COMPACTIFICATION OF A SOFT FUZZY PRODUCT C-SPACE

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Abstract. In this paper, the soft fuzzy product C-structure is introduced and some of the relevant properties of the associated product map on soft fuzzy product C-space are studied. Moreover, compactifying the soft fuzzy product C-space through the soft fuzzy product generalized topological space on $\mathcal{Q}(X_1 \times X_2)$ is established.

Keywords: Soft fuzzy product C-structure, functor, associated product map, weakly induced soft fuzzy product $C\#$-space, soft fuzzy $C^{\#}$-quotient product map, soft fuzzy product dense set, soft fuzzy product strong generalized topological space, soft fuzzy $\mathfrak{G}$-compact space.

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1. Introduction

The fuzzy concept has penetrated almost all branches of Mathematics since the introduction of the concept of fuzzy set by Zadeh [8]. Fuzzy sets have applications in many fields such as information [4] and control [5]. The theory of fuzzy topological spaces was introduced and developed by C.L.Chang [2]. Several properties on fuzzy product topological spaces were discussed by K.K. Azad [1].

The notions of Soft fuzzy set over a poset I and soft fuzzy topological space was introduced by Ismail U. Tiryaki [6]. The notion of C-set in general topology was introduced by E. Hatir, T. Noiri and S. Yuksel [3]. The concept of a soft fuzzy C-open set in a soft fuzzy topological space is introduced by T. Yogalakshmi, E. Roja, M.K. Uma [7].

In this paper, a new structure, called soft fuzzy product C-structure on the soft fuzzy product space is introduced. An associated product map is defined and some of its properties are studied. Moreover, a compactification of the soft fuzzy product C-space through the soft fuzzy product strong generalized topological space is established.

2. Preliminaries

Definition 2.1. [7] Let X be a nonempty set. Let μ be a fuzzy subset of X such that $\mu : X \to [0,1]$ and M be any crisp subset of X. Then, the ordered pair (μ, M) is called as a *soft fuzzy set* in X. The family of all soft fuzzy subsets of X,will be denoted by $\mathbf{SF(X)}$.

Definition 2.2. [7] Let X be a non-empty set. Then, the *complement* of a soft fuzzy set (μ, M) is defined as $(\mu, M)' = (1 - \mu, X|M)$

Definition 2.3. [7] Let X be a non-empty set and the soft fuzzy sets A and B be in the form,

$$A = \{(\mu, M) : \mu(x) \in I^X, \forall x \in X, M \subseteq X\}$$
$$B = \{(\lambda, N) : \lambda(x) \in I^X, \forall x \in X, N \subseteq X\}$$

Then,

(1) $A \sqsubseteq B \Leftrightarrow \mu(x) \leq \lambda(x), \forall x \in X, M \subseteq N$.

(2) $A = B \Leftrightarrow \mu(x) = \lambda(x), \forall x \in X, M = N$.

(3) $A \sqcap B \Leftrightarrow \mu(x) \wedge \lambda(x), \forall x \in X, M \cap N$.

(4) $A \sqcup B \Leftrightarrow \mu(x) \vee \lambda(x), \forall x \in X, M \cup N$.

Definition 2.4. [7] A soft fuzzy topology on a non-empty set X is a family τ of soft fuzzy sets in X satisfying the following axioms:

(1) $(0, \phi), (1, X) \in \tau$.

(2) For any family of soft fuzzy sets $(\lambda_j, N_j) \in \tau$, $j \in J$, $\Rightarrow \sqcup_{j \in J}(\lambda_j, N_j) \in \tau$.

(3) For any finite number of soft fuzzy sets $(\lambda_j, N_j) \in \tau, j = 1, 2, 3, ...n$, $\Rightarrow \sqcap_{j=1}^{n}(\lambda_j, N_j) \in \tau$.

Then, the pair $(X, \tau)$is called as a *soft fuzzy topological space.* (in short, **SFTS**)

Any soft fuzzy set in τ is said to be a *soft fuzzy open set* (in short, **SFOS**) in X.

The complement of SFOS in a SFTS (X, τ) is called as a *soft fuzzy closed set*, denoted **SFCS** in X.

Definition 2.5. [1] The *product* $\lambda \times \mu$ *of a fuzzy set* λ *of* X *and a fuzzy set* μ *of* Y is a fuzzy set of $X \times Y$, defined by $(\lambda \times \mu)\langle x, y\rangle = \min(\lambda(x), \mu(y))$, for each $\langle x, y\rangle \in X \times Y$.

Definition 2.6. [1] The *product* $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ *of mappings* $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, defined by $(f_1 \times f_2)\langle x_1, x_2\rangle = (f_1(x_1), f_2(x_2))$, for each $\langle x_1, x_2\rangle \in X_1 \times X_2$.

3. On soft fuzzy product C-space

Definition 3.1. Let $\langle x_1, x_2\rangle \in X_1 \times X_2$ and $\lambda : X_1 \times X_2 \to [0, 1]$. Define,

$$\langle x_1, x_2\rangle_\lambda \langle y_1, y_2\rangle = \begin{cases} \lambda \ (0 < \lambda \le 1), & \text{if } \langle x_1, x_2\rangle = \langle y_1, y_2\rangle \\ 0, & \text{otherwise} \end{cases}$$

Then, the soft fuzzy set $(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})$ is called as the *soft fuzzy point* (in short, **SFP**) in $SF(X_1 \times X_2)$, with *support,* $\langle x_1, x_2\rangle$ and *value,* λ.

Definition 3.2. Let (λ_1, N_1) and (λ_2, N_2) be any two soft fuzzy sets. Then, the *soft fuzzy product set* is defined as $(\lambda_1, N_1) \times (\lambda_2, N_2) = (\lambda_1 \times \lambda_2, N_1 \times N_2)$

Definition 3.3. Let (X_1, τ_1) and (X_2, τ_2) be any two soft fuzzy topological spaces. The collection $\mathcal{B} = \{(\lambda_1 \times \lambda_2, N_1 \times N_2) : (\lambda, N) \in \tau_1, (\mu, M) \in \tau_2 \ \& \ N \times M \subseteq X_1 \times X_2\}$ forms *an open base* of a soft fuzzy topology in $X_1 \times X_2$.

The soft fuzzy topology in $X_1 \times X_2$, induced by $\mathcal{B}$ is called as the *soft fuzzy product topology* of τ_1 and τ_2, denoted by $\tau_1 \times \tau_2$.

The ordered pair $(X_1 \times X_2, \tau_1 \times \tau_2)$, which means the product of (X_1, τ_1) and (X_2, τ_2), is called the *soft fuzzy product topological space* (in short, $SFPTS$).

Moreover, the member of a soft fuzzy product topology is called as a *soft fuzzy product open set.*

Definition 3.4. Let $(X_1 \times X_2, \tau_1 \times \tau_2)$ be a SFPTS and (λ, N) be a soft fuzzy product set in $X_1 \times X_2$. Then, the *soft fuzzy product interior* and *soft fuzzy product closure* of (λ, N) are defined by,

$$\begin{aligned} cl(\lambda, N) &= \sqcap\{(\mu, M) : (\mu, M) \text{ is a soft fuzzy closed set in } X_1 \times X_2 \\ &\quad \text{and } (\lambda, N) \sqsubseteq (\mu, M)\} \\ int(\lambda, N) &= \sqcup\{(\gamma, L) : (\gamma, L) \text{ is a soft fuzzy open set in } X_1 \times X_2 \\ &\quad \text{and } (\lambda, N) \sqsupseteq (\gamma, L)\} \end{aligned}$$

Definition 3.5. Let $(X_1 \times X_2, \tau_1 \times \tau_2)$ be a SFPTS. A soft fuzzy product set (λ, N) is said to be *soft fuzzy product* α^**-open* , if $int(\lambda, N) = int(cl(int(\lambda, N)))$.

Definition 3.6. Let $(X_1 \times X_2, \tau_1 \times \tau_2)$ be a SFPTS. A soft fuzzy product set (λ, N) is said to be *soft fuzzy product C-open* (in short, $SFPcOS$), if

$$(\lambda, N) = (\mu, M) \sqcap (\gamma, K)$$

where, (μ, M) is a soft fuzzy product open set and (γ, K) is a soft fuzzy product α^*-open set.

The complement of soft fuzzy product C-open set is called as a *soft fuzzy product C-closed set* (in short, $SFPcCS$).

Definition 3.7. A *soft fuzzy product C-structure* on a non-empty set $X_1 \times X_2$ is a family $\mathfrak{st}(\tau_1 \times \tau_2)$ of soft fuzzy product C-open sets in $X_1 \times X_2$ satisfying the following axioms:

(1) $(0, \phi), (1, X) \in \tau$.

(2) For any finite number of soft fuzzy C-open sets $(\lambda_j, N_j) \in \tau, j = 1, 2, 3, ...n$, $\Rightarrow \sqcap_{j=1}^{n}(\lambda_j, N_j) \in \tau$.

Then, the pair $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$ is called as a *soft fuzzy product C-space.* (in short, $SFPC\mathfrak{st}(X_1 \times X_2)$)

Any soft fuzzy product set (λ, N) in $\mathfrak{st}(\tau_1 \times \tau_2)$ is said to be a *soft fuzzy product C-open set* in $X_1 \times X_2$.

The complement of a soft fuzzy product C-open set in $SFPC\mathfrak{st}(X_1 \times X_2)$ is called as a *soft fuzzy product C-closed set.*

Definition 3.8. Let $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$ and $(Y_1 \times Y_2, \mathfrak{st}(\sigma_1 \times \sigma_2))$ be any soft fuzzy product C-spaces. A function $f : X \to Y$ is said to be *soft fuzzy $C^{\#}$-continuous product map*, if the inverse image of every soft fuzzy product C-open set in $(Y_1 \times Y_2, \mathfrak{st}(\sigma_1 \times \sigma_2))$ is a soft fuzzy product C-open set in $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$.

Definition 3.9. Let $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$ and $(Y_1 \times Y_2, \mathfrak{st}(\sigma_1 \times \sigma_2))$ be any soft fuzzy product topological spaces. A surjective function $f : X \to Y$ is said to be *soft fuzzy $C^{\#}$-quotient product map*, if the inverse image of every soft fuzzy product C-open set in $(Y_1 \times Y_2, \mathfrak{st}(\sigma_1 \times \sigma_2))$ is a soft fuzzy product C-open set in $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$.

Definition 3.10. Let $(X_1 \times X_2, T_1 \times T_2)$ be a product topological space and $I = [0, 1]$ equipped with the usual topology, *a lower semi $C^{\#}$-continuous pair* (μ, M), where $\mu : (X_1 \times X_2, T_1 \times T_2) \to I$ with a C-open set $\mu^{-1}((\alpha, 1])$ and $M \subseteq X_1 \times X_2$ is also a C-open set in $X_1 \times X_2$, for all $\alpha \in [0, 1]$.

Definition 3.11. A soft fuzzy product C-space $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$ is said to be *a weakly induced soft fuzzy product $C^{\#}$-space*, which is the soft fuzzy product C-space induced by a topological space $(X_1 \times X_2, T_1 \times T_2)$ if the following conditions hold :

(a) $T_1 \times T_2 = \{A \subset X_1 \times X_2$ is a product C-open set $| \ (\chi_A, A) \in St(\tau_1 \times \tau_2)\}$

(b) Every $(\mu, M) \in \mathfrak{st}(\tau_1 \times \tau_2)$ is a lower semi $C^{\#}$-continuous pair.

Definition 3.12. Let $PrTop$ be the category of all the product topological spaces and the continuous product maps. Let $SFPrCst$ be the category of all the soft fuzzy product C-space and SF $C\#$-continuous product maps. Define a *functor*, ω :

$PrTop \rightarrow SFPrCst$, which associates to any product topological space $(X_1 \times X_2, \tau_1 \times \tau_2)$, the soft fuzzy product C-space $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$, where $\omega(\tau_1 \times \tau_2)$ is the totality of all lower semi $C^{\#}$-continuous pair. Then, $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$ is called as the weakly induced soft fuzzy product $C^{\#}$-space by $(X_1 \times X_2, \tau_1 \times \tau_2)$.

Proposition 3.1. *For mappings $f_i : X_i \rightarrow Y_i$ and soft fuzzy sets (λ_i, N_i) of Y_i, $(i = 1, 2)$; we have $(f_1 \times f_2)^{-1}(\lambda_1 \times \lambda_2, N_1 \times N_2) = f_1^{-1}(\lambda_1, N_1) \times f_2^{-1}(\lambda_2, N_2)$.*

Proof. The proof is clear.

Proposition 3.2. *For mappings $f_i : X_i \rightarrow Y_i$ and soft fuzzy sets (λ_i, N_i) of Y_i, $(i = 1, 2)$; we have $(f_1 \times f_2)(\lambda_1 \times \lambda_2, N_1 \times N_2) \sqsubseteq f_1(\lambda_1, N_1) \times f_2(\lambda_2, N_2)$.*

Proof. The proof is clear.

4. Properties of the associated product map on soft fuzzy product C-space

Definition 4.1. Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be any two maps. Let $X_1 \times X_2$ and $Y_1 \times Y_2$ be two product sets and $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ be a product map. Then, define the product associated map $\widetilde{f_1 \times f_2}$ as $\widetilde{f_1 \times f_2}(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})$ $= f_1 \times f_2(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})$, for each soft fuzzy point $(\langle x_1, x_2\rangle\, \lambda, \{\langle x_1, x_2\rangle\})$ in $SFPC\mathfrak{st}(X_1 \times X_2)$.

Proposition 4.1. *Let $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ be any two onto maps. Let $X_1 \times X_2$ and $Y_1 \times Y_2$ be two product sets. If $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$ is a product onto map, then for each soft fuzzy point $(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$ in $SFPC\mathfrak{st}(X_1 \times X_2($, $\widetilde{f_1 \times f_2}(< x_1, x_2 >_\lambda, \{< x_1, x_2 >\})$ is the soft fuzzy point in $SFPC\mathfrak{st}(Y_1 \times Y_2)$ that takes the value λ in $f_1 \times f_2 < x_1, x_2 >$.*

Proof. For $0 < \lambda \leq 1$,

$$\begin{aligned}\widetilde{f_1 \times f_2}(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\}) &= f_1 \times f_2(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\}) \\ &= (f_1 \times f_2 \langle x_1, x_2\rangle_\lambda, f_1 \times f_2(\{\langle x_1, x_2\rangle\})) \\ &= (f_1 \times f_2(\langle x_1, x_2\rangle_\lambda), \{\langle f_1(x_1), f_2(x_2)\rangle\})\end{aligned}$$

where, $f_1 \times f_2(\langle x_1, x_2\rangle_\lambda)(\langle y_1, y_2\rangle)$

$$= \begin{cases} \sup\limits_{\langle x,y\rangle \in (f_1 \times f_2)^{-1}(\langle y_1, y_2\rangle)} \langle x_1, x_2\rangle_\lambda (\langle x, y\rangle), & \text{if } (f_1 \times f_2)^{-1}(\langle y_1, y_2\rangle) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} \lambda, & \text{if} (f_1 \times f_2)^{-1}(\langle y_1, y_2\rangle) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

Then, $\widetilde{f_1 \times f_2}(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})$ is the soft fuzzy point in $SFPC\mathfrak{st}(Y_1 \times Y_2)$ that takes the value λ in $f_1 \times f_2(\langle x_1, x_2\rangle)$. ■

Proposition 4.2. *Let $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$, $g_1 : Y_1 \to Z_1$ and $g_2 : Y_2 \to Z_2$ be any maps. Let $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ and $g_1 \times g_2 : Y_1 \times Y_2 \to Z_1 \times Z_2$ be the two product onto maps. Then, $\widetilde{(g_1 \times g_2) \circ (f_1 \times f_2)} = (\widetilde{g_1 \times g_2}) \circ (\widetilde{f_1 \times f_2})$.*

Proof. By using the above Property 4.1, we have for each soft fuzzy point $(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})$ in $SFPC\mathfrak{st}(X_1 \times X_2)$

$$
\begin{aligned}
&\widetilde{(g_1 \times g_2) \circ (f_1 \times f_2)}(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\}) \\
&\quad = (g_1 \times g_2) \circ (f_1 \times f_2)(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\}) \\
&\quad = (g_1 \times g_2)((f_1 \times f_2)(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})) \\
&\quad = (g_1 \times g_2)((\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})) \\
&\quad = (\widetilde{g_1 \times g_2})((\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})) \\
&\quad = (\widetilde{g_1 \times g_2}) \circ (\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})
\end{aligned}
$$

Thus, $\widetilde{(g_1 \times g_2) \circ (f_1 \times f_2)} = (\widetilde{g_1 \times g_2}) \circ (\widetilde{f_1 \times f_2})$. ∎

Proposition 4.3. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two onto maps. Let $X_1 \times X_2$ and $Y_1 \times Y_2$ be two product sets. Let $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a product onto map. If $(\widetilde{f_1 \times f_2})$ is the identity map, then $f_1 \times f_2$ is also the identity map.*

Proof. Since $(\widetilde{f_1 \times f_2})$ is the identity map, $(\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\}) = \langle (x_1, x_2)_\lambda, \{(x_1, x_2)\}\rangle$, for each soft fuzzy point $(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})$ in $SFPC\mathfrak{st}(X_1 \times X_2)$. This implies that, $(f_1 \times f_2)(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\}) = \langle (x_1, x_2)_\lambda, \{(x_1, x_2)\}\rangle$, for each soft fuzzy point $(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})$. Now, by the definition of the associated product map, $(\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\}) = (f_1 \times f_2)(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\}) = \langle (f_1 \times f_2)(\langle x_1, x_2\rangle_\lambda), \{\langle f_1(x_1), f_2(x_2)\rangle\}\rangle$. This implies that, $((f_1 \times f_2)(\langle x_1, x_2\rangle_\lambda), \{\langle f_1(x_1), f_2(x_2)\rangle\})$ is the soft fuzzy point which takes the value λ in $(f_1 \times f_2)\langle x_1, x_2\rangle$ and $(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})$ is also the soft fuzzy point which takes the value λ in $\langle x_1, x_2\rangle$. Thus, $(f_1 \times f_2)(\langle x_1, x_2\rangle) = \langle x_1, x_2\rangle$, for each $\langle x_1, x_2\rangle \in X_1 \times X_2$. Hence, $f_1 \times f_2$ is the identity map. ∎

Proposition 4.4. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two maps.*

(1) *If $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a product onto map, then $(\widetilde{f_1 \times f_2})$ is also the product onto map.*

(2) *If $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a product one- to-one map, then $(\widetilde{f_1 \times f_2})$ is also the product one-to-one map.*

Proof. (1) For each $(\langle y_1, y_2\rangle_\alpha, \{\langle y_1, y_2\rangle\})$ soft fuzzy point in $SFPC\mathfrak{st}(Y_1 \times Y_2)$, we have $\langle y_1, y_2\rangle \in Y_1 \times Y_2$, then there exists at least $\langle x_1, x_2\rangle \in X_1 \times X_2$ such that $(f_1 \times f_2)\langle x_1, x_2\rangle = \langle y_1, y_2\rangle$. Now, $(\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) = (f_1 \times f_2)(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) = ((f_1 \times f_2)(\langle x_1, x_2\rangle_\alpha), (f_1 \times f_2)(\{\langle x_1, x_2\rangle\}))$ which takes the value α in $(f_1 \times f_2)\langle x_1, x_2\rangle$ and since $(f_1 \times f_2)\langle x_1, x_2\rangle = \langle y_1, y_2\rangle$, this shows that $((f_1 \times f_2)(\langle x_1, x_2\rangle_\alpha), (f_1 \times f_2)(\{\langle x_1, x_2\rangle\})) = (\langle y_1, y_2\rangle_\alpha, \{\langle y_1, y_2\rangle\})$. Therefore, $(\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) = (\langle y_1, y_2\rangle_\alpha, \{\langle y_1, y_2\rangle\})$. Thus, $\widetilde{f_1 \times f_2}$ is the product onto map.

(2) If $(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}), (\langle x'_1, x'_2\rangle_\beta, \{\langle x'_1, x'_2\rangle\})$ are the two soft fuzzy points in $SFPC\mathfrak{st}(X_1 \times X_2)$ such that $(\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) = (\widetilde{f_1 \times f_2})(\langle x'_1, x'_2\rangle_\beta, \{\langle x'_1, x'_2\rangle\})$. This implies that $(f_1 \times f_2)(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) = (f_1 \times f_2)(\langle x'_1, x'_2\rangle_\beta, \{\langle x'_1, x'_2\rangle\})$. This shows $(f_1 \times f_2)\langle x_1, x_2\rangle = (f_1 \times f_2)\langle x'_1, x'_2\rangle$ and $\alpha = \beta$. Since $(f_1 \times f_2)$ is a one-to-one map, $\langle x_1, x_2\rangle = \langle x'_1, x'_2\rangle$ and $\alpha = \beta$, it follows

$$(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) = (\langle x'_1, x'_2\rangle_\beta, \{\langle x'_1, x'_2\rangle\}).$$

Thus, $\widetilde{f_1 \times f_2}$ is one-to-one. ■

Proposition 4.5. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two maps. If $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is a product one- to-one map, then $\widetilde{(f_1 \times f_2)^{-1}} = (\widetilde{f_1 \times f_2})^{-1}$.*

Proof. For each soft fuzzy point $(\langle y_1, y_2\rangle_\alpha, \{\langle y_1, y_2\rangle\})$ in $SFPC\mathfrak{st}(Y_1 \times Y_2)$ and by the hypothesis, there exists a unique $\langle x_1, x_2\rangle \in X_1 \times X_2$ such that $(f_1 \times f_2)\langle x_1, x_2\rangle = \langle y_1, y_2\rangle$. It must be shown that $\widetilde{(f_1 \times f_2)^{-1}}$ is well-defined. It is enough to show that $\widetilde{(f_1 \times f_2)^{-1}}(\langle y_1, y_2\rangle_\alpha, \{\langle y_1, y_2\rangle\}) = (\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\})$. Otherwise, let $\widetilde{(f_1 \times f_2)^{-1}}(\langle y_1, y_2\rangle_\alpha, \{\langle y_1, y_2\rangle\}) = (\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})$ and $\alpha \neq \lambda$. Then, $(\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\}) = (f_1 \times f_2)(\widetilde{(f_1 \times f_2)^{-1}}(\langle y_1, y_2\rangle_\alpha, \{\langle y_1, y_2\rangle\}))$ $= (\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\})$. Since $(f_1 \times f_2)$ is a one-to-one map, $(\widetilde{f_1 \times f_2})$ is one-to-one. Thus, $(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) = (\langle x_1, x_2\rangle_\lambda, \{\langle x_1, x_2\rangle\})$. But, $\lambda \neq \alpha$. Thus, it leads to a contradiction. Therefore, $\widetilde{(f_1 \times f_2)^{-1}}(\langle y_1, y_2\rangle_\alpha, \{\langle y_1, y_2\rangle\})$ is uniquely the soft fuzzy point in $SFPC\mathfrak{st}(X_1 \times X_2)$ which takes the value α in $(f_1 \times f_2)^{-1} < y_1, y_2 >$. Thus, it is well defined.

Next, $(\widetilde{f_1 \times f_2})^{-1}(\langle y_1, y_2\rangle_\alpha, \{\langle y_1, y_2\rangle\}) = (f_1 \times f_2)^{-1}(\langle y_1, y_2\rangle_\alpha, \{\langle y_1, y_2\rangle\}) = \widetilde{(f_1 \times f_2)^{-1}}(\langle y_1, y_2\rangle_\alpha, \{\langle y_1, y_2\rangle\})$. Hence, $\widetilde{(f_1 \times f_2)^{-1}} = (\widetilde{f_1 \times f_2})^{-1}$. ■

Proposition 4.6. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two maps. Let $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ be a product map.*

(a) *If $(\widetilde{f_1 \times f_2})$ is onto, then $(f_1 \times f_2)$ is also onto.*

(b) *If $(\widetilde{f_1 \times f_2})$ is one-to-one, then $(f_1 \times f_2)$ is also one-to-one.*

Proof. (a) For each $\langle y_1, y_2\rangle \in Y_1 \times Y_2$, let $(\langle y_1, y_2\rangle_1, \{\langle y_1, y_2\rangle\})$ be the soft fuzzy point in $SFPC\mathfrak{st}(Y_1 \times Y_2)$ which takes the value 1 in $\langle y_1, y_2\rangle$. By hypothesis, there exists a soft fuzzy point $(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\})$ in $SFPC\mathfrak{st}(X_1 \times X_2)$ such that $\widetilde{f_1 \times f_2}(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) = (\langle y_1, y_2\rangle_1, \{\langle y_1, y_2\rangle\})$. Then,

$f_1 \times f_2(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) = (\langle y_1, y_2\rangle_1, \{\langle y_1, y_2\rangle\})$ and $(f_1 \times f_2)^{-1}(\langle y_1, y_2\rangle) \neq \phi$.

Hence, $(f_1 \times f_2)$ is a onto map.

(b) Let $\langle x_1, x_2\rangle, \langle x_1', x_2'\rangle \in X_1 \times X_2$ with $(f_1 \times f_2)(\langle x_1, x_2\rangle) = (f_1 \times f_2)(\langle x_1', x_2'\rangle)$. Now, $(\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_1, \{\langle x_1, x_2\rangle\}) = (f_1 \times f_2)(\langle x_1, x_2\rangle_1, \{\langle x_1, x_2\rangle\}) = \langle (f_1 \times f_2)(\langle x_1, x_2\rangle)_1, (f_1 \times f_2)(\{\langle x_1, x_2\rangle\})\rangle$, where, for λ=1

$$(f_1 \times f_2)(\langle x_1, x_2\rangle_1)(\langle y_1, y_2\rangle)$$

$$= \begin{cases} \sup\limits_{\langle x,y\rangle \in (f_1\times f_2)^{-1}(\langle y_1,y_2\rangle)} \langle x_1, x_2\rangle_\lambda (< x, y >), & \text{if } (f_1 \times f_2)^{-1}(\langle y_1, y_2\rangle) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & \text{if } \langle x_1, x_2\rangle = \langle x, y\rangle \text{ and} (f_1 \times f_2)^{-1}(\langle y_1, y_2\rangle) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$= \begin{cases} 1, & \text{if} (f_1 \times f_2)(\langle x_1', x_2'\rangle) = \langle y_1, y_2\rangle \\ 0, & \text{otherwise} \end{cases}$$

$$= (f_1 \times f_2)(\langle x_1', x_2'\rangle_1)(\langle y_1, y_2\rangle).$$

This implies that, $(\widetilde{f_1 \times f_2})(\langle x_1, x_2\rangle_1, \{\langle x_1, x_2\rangle\}) = (\widetilde{f_1 \times f_2})(\langle x_1', x_2'\rangle_1, \{\langle x_1', x_2'\rangle\})$. Since $(\widetilde{f_1 \times f_2})$ is a one-to-one map, $(\langle x_1', x_2'\rangle_1, \{\langle x_1', x_2'\rangle\}) = (\langle x_1, x_2\rangle_1, \{\langle x_1, x_2\rangle\})$. This implies that, $\langle x_1, x_2\rangle = \langle x_1', x_2'\rangle$. Thus, $(f_1 \times f_2)$ is a one-to-one map. ■

Definition 4.2. Let $(X_1 \times X_2, T_1 \times T_2)$ and $(Y_1 \times Y_2, S_1 \times S_2)$ be any two product topological spaces. A function $f : X \to Y$ is said to be a *C-irresolute product map*, if the inverse image of every product C-open set in $(Y_1 \times Y_2, S_1 \times S_2)$ is a product C-open set in $(X_1 \times X_2, T_1 \times T_2)$.

Proposition 4.7. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two maps. Let $(X_1 \times X_2, \tau_1 \times \tau_2)$ and $(Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ be any two product topological spaces. If $f_1 \times f_2 : (X_1 \times X_2, \tau_1 \times \tau_2) \to (Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ is a C-irresolute product map iff $\widetilde{f_1 \times f_2} : (X_1 \times X_2, \omega(\tau_1 \times \tau_2)) \to (Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ is a soft fuzzy $C^\#$-continuous product map.*

Proof. For each soft fuzzy C-open set (μ, M) in $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$, we have $\mu^{-1}((\alpha, 1])$ is a C-open set in $\sigma_1 \times \sigma_2$ for all $\alpha \in [0, 1]$ and by hypothesis $(f_1 \times f_2)^{-1}(\mu^{-1}(\alpha, 1])$ is a C-open set in $\tau_1 \times \tau_2$. Then, $(\mu \circ (f_1 \times f_2))^{-1}(\alpha, 1]$ is a C-open set in $\tau_1 \times \tau_2$, and also $(f_1 \times f_2)^{-1}(M) \subseteq X_1 \times X_2$ is a C-open in $X_1 \times X_2$. Therefore, $< (\mu \circ (f_1 \times f_2)), (f_1 \times f_2)^{-1}(M) >$ is a soft fuzzy C-open set in $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$. Now,

$$\begin{aligned} (\widetilde{f_1 \times f_2})^{-1}(\mu, M) &= (f_1 \times f_2)^{-1}(\mu, M) \\ &= ((f_1 \times f_2)^{-1}(\mu), (f_1 \times f_2)^{-1}(M)) \\ &= ((\mu \circ (f_1 \times f_2)), (f_1 \times f_2)^{-1}(M)) \end{aligned}$$

Thus, $(\widetilde{f_1 \times f_2})^{-1}(\mu, M)$ is a soft fuzzy C-open set in $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$. Hence, $(\widetilde{f_1 \times f_2})$ is a soft fuzzy $C^\#$-continuous product map.

The converse part is clear by using the definition of the weakly induced soft fuzzy product $C^{\#}$-space. ■

Definition 4.3. Let $(X_1 \times X_2, T_1 \times T_2)$ and $(Y_1 \times Y_2, S_1 \times S_2)$ be any two product topological spaces. A surjective function $f : X \to Y$ is said to be *C-quotient product map*, if the inverse image of every product C-open set in $(Y_1 \times Y_2, S_1 \times S_2)$ is a product C-open set in $(X_1 \times X_2, T_1 \times T_2)$.

Proposition 4.8. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be any two maps. Let $(X_1 \times X_2, \tau_1 \times \tau_2)$ and $(Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ be any two product topological spaces. Then, $f_1 \times f_2 : (X_1 \times X_2, \tau_1 \times \tau_2) \to (Y_1 \times Y_2, \sigma_1 \times \sigma_2)$ is a C-quotient product map iff $\widetilde{f_1 \times f_2} : (X_1 \times X_2, \omega(\tau_1 \times \tau_2)) \to (Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ is also a soft fuzzy $C^{\#}$-quotient product map.*

Proof. By the definition of a weakly induced soft fuzzy product C-space, A is a C-open set in $(\sigma_1 \times \sigma_2)$ iff (χ_A, A) is a soft fuzzy C-open set in $((Y_1 \times Y_2), \omega(\sigma_1 \times \sigma_2))$. Now,

$$\begin{aligned}
(\widetilde{f_1 \times f_2})^{-1}(\chi_A, A) &= (f_1 \times f_2)^{-1}(\chi_A, A) \\
&= ((f_1 \times f_2)^{-1}(\chi_A), (f_1 \times f_2)^{-1}(A)) \\
&= (\chi_A \circ (f_1 \times f_2), (f_1 \times f_2)^{-1}(A)) \\
&= (\chi_{(f_1 \times f_2)^{-1}(A)}, (f_1 \times f_2)^{-1}(A))
\end{aligned}$$

That is, $(\widetilde{f_1 \times f_2})^{-1}(\chi_A, A) = (\chi_{(f_1 \times f_2)^{-1}(A)}, (f_1 \times f_2)^{-1}(A))$ is the soft fuzzy C-open set in $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$. Hence, $(f_1 \times f_2)^{-1}(A)$ is a C-open set in $\tau_1 \times \tau_2$. Therefore, $f_1 \times f_2$ is a C-quotient product map.

Conversely, let (μ, M) be a soft fuzzy C-open set in $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ iff $\mu^{-1}(\alpha, 1]$ is a C-open set in $\sigma_1 \times \sigma_2$, and $M \subseteq Y_1 \times Y_2$ is a C-open set in $\sigma_1 \times \sigma_2$, for all $\alpha \in [0, 1]$. By the hypothesis, $(f_1 \times f_2)^{-1}(\mu^{-1}(\alpha, 1])$ is a C-open set in $\tau_1 \times \tau_2$ and $(f_1 \times f_2)^{-1}(M) \subseteq X_1 \times X_2$ is a C-open set in $\tau_1 \times \tau_2$, for each $\alpha \in [0, 1]$. That is, $((\widetilde{f_1 \times f_2})^{-1}(\mu), (\widetilde{f_1 \times f_2})^{-1}(M)) = (\widetilde{f_1 \times f_2})^{-1}(\mu, M)$ is a soft fuzzy C-open set in $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$. Hence, $\widetilde{f_1 \times f_2}$ is also a soft fuzzy $C^{\#}$-quotient product map. ■

Definition 4.4. Let $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$ and $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ be any two soft fuzzy product C-spaces and $f_1 \times f_2$ be a soft fuzzy product map from $X_1 \times X_2$ to $Y_1 \times Y_2$. Then, $f_1 \times f_2$ is called as *a soft fuzzy $C^{\#}$-homeomorphism* from $X_1 \times X_2$ to $Y_1 \times Y_2$, if

(i) $f_1 \times f_2$ is a soft fuzzy $C^{\#}$-continuous product function.

(ii) $f_1 \times f_2$ is a soft fuzzy bijective product function.

(iii) Inverse of $f_1 \times f_2$ is also soft fuzzy $C^{\#}$-continuous product function.

Proposition 4.9. *Let $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$ and $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ be two weakly induced soft fuzzy product C-spaces, and $(\widetilde{f_1 \times f_2})$ be a soft fuzzy $C^{\#}$-continuous map from $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$ onto $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$. If there exists a soft fuzzy $C^{\#}$-continuous map $(\widetilde{g_1 \times g_2})$ from $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ to $(X_1 \times X_2, \omega(\tau_1 \times \tau_2))$ such that $(\widetilde{f_1 \times f_2}) \circ (\widetilde{g_1 \times g_2}) = 1_{Y_1 \times Y_2}$, then $(Y_1 \times Y_2, \omega(\sigma_1 \times \sigma_2))$ is soft fuzzy $C^{\#}$-homeomorphic with $(X_1 \times X_2) \mid R$, where R is the equivalence relation.*

Proof. Since $(\widetilde{f_1 \times f_2}) \circ (\widetilde{g_1 \times g_2}) = 1_{Y_1 \times Y_2}$, then by using all the above propositions, we have $(f_1 \times f_2) \circ (g_1 \times g_2) = 1_{Y_1 \times Y_2}$. Then, the map $h_1 \times h_2 : (X_1 \times X_2) \mid R \to Y_1 \times Y_2$ induced by $f_1 \times f_2$ is a $C^{\#}$-homeomorphism. Finally, by the above all propositions , $\widetilde{h_1 \times h_2}$ is clearly a soft fuzzy $C^{\#}$-homeomorphism. ∎

5. Compactification of $SFPC\mathfrak{st}(X_1 \times X_2)$

Definition 5.1. Let $X_1 \times X_2$ be a product space. Let $(X_1 \times X_2) \mid R$ be a quotient set on $(X_1 \times X_2$ with R, an equivalence relation. Then, the collection of all quotient sets on $X_1 \times X_2$, denoted by $\mathcal{Q}(X_1 \times X_2)$.

Definition 5.2. Let R_x be an equivalence relation. Then, $X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle} = \{[\langle x_1, x_2 \rangle], [\langle y_1, y_2 \rangle] : \langle x_1, x_2 \rangle \not R \langle z_1, z_2 \rangle, \langle y_1, y_2 \rangle R \langle z_1, z_2 \rangle, \forall \langle z_1, z_2 \rangle \in X_1 \times X_2\}$ is also a *quotient product set* on $X_1 \times X_2$.

Definition 5.3. Let $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$ be a soft fuzzy product C-space and A be a subset of $X_1 \times X_2$. If χ_A is a characteristic function of A in $X_1 \times X_2$, then

$$\mathfrak{st}(\tau_1 \times \tau_2)_A = \{(\lambda, N) \sqcap (\chi_A, A) : (\lambda, N) \in \mathfrak{st}(\tau_1 \times \tau_2)\}$$

is called as a *soft fuzzy product C-substructure*. Now, the pair $(A, \mathfrak{st}(\tau_1 \times \tau_2)_A)$ is called as a *soft fuzzy product C-subspace*.

Let $(X_1 \times X_2, \mathfrak{st}(\tau_1 \times \tau_2))$ be a non compact soft fuzzy product C-space. Associated with each $(\mu, M) \in \mathfrak{st}(\tau_1 \times \tau_2)$, we define $(\mu, M)^* = (\mu^*, M^*) \in SF(\mathcal{Q}(X_1 \times X_2))$. For each $(X_1 \times X_2) \mid R \in \mathcal{Q}(X_1 \times X_2)$

$$\mu^*((X_1 \times X_2) \mid R) = \begin{cases} \mu(< x_1, x_2 >), & \text{if } \exists < x_1, x_2 > \in X_1 \times X_2 \\ & \text{such that } X_1 \times X_2 \mid R = X_1 \times X_2 \mid R_{< x_1, x_2 >}; \\ \bigvee_{[< x_1, x_2 >] \in X_1 \times X_2 \mid R} \mu(< x_1, x_2 >), & \text{otherwise.} \end{cases}$$

$$M^* = \begin{cases} \phi, & \text{if } M = \phi; \\ \mathcal{Q}(X_1 \times X_2), & \text{if } M = X_1 \times X_2; \\ \{(X_1 \times X_2) \mid R_{< x_1, x_2 >_i}\}, & \text{if } < x_1, x_2 >_i \in M \subset X_1 \times X_2, i \in I. \end{cases}$$

Proposition 5.1. *Under the previous conditions the following identities hold.*

(i) $(0, \phi)^* = (0, \phi)$.

(ii) $(1_{X_1 \times X_2}, X_1 \times X_2)^* = (1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2))$.

Definition 5.4. A soft fuzzy product strong generalized topology on a non-empty set X is a family $\mathfrak{G}$ of soft fuzzy product sets in X satisfying the following axioms:

(1) $(0,\phi),(1,X)\in\tau$.

(2) For any family of soft fuzzy sets $(\lambda_j,N_j)\in\tau, j\in J$, $\Rightarrow \sqcup_{j\in J}(\lambda_j,N_j)\in\tau$.

Then, the pair $(X,\mathfrak{G})$is called as a *soft fuzzy product strong generalized topological space.* (in short, **SFPsGTS**)

Any soft fuzzy product set in $\mathfrak{G}$ is said to be a *soft fuzzy product $\mathfrak{G}$-open set* (in short, **SFPGOS**) in X.

The complement of SFPGOS in a SFPsGTS (X,τ) is called as a *soft fuzzy product $\mathfrak{G}$-closed set*, denoted **SFPGCS** in X.

Proposition 5.2. *Under the previous conditions the collection*

$$\mathfrak{B}^*=\{(\mu,M)^*:(\mu,M)\in\mathfrak{st}(\tau_1\times\tau_2)\}$$

is a base for some soft fuzzy product strong generalized topology on $\mathcal{Q}(X_1\times X_2)$.

Proof.

(i) For $(\mu_1,M_1),(\mu_2,M_2)\in\mathfrak{st}(\tau_1\times\tau_2)$ and $X_1\times X_2\mid R\in\mathcal{Q}(X_1\times X_2)$, we have $(\bigsqcup_{i\in I}(\mu_i,M_i))^*=(\bigvee_{i\in I}\mu_i,\bigcup_{i\in I}M_i)^*=((\bigvee_{i\in I}\mu_i)^*,(\bigcap_{i\in I}M_i)^*)$

$$(\bigvee_{i\in I}\mu_i)^*((X_1\times X_2)\mid R)$$

$$=\begin{cases}(\bigvee\mu_i)(\langle x_1,x_2\rangle), & \text{if } \exists\,\langle x_1,x_2\rangle\in X_1\times X_2 \\ & \text{such that } (X_1\times X_2)\mid R=(X_1\times X_2)\mid R_{\langle x_1,x_2\rangle}\ ; \\ \bigvee_{[\langle x_1,x_2\rangle]\in(X_1\times X_2)\mid R}(\bigvee_{i\in I}\mu_i)(\langle x_1,x_2\rangle), & \text{otherwise.}\end{cases}$$

$$=\begin{cases}\bigvee_{i\in I}\mu_i(\langle x_1,x_2\rangle), & \text{if } \exists\,\langle x_1,x_2\rangle\in X_1\times X_2 \\ & \text{such that } (X_1\times X_2)\mid R=(X_1\times X_2)\mid R_{\langle x_1,x_2\rangle}\ ; \\ \bigvee_{i\in I}\bigvee_{[\langle x_1,x_2\rangle]\in(X_1\times X_2)\mid R}\mu_i(\langle x_1,x_2\rangle), & \text{otherwise.}\end{cases}$$

$$=\bigvee_{i\in I}\mu_i^*(X_1\times X_2)\mid R).$$

Thus $(\bigvee_{i\in I}\mu_i)^*=\bigvee_{i\in I}\mu_i^*$.

Now,

$$(M_1\cap M_2)^*=\begin{cases}\phi, & \text{if } M_1\cap M_2=\phi; \\ \mathcal{Q}(X_1\times X_2), & \text{if } M_1\cap M_2=X_1\times X_2; \\ X_1\times X_2\mid R_{\langle x_1,x_2\rangle}, & \text{if } \langle x_1,x_2\rangle\in M_1\cap M_2\subset X_1\times X_2.\end{cases}$$

$$= \begin{cases} \phi, & \text{if } M_1 = \phi \text{ and } M_2 = \phi \text{ or } M_1 \neq \phi \text{ and } M_2 \neq \phi; \\ \mathcal{Q}(X_1 \times X_2), & \text{if } M_1 = X_1 \times X_2 \text{ and } M_2 = X_1 \times X_2; \\ X_1 \times X_2 \mid R_{<x_1,x_2>}, & \text{if } \langle x_1, x_2 \rangle \in M_1 \subset X_1 \times X_2 \\ & \text{and } \langle x_1, x_2 \rangle \in M_2 \subset X_1 \times X_2. \end{cases}$$

$$= M_1^* \cap M_2^*$$

Therefore $((\mu_1, M_1) \sqcap (\mu_2, M_2))^* = (\mu_1, M_1)^* \sqcap (\mu_2, M_2)^*$.

Thus, $\mathfrak{b}^*$ forms a base for $\mathcal{Q}(X_1 \times X_2)$. ■

Definition 5.5. The soft fuzzy product C-space, generated by the base $\mathfrak{b}^*$ of soft fuzzy product C-open sets, is denoted by $(\tau_1 \times \tau_2)^* = \tau_1^* \times \tau_2^*$.

Definition 5.6. Let $\mathfrak{q} : X_1 \times X_2 \to Q(X_1 \times X_2)$ defined by

$$\mathfrak{q}(< x_1, x_2 >) = X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}$$

for each $\langle x_1, x_2 \rangle \in X_1 \times X_2$.

Proposition 5.3. *Under the previous conditions, $\mathfrak{q}(X_1 \times X_2)$ is soft fuzzy Cst-dense in $(\mathcal{Q}\ (X_1 \times X_2), (\tau_1 \times \tau_2)^*)$, that is $C\text{-}cl_{(\tau_1 \times \tau_2)^*}(\mathfrak{q}(1_{X_1 \times X_2}), X_1 \times X_2) = (1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2))$.*

Proof. Given $(\mu, M) \in SFPCst(X_1 \times X_2)$, we have $\mathfrak{q}(\mu, M) \in SF(\mathcal{Q}(X_1 \times X_2))$. Then for each $(\mu, M) \in SFPCst(X_1 \times X_2)$. Now $\mathfrak{q}(\mu, M) = (\mathfrak{q}(\mu), \mathfrak{q}(M))$

$$q(\mu)(X_1 \times X_2 \mid R) = \begin{cases} \sup\limits_{\langle x_1, x_2 \rangle \in \mathfrak{q}^{-1}(X_1 \times X_2 \mid R)} \mu(\langle x_1, x_2 \rangle), & \text{if } \mathfrak{q}^{-1}(X_1 \times X_2 \mid R) \neq \phi\ ; \\ 0, & \text{if } \mathfrak{q}^{-1}(X_1 \times X_2 \mid R) = \phi. \end{cases}$$

$$= \begin{cases} \mu(< x_1, x_2 >), & \text{if } \exists < x_1, x_2 > \in X_1 \times X_2 \text{ such that} \\ & X_1 \times X_2 \mid R = X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}; \\ 0, & \text{if } \forall \langle x_1, x_2 \rangle \in X_1 \times X_2, X_1 \times X_2 \mid R \neq X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}. \end{cases}$$

$$\mathfrak{q}(M) = \{\mathfrak{q}(< x_1, x_2 >), \forall < x_1, x_2 > \in M\}.$$

Now $C\text{-}cl_{(\tau_1 \times \tau_2)^*}(\mathfrak{q}(1_{X_1 \times X_2}, X_1 \times X_2))$

$$= \begin{cases} \text{C-cl}_{(\tau_1 \times \tau_2)^*}(1_{\mathcal{Q}(X_1, X_2)}, \mathcal{Q}(X_1 \times X_2)), & \text{if } \exists \langle x_1, x_2 \rangle \in X_1 \times X_2 \text{ such that} \\ & X_1 \times X_2 \mid R = X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}\ ; \\ \text{C-cl}_{(\tau_1 \times \tau_2)^*}(0, \phi), & \text{if } \forall \langle x_1, x_2 \rangle \in X_1 \times X_2, X_1 \times X_2 \mid \\ & R \neq X_1 \times X_2 \mid R_{\langle x_1, x_2 \rangle}. \end{cases}$$

Let $(\delta, L)^* = \sqcap_{j \in J} (\mu_j, M_j)^* = C\text{-}cl_{(\tau_1 \times \tau_2)^*}(\mathfrak{q}(1_{X_1 \times X_2}, X_1 \times X_2))$.

Since $\mathfrak{q}(1_{X_1 \times X_2}, X_1 \times X_2) \sqsubseteq (\delta, L)^*$, we have for each $\langle x_1, x_2 \rangle \in X_1 \times X_2$, $\delta^*((X_1 \times X_2) \mid R_{\langle x_1, x_2 \rangle}) \geq \mathfrak{q}(1_{X_1 \times X_2})((X_1 \times X_2) \mid R_{\langle x_1, x_2 \rangle}) = 1$ and $L^* \supseteq \mathfrak{q}(X_1 \times X_2)$. This implies that, $\delta^*((X_1 \times X_2) \mid R_{<x_1, x_2>}) = 1$ and $L^* = \mathcal{Q}(X_1 \times X_2)$. Now, for each $\langle x_1, x_2\rangle \in X_1 \times X_2$ and $j \in J$, $\wedge\mu_j^*((X_1 \times X_2) \mid R_{\langle x_1,x_2\rangle}) = 1$ and $\cap M_j^* = \mathcal{Q}(X_1 \times X_2)$. It implies $\mu_j^*((X_1 \times X_2) \mid R_{\langle x_1,x_2\rangle}) = 1$ and $M_j^* = \mathcal{Q}(X_1 \times X_2)$. Thus, for each $\langle x_1, x_2\rangle \in X_1 \times X_2$ and $j \in J$, $\mu_j(\langle x_1, x_2\rangle) = 1$ and $M_j = X_1 \times X_2$. That is, for each $j \in J$, $(\mu_j, M_j) = (1_{X_1\times X_2}, X_1 \times X_2))$. Now, we conclude that, $(\delta^*, L^*) = \sqcap_{j\in J}(\mu_j, M_j)^* = \sqcap_{j\in J}(1_{X_1\times X_2}, X_1 \times X_2))^* = (1_{\mathcal{Q}(X_1\times X_2)}, \mathcal{Q}(X_1 \times X_2))$. This implies that, $\mathfrak{q}(1_{X_1\times X_2}, X_1 \times X_2)$ is a soft fuzzy Cst-dense set in $(\mathcal{Q}\ (X_1 \times X_2), (\tau_1 \times \tau_2)^*)$. ■

Proposition 5.4. *The function* $\mathfrak{q}$ *is a soft fuzzy Cst-embedding of* $X_1 \times X_2$ *into* $\mathcal{Q}(X_1 \times X_2)$.

Proof.

(i) $\mathfrak{q}$ *is a one to one function*:

If $\langle x_1, x_2\rangle \neq \langle y_1, y_2\rangle$, we have $R_{\langle x_1,x_2\rangle} \neq R_{\langle y_1,y_2\rangle}$. Let $(\langle x_1, x_2\rangle)_\alpha, \{< x_1, x_2 >\}) \neq (\langle y_1, y_2\rangle_\beta, \{\langle y_1, y_2\rangle\})$ be two soft fuzzy points.

(a) If $\langle x_1, x_2\rangle \neq \langle y_1, y_2\rangle$ for each $X_1 \times X_2 \mid R \in \mathcal{Q}(X_1 \times X_2)$. We have

$$\mathfrak{q}(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) = (X_1 \times X_2 \mid R_{\langle x_1,x_2\rangle_\alpha}, \{X_1 \times X_2 \mid R_{\langle x_1,x_2\rangle}\}).$$

Similarly, $\mathfrak{q}(\langle y_1, y_2\rangle_\beta, \{\langle y_1, y_2\rangle\})$ and it is clear that

$$\mathfrak{q}(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) \neq \mathfrak{q}(\langle y_1, y_2\rangle_\beta, \{\langle y_1, y_2\rangle\}).$$

(b) If $\langle x_1, x_2\rangle = \langle y_1, y_2\rangle$, then $\alpha \neq \beta$ and therefore, clearly

$$\mathfrak{q}(\langle x_1, x_2\rangle_\alpha, \{\langle x_1, x_2\rangle\}) \neq \mathfrak{q}(\langle y_1, y_2\rangle_\beta, \{\langle y_1, y_2\rangle\}).$$

Hence $\mathfrak{q}$ is one to one.

(ii) $\mathfrak{q}$ *is soft fuzzy product Cst-continuous*:

For each $(\mu, M)^* \in \mathfrak{b}^*$ and $\langle x_1, x_2\rangle \in X_1 \times X_2$, we have

$$\begin{aligned}\mathfrak{q}^{-1}(\mu, M)^* &= \mathfrak{q}^{-1}(\mu^*, M^*)\\ &= (\mathfrak{q}^{-1}(\mu^*), \mathfrak{q}^{-1}(M^*))\\ &= (\mu^* \circ \mathfrak{q}, \mathfrak{q}^{-1}(M^*))\end{aligned}$$

where

$$\begin{aligned}\mu^* \circ \mathfrak{q}(< x_1, x_2 >) &= \mu^*(\mathfrak{q}(< x_1, x_2 >))\\ &= \mu^*(X_1 \times X_2 \mid R_{<x_1,x_2>})\\ &= \mu(< x_1, x_2 >)\end{aligned}$$

and $\mathfrak{q}^{-1}(M^*) = M$. Thus $\mathfrak{q}^{-1}(\mu, M)^* = (\mu, M) \in St(\tau_1 \times \tau_2)$. Hence $\mathfrak{q}$ is soft fuzzy product Cst-continuous.

(iii) $\mathfrak{q}$ is a soft fuzzy product Cst-open function on $\mathcal{Q}(X_1 \times X_2)$:
For each $X_1 \times X_2 \mid R \in \mathcal{Q}(X_1 \times X_2)$ and $(\mu, M) \in St(\tau_1 \times \tau_2)$.

$$(\mu^*, M^*) \sqcap (\chi_{\mathfrak{q}(X_1 \times X_2)}, \{\mathfrak{q}(X_1 \times X_2)\}) = (\mu^* \wedge \chi_{\mathfrak{q}(X_1 \times X_2)}, M^* \cap \{\mathfrak{q}(X_1 \times X_2)\}) = (\mathfrak{q}(\mu), \mathfrak{q}(M)) = \mathfrak{q}(\mu, M) \in (\tau_1 \times \tau_2)^*$$

Thus $\mathfrak{q}$ is a soft fuzzy product Cst-open function. Hence $\mathfrak{q}$ is a soft fuzzy Cst-embedding of $X_1 \times X_2$ into $\mathcal{Q}(X_1 \times X_2)$. ∎

Proposition 5.5. *The soft fuzzy quotient product space* $(\mathcal{Q}(X_1 \times X_2), (\tau_1 \times \tau_2)^*)$ *is soft fuzzy product Cst-compact.*

Proof. Let $\mathfrak{F} = \{(\lambda_i^*, N_i^*) \in \tau_1^* \times \tau_2^* : (\lambda_i, N_i) \in St(\tau_1 \times \tau_2) \text{ for } i \in J\}$ be a soft fuzzy product Cst-open cover of $\mathcal{Q}(X_1 \times X_2)$. That is,

$$\sqcup_{i \in J}(\lambda_i^*, N_i^*) = (1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2)).$$

By definition of (λ_i^*, N_i^*), $\sqcup_{i \in F}(\lambda_i^*, N_i^*) \sqsubseteq (1_{\mathcal{Q}(X_1 \times X_2)}, \mathcal{Q}(X_1 \times X_2))$, for some finite subfamily F of J. Thus $\mathfrak{F}$ has a soft fuzzy product finite subcover.

Hence, $(\mathcal{Q}(X_1 \times X_2), (\tau_1 \times \tau_2)^*)$ is soft fuzzy product Cst-compact. ∎

Conclusion. Cst-compactification of the category of all product topological space can be done, using section- 5.

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