(x)$. Thus $f(x) \leq g(x)$ for all $g \in R_f$, so

$$f(x) \leq \inf\{g(x) \mid g \in R_f\} := (\bigwedge_{g\in R_f} g)(x).$$

Hence $\bigwedge_{g\in R_f} g \supseteq f$, so $\bigwedge_{g\in R_f} g \in R_f$.
(ii) It follows from Remark 2.3. ■

Remark 2.9. By Proposition 2.8, we have the following:

(i) $\bigwedge_{g\in R_f} g$ is a fuzzy ordered right ideal of M generated by f,

(ii) $\bigwedge_{g\in L_f} g$ is a fuzzy ordered left ideal of M generated by f,

(iii) $\bigwedge_{g\in I_f} g$ is a fuzzy ordered ideal of M generated by f,

(iv) $\bigwedge_{g\in N_f} g$ is a fuzzy ordered filter of M generated by f.

Proposition 2.10. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and f a fuzzy subsets of M. Let h be a fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) of M generated by f. Then $h = \bigwedge_{g\in R_f} g$ (resp. $h = \bigwedge_{g\in L_f} g, h = \bigwedge_{g\in I_f} g$ and $h = \bigwedge_{g\in N_f} g$).*

Proof. Since $h \in R_f$ and by Remark 2.3, we have $\bigwedge_{g\in R_f} g \subseteq h$. By Proposition 2.8, we have $\bigwedge_{g\in R_f} g \in R_f$. Since h is a fuzzy ordered right ideal of M generated by f, we have $h \subseteq \bigwedge_{g\in R_f} g$. Hence $h = \bigwedge_{g\in R_f} g$. The other cases are proved similarly. ■

Notation 2.11. If $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and f a fuzzy subsets of M, we denote by $R(f)$ (resp. $L(f), I(f)$ and $N(f)$) the fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) of M generated by f. That is,

$$R(f) := \bigwedge_{g\in R_f} g \text{ (resp. } L(f) := \bigwedge_{g\in L_f} g, I(f) := \bigwedge_{g\in I_f} g \text{ and } N(f) := \bigwedge_{g\in N_f} g).$$

Remark 2.12. Clearly, $1_F \in R_x, 1_F \in L_x, 1_F \in I_x$ and $1_F \in N_x$. Since R_x, L_x, I_x and N_x are nonempty subsets of $F(M)$, by Proposition 2.1, the elements $\bigwedge_{g\in R_x} g$, $\bigwedge_{g\in L_x} g$, $\bigwedge_{g\in I_x} g$ and $\bigwedge_{g\in N_x} g$ exist in $F(M)$. Moreover, by Proposition 2.5, we have the following:

(i) $\bigwedge_{g\in R_x} g$ is a fuzzy ordered right ideal of M,

(ii) $\bigwedge_{g\in L_x} g$ is a fuzzy ordered left ideal of M,

(iii) $\bigwedge_{g\in I_x} g$ is a fuzzy ordered ideal of M,

(iv) $\bigwedge_{g\in N_x} g$ is a fuzzy ordered filter of M.

Theorem 2.13. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and $x \in M$. Then we have the following:*

(i) $R_x = R_{f_{\{x\}}}$,

(ii) $L_x = L_{f_{\{x\}}}$,

(iii) $I_x = I_{f_{\{x\}}}$,

(iv) $N_x = N_{f_{\{x\}}}$.

Proof. (i) Let $f \in R_x$. Then f is a fuzzy ordered right ideal of M and $f(x) = 1$. Let $y \in M$. If $y = x$, then $1 = f(x) = f(y)$ and $f_{\{x\}}(y) = f_{\{x\}}(x) = 1$, so $f_{\{x\}}(y) = f(y)$. If $y \neq x$, then $f_{\{x\}}(y) = 0 \leq f(y)$. Since $f_{\{x\}}(y) \leq f(y)$ for all $y \in M$, we have $f \supseteq f_{\{x\}}$. Thus $f \in R_{f_{\{x\}}}$. On the other hand, let $g \in R_{f_{\{x\}}}$. Then g is a fuzzy ordered right ideal of M and $g \supseteq f_{\{x\}}$. Thus $1 = f_{\{x\}}(x) \leq g(x)$, so $g(x) = 1$. Hence $g \in R_x$. Therefore $R_x = R_{f_{\{x\}}}$. The other cases can be proved similarly. ■

Proposition 2.14. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and f a fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) of M. Then we have the following:*

(i) *$f^{-1}(1) = \emptyset$ or $f^{-1}(1)$ is an ordered right ideal (resp. ordered left ideal, ordered ideal and ordered filter) of M,*

(ii) *$f_{h^{-1}(1)} \subseteq h$ for all $h \in R_x$ (resp. for all $h \in L_x, h \in I_x$ and $h \in N_x$).*

Proof. (i) Assume that $f^{-1}(1) \neq \emptyset$. Let $a \in f^{-1}(1), b \in M$ and $\gamma \in \Gamma$. Then $f(a) = 1$. Since f is a fuzzy ordered right ideal of M, we have $f(a\gamma b) \geq f(a) = 1$. Thus $f(a\gamma b) = 1$, so $a\gamma b \in f^{-1}(1)$. Let $a \in M$ and $b \in f^{-1}(1)$ be such that $a \leq b$. Then $f(b) = 1$. Since f is a fuzzy ordered right ideal of M and $a \leq b$, we have $f(a) \geq f(b) = 1$. Thus $f(a) = 1$, so $a \in f^{-1}(1)$. Hence $f^{-1}(1)$ is an ordered right ideal of M. The rest of the proof is similar.
(ii) Let $h \in R_x$ and let $x \in M$. If $x \in h^{-1}(1)$, then $f_{h^{-1}(1)}(x) = 1$ and $h(x) = 1$, so $f_{h^{-1}(1)}(x) = h(x)$. If $x \notin h^{-1}(1)$, then $f_{h^{-1}(1)}(x) = 0 \leq h(x)$. Since $f_{h^{-1}(1)}(x) \leq h(x)$ for all $x \in M$, we have $f_{h^{-1}(1)} \subseteq h$. ∎

Lemma 2.15. [4] *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. A nonempty subset R (resp. L) of M is an ordered right ideal (resp. ordered left ideal) of M if and only if the characteristic function f_R (resp. f_L) is a fuzzy ordered right ideal (resp. fuzzy ordered left ideal) of M. A nonempty subset I of M is an ordered ideal (resp. ordered filter) of M if and only if the characteristic function f_I is a fuzzy ordered ideal (resp. fuzzy ordered filter) of M.*

Remark 2.16. If $(M, \Gamma, \leq)$ is an ordered Γ-groupoid and A an ordered right ideal (resp. ordered left ideal, ordered ideal and ordered filter) of M, then there exists a fuzzy ordered right ideal (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) f of M such that $f^{-1}(1) = A$. Indeed: If A an ordered right ideal of M, then the characteristic function f_A is a fuzzy ordered right ideal of M. On the other hand, we have $x \in A$ if and only if $f_A(x) = 1$ if and only if $x \in f_A^{-1}(1)$. Hence $f_A^{-1}(1) = A$.

Theorem 2.17. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and $x \in M$. Then we have the following:*

(i) $R(f_{\{x\}}) = f_{R(x)}$,

(ii) $L(f_{\{x\}}) = f_{L(x)}$,

(iii) $I(f_{\{x\}}) = f_{I(x)}$,

(iv) $N(f_{\{x\}}) = f_{N(x)}$.

Proof. (i) According to Notation 2.11, Theorem 2.13 and Remark 2.3, we have

$$(\star) \qquad R(f_{\{x\}}) := \bigwedge_{h \in R_{f_{\{x\}}}} h = \bigwedge_{h \in R_x} h \subseteq h \text{ for all } h \in R_x.$$

Since $R(x)$ is an ordered right ideal of M, it follows from Lemma 2.15 that $f_{R(x)}$ is a fuzzy ordered right ideal of M. Since $x \in R(x)$, we get $f_{R(x)}(x) = 1$. Thus $f_{R(x)} \in R_x$. Then, by $(\star)$, we have $R(f_{\{x\}}) \subseteq f_{R(x)}$. On the other hand, let $h \in R_x$. Since $h(x) = 1$, we have $x \in h^{-1}(1)$. By Proposition 2.14(i), we have $h^{-1}(1)$ is an ordered right ideal of M containing x, so $R(x) \subseteq h^{-1}(1)$. By Proposition 2.14(ii), we have $f_{R(x)} \subseteq f_{h^{-1}(1)} \subseteq h$. Thus

$$(\star\star) \qquad f_{R(x)} \subseteq h \text{ for all } h \in R_x.$$

By ($\star\star$), Proposition 2.4 and ($\star$), we have

$$f_{R(x)} \subseteq \inf\{h \mid h \in R_x\} = \bigwedge_{h \in R_x} h = R(f_{\{x\}}).$$

Hence $R(f_{\{x\}}) = f_{R(x)}$.

The other cases can be proved similarly. ∎

As we have already seen, $R(f_{\{a\}})$ is the fuzzy ordered right ideal of M generated by $f_{\{a\}}$, $L(f_{\{a\}})$ is the fuzzy ordered left ideal of M generated by $f_{\{a\}}$ and $I(f_{\{a\}})$ (resp. $N(f_{\{a\}})$ is the fuzzy ordered ideal (resp. fuzzy ordered filter) of M generated by $f_{\{a\}}$.

Notation 2.18. For an element $a \in M$, we denote:

$$\begin{aligned} R^F(a) &= R(f_{\{a\}}), \\ L^F(a) &= L(f_{\{a\}}), \\ I^F(a) &= I(f_{\{a\}}), \\ N^F(a) &= N(f_{\{a\}}). \end{aligned}$$

Definition 2.19. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid, f a fuzzy subset of M and $x \in M$. We say that f contains x if $f(x) = 1$.

Remark 2.20. According to Notation 2.18, Notation 2.11, and Theorem 2.13, for any $x \in M$, we have the following:

(i) $R^F(x) = R(f_{\{x\}}) = \bigwedge_{g \in R_{f_{\{x\}}}} g = \bigwedge_{g \in R_x} g$,

(ii) $L^F(x) = L(f_{\{x\}}) = \bigwedge_{g \in L_{f_{\{x\}}}} g = \bigwedge_{g \in L_x} g$,

(iii) $I^F(x) = I(f_{\{x\}}) = \bigwedge_{g \in I_{f_{\{x\}}}} g = \bigwedge_{g \in I_x} g$,

(iv) $N^F(x) = N(f_{\{x\}}) = \bigwedge_{g \in N_{f_{\{x\}}}} g = \bigwedge_{g \in N_x} g$.

Proposition 2.21. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and $x \in M$. Then we have the following:*

(i) $R^F(x) \in R_x$ *and* $R^F(x) \subseteq h$ *for all* $h \in R_x$,

(ii) $L^F(x) \in L_x$ *and* $L^F(x) \subseteq h$ *for all* $h \in L_x$,

(iii) $I^F(x) \in I_x$ *and* $I^F(x) \subseteq h$ *for all* $h \in I_x$,

(iv) $N^F(x) \in N_x$ *and* $N^F(x) \subseteq h$ *for all* $h \in N_x$.

Proof. (i) Since $R^F(x) = \bigwedge_{g \in R_x} g, R_x \neq \emptyset$ and each $g \in R_x$ is a fuzzy ordered right ideal of M and by Proposition 2.5, we have $R^F(x)$ is a fuzzy ordered right ideal of M. On the other hand, $(\bigwedge_{g \in R_x} g)(x) := \inf\{g(x) \mid g \in R_x\} = 1$. According to Definition 2.19, $R^F(x)$ is a fuzzy ordered right ideal of M containing x and so $R^F(x) \in R_x$. By Remark 2.3, we have $R^F(x) = \bigwedge_{g \in R_x} g \subseteq h$ for all $h \in R_x$.

The other cases can be proved similarly. ∎

Definition 2.22. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid and $x \in M$. Then $R^F(x)$ (resp. $L^F(x), I^F(x)$ and $N^F(x)$) is called the *fuzzy ordered right ideal* (resp. fuzzy ordered left ideal, fuzzy ordered ideal and fuzzy ordered filter) *of M generated by x*.

Definition 2.23. Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. We define relations $\mathcal{R}^F, \mathcal{L}^F, \mathcal{I}^F, \mathcal{N}^F$ and $\mathcal{N}^F$ on M as follows:

$$\begin{aligned} a\mathcal{R}^F b &\Leftrightarrow R^F(a) = R^F(b), \\ a\mathcal{L}^F b &\Leftrightarrow L^F(a) = L^F(b), \\ a\mathcal{I}^F b &\Leftrightarrow I^F(a) = I^F(b), \\ a\mathcal{N}^F b &\Leftrightarrow N^F(a) = N^F(b). \end{aligned}$$

Theorem 2.24. *Let $(M, \Gamma, \leq)$ be an ordered Γ-groupoid. Then we have the following:*

(i) $\mathcal{R}^F = \mathcal{R}$,

(ii) $\mathcal{L}^F = \mathcal{L}$,

(iii) $\mathcal{I}^F = \mathcal{I}$,

(iv) $\mathcal{N}^F = \mathcal{N}$.

Proof. (i) For all $a, b \in M$, we have

$$\begin{aligned} a\mathcal{R}^F b &\Leftrightarrow R^F(a) = R^F(b) \\ &\Leftrightarrow R(f_{\{a\}}) = R(f_{\{b\}}) \\ &\Leftrightarrow f_{R(a)} = f_{R(b)} \text{ (by Theorem 2.17)} \\ &\Leftrightarrow R(a) = R(b) \\ &\Leftrightarrow a\mathcal{R}b. \end{aligned}$$

Hence $\mathcal{R}^F = \mathcal{R}$. The other cases can be proved similarly. ∎

Proposition 2.25. *Let $(M,\Gamma,\leq)$ be an ordered Γ-groupoid and $\{A_i \mid i\in I\}$ a nonempty family of fuzzy subsets of M. Then*

$$\bigwedge_{i\in I} f_{A_i} = f_{\bigcap\limits_{i\in I} A_i}.$$

Proof. Since f_{A_i} is a fuzzy subset of M for all $i\in I$ and by Proposition 2.1, we have $\bigwedge\limits_{i\in I} f_{A_i}$ is a fuzzy subset of M. Let $x\in M$. Then

$$(\bigwedge_{i\in I} f_{A_i})(x) := \inf\{f_{A_i}(x) \mid i\in I\}.$$

If $x\in A_i$ for all $i\in I$, then $f_{A_i}(x)=1$ for all $i\in I$ and $(f_{\bigcap\limits_{i\in I} A_i})(x)=1$. Thus

$$(\bigwedge_{i\in I} f_{A_i})(x) := \inf\{f_{A_i}(x) \mid i\in I\} = 1 = (f_{\bigcap\limits_{i\in I} A_i})(x).$$

If $x\notin A_j$ for some $j\in I$, then $x\notin \bigcap\limits_{i\in I} A_i$. Thus

$$(f_{\bigcap\limits_{i\in I} A_i})(x) = 0.$$

Since $x\notin A_j$, we have

$$f_{A_j}(x)=0.$$

Since $0\leq f_{A_i}(x)$ for all $i\in I$, we have

$$0\leq \inf\{f_{A_i}(x) \mid i\in I\} \leq f_{A_j}(x) = 0.$$

Thus

$$(\bigwedge_{i\in I} f_{A_i})(x) := \inf\{f_{A_i}(x) \mid i\in I\} = 0 = (f_{\bigcap\limits_{i\in I} A_i})(x).$$

Hence $\bigwedge\limits_{i\in I} f_{A_i} = f_{\bigcap\limits_{i\in I} A_i}$. ∎

In a similar way we prove the following.

Proposition 2.26. *Let $(M,\Gamma,\leq)$ be an ordered Γ-groupoid and $\{A_i \mid i\in I\}$ a nonempty family of fuzzy subsets of M. Then*

$$\bigvee_{i\in I} f_{A_i} = f_{\bigcup\limits_{i\in I} A_i}.$$

Remark 2.27. For an ordered Γ-groupoid $(M,\Gamma,\leq)$, we have

$$R(a) = \bigcap\{R \mid R \text{ ordered right ideal of } M \text{ and } a\in R\}.$$

Thus, by Proposition 2.25, we have

$$f_{R(a)} = \bigwedge\{f_R \mid R \text{ ordered right ideal of } M \text{ and } a\in R\}.$$

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